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INTUITIONISTIC FUZZY EQUIVALENCE RELATIONS

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Abstract. We study some properties of intuitionistic fuzzy equivalence relations. Also we introduce the concepts of intuitionistic fuzzy transitive closures and level sets of an intuitionistic fuzzy relation and we investigate some of their properties.

0. Introduction

The subject of fuzzy sets as an approach to a mathematical representation of vagueness in everyday language was introduced by L.A.Zadeh in 1965[15]. He generalized the idea of the characteristic function of a subset of a set X by defining a fuzzy subset of X as a map from X into $[0,1]$.

L.A.Zadeh[16] introduced also a concept of a fuzzy relation naturally, as a generalization of crisp relations in fuzzy set theory. In particular, fuzzy analogs have been defined for the reflexivity, (anti-)symmetry and transitivity of relations ; in terms of these, fuzzy analogs of equivalence and order relations have been introduced. The notion of a fuzzy relation can model situations where interactions between elements are more or less strong. It plays an important role in the theory of fuzzy sets and their applications. Also, Murali [13] and Nemitz [14] investigated many properties of fuzzy relations.

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As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues [6,7,9], and Lee and Lee [12] introduced the concept of intuitionistic fuzzy topological spaces. Also, Banerjee and Basnet [2], Biswas [4], Hur and his colleagues [10,11] applied to group theory using intuitionistic fuzzy sets. In 1996, Bustince and Burillo [5] introduced the concept of intuitionistic fuzzy relations and studied some of it's properties. In 2003, Deschrijver and Kerre [8] investigated some properties of the composition of intuitionistic fuzzy relations.

In this paper, we study some properties of intuitionistic fuzzy equivalence relations. Also we introduce the concepts of intuitionistic fuzzy transitive closures and level sets of an intuitionistic fuzzy relation and we investigate some of their properties.

1. Preliminaries

In this section, we list some basic concepts which are needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I. And for a lattice, refer to [3]. For any ordinary relation R on a set X, we will denote the characteristic function of R as χ_R.

Definition 1.1. [1,6]. Let X be a nonempty set. A complex mapping $A = (\mu_A,\ \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively.In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic* *fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2. [1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ iff $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3. [6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.
(2) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

2. Intuitionistic fuzzy equivalence relations

Definition 2.1. [5]. Let X be a set. Then a complex mapping $R = (\mu_R,\ \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x,y) + \nu_R(x,y) \leq 1$ for each $(x,y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 2.2. [5]. Let $R \in$ IFR(X). Then the *inverse* of R, denoted by R^{-1}, is defined by $R^{-1}(x,y) = R(y,x)$ for any $x, y \in X$.

Definition 2.3. [5,8]. Let X be a set and let $P, Q \in \text{IFR}(X)$. Then the *composition* $Q \circ P$ of P and Q, is defined as follows : for any $x, y \in X$,

$$\mu_{Q\circ P}(x,y) = \bigvee_{z\in X} [\mu_P(x,z) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q\circ P}(x,y) = \bigwedge_{z\in X} [\nu_P(x,z) \vee \nu_Q(z,y)].$$

The following is the immediate result of Definitions 2.2 and 2.3.

Definition 2.4. Let X be a set and let R_1, R_2, R_3, Q_1, $Q_2 \in \text{IFR}(X)$. Then

(1) $(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3)$.
(2) If $R_1 \subset R_2$ and $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_2 \circ Q_2$.
In particular, if $Q_1 \subset Q_2$, then $R_1 \circ Q_1 \subset R_1 \circ Q_2$.
(3) $R_1 \circ (R_2 \cup R_3) = R_1 \circ R_2 \cup R_1 \circ R_3$.
(4) $R_1 \circ (R_2 \cap R_3) = R_1 \circ R_2 \cap R_1 \circ R_3$.
(5) If $R_1 \subset R_2$ then $R_1^{-1} \subset R_2^{-1}$.
(6) $(R^{-1})^{-1} = R$ and $(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$.
(7) $(R_1 \cup R_2)^{-1} = R_1^{-1} \cup R_2^{-1}$.
(8) $(R_1 \cap R_2)^{-1} = R_1^{-1} \cap R_2^{-1}$.

The following is the immediate result of Proposition 2.4(1).

Proposition 2.5. Let P and Q be any intuitionistic fuzzy relations on a set X. If $Q \circ P = P \circ Q$, then $(Q \circ P) \circ (Q \circ P) = (Q \circ Q) \circ (P \circ P)$.

Definition 2.6. [4]. An Intuitionistic fuzzy relation R on a set X is called an *intuitionistic fuzzy equivalence relation* (in short, IFER) on X if it satisfies the following conditions :

(i) it is *intuitionistic fuzzy reflexive*, i.e., $R(x,x) = (1,0)$ for each $x \in X$.
(ii) it is *intuitionistic fuzzy symmetric*, i.e., $R^{-1} = R$.
(iii) it is *intuitionistic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

The following is the immediate result of Definition 2.6.

Proposition 2.7. Let X be a set and let $R, Q \in$ IFR(X).

(1) If R is intuitionistic fuzzy reflexive [resp. symmetric, transitive], then R^{-1} is intuitionistic fuzzy reflexive [resp. symmetric, transitive].
(2) If R is intuitionistic fuzzy reflexive [resp. symmetric, transitive], then $R \circ R$ is intuitionistic fuzzy reflexive [resp. symmetric, transitive].
(3) If R is intuitionistic fuzzy reflexive, then $R \subset R \circ R$.
(4) If R is intuitionistic fuzzy symmetric, then $R \cup R^{-1}$ and $R \cap R^{-1}$ are symmetric and $R \circ R^{-1} = R^{-1} \circ R$.
(5) If R and Q are intuitionistic fuzzy reflexive [resp. symmetric, transitive], then $R \cap Q$ is intuitionistic fuzzy reflexive [resp. symmetric, transitive].
(6) If R and Q are intuitionistic fuzzy symmetric, then $R \cup Q$ is intuitionistic fuzzy symmetric.

Remark 2.8. (1) By (1), (2) and (5) of Proposition 2.7, it is clear that if $R, Q \in$ IFE(X), then $R^{-1}, R \circ R$, $R \cap Q \in$ IFE(X).
(2) If $R \in$ IFE(X), then $<> R$, $[\,]R \in$ IFE(X).
(3) If μ_R is a fuzzy equivalence relation on X, then $(\mu_R, \mu_R^c) \in$ IFE(X).
(3′) If $R \in$ IFE(X), then μ_R and ν_R^c are fuzzy equivalence relations on X.
(4) Let R be an ordinary relation on a set X. Then R is an equivalence relation on X if and only if $(\chi_R, \chi_{R^c}) \in$ IFE(X).

The following two results are easily seen.

Proposition 2.9. Let X be a set. If $R \in$ IFE(X), then $R \circ R = R$.

Proposition 2.10. Let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a nonempty family of IFERs on a set X. Then $\bigcap_{\alpha\in\Gamma} R_\alpha \in \text{IFE}(X)$. However, in general, $\bigcup_{\alpha\in\Gamma} R_\alpha$ need not be an IFER on X.

Example 2.11. Let $X = \{a, b, c\}$. Let P and Q be the IFRs on X represented by matrices shown in Figures, respectively :

P	a	b	c	Q	a	b	c
a	(1,0)	(0.8,0.1)	(0.7,0.2)	a	(1,0)	(0.8,0.2)	(1,0)
b	(0.8,0.1)	(1,0)	(0.7,0.2)	b	(0.8,0.2)	(1,0)	(0.7,0.2)
c	(0.7,0.2)	(0.7,0.2)	(1,0)	c	(1,0)	(0.7,0.2)	(1,0)

Then clearly $P, Q \in$ IFE(X) and $P \cup Q$ is the IFR on X represented by the following matrix :

$P \cup Q$	a	b	c
a	(1,0)	(0.8,0.1)	(1,0)
b	(0.8,0.1)	(1,0)	(0.7,0.2)
c	(1,0)	(0.7,0.2)	(1,0)

On the other hand,

$$\mu_{(P\cup Q)\circ(P\cup Q)}(b, c) = 0.8 > 0.7 = \mu_{P\cup Q}(b, c)$$

and

$$\nu_{(P\cup Q)\circ(P\cup Q)}(b, c) = 0.1 < 0.2 = \nu_{P\cup Q}(b, c).$$

Thus $(P \cup Q) \circ (P \cup Q) \not\subset P \cup Q$. So $P \cup Q$ is not intutionistic fuzzy transitive. Hence $P \cup Q \notin$ IFE(X). □

Definition 2.12. We define two IFRs $\triangle$ and $\bigtriangledown$ on a set X as follows, respectively : for any $x, y \in X$,

$$\triangle(x, y) = \begin{cases} (1, 0) & \text{if } x = y \\ (0, 1) & \text{if } x \neq y, \end{cases}$$

and

$$\bigtriangledown(x, y) = (1, 0).$$

It is clear that $\triangle$, $\bigtriangledown \in$ IFE(X).

Proposition 2.13. Let P and Q be intuitionistic fuzzy reflexive relations on a set X. Then $Q \circ P$ is also an intuitionistic fuzzy reflexive relation on X.

Proof. Let $x \in X$. Then

$$\begin{aligned} \mu_{Q\circ P}(x, x) &= \bigvee_{t\in X} [\mu_P(x, t) \wedge \mu_Q(t, x)] \\ &\geq \mu_P(x, x) \wedge \mu_Q(x, x) \\ &\quad \text{(Since } P \text{ and } Q \text{ are intuitionistic fuzzy reflexive)} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \nu_{Q\circ P}(x, x) &= \bigwedge_{t\in X} [\nu_P(x, t) \vee \nu_Q(t, x)] \\ &\leq \nu_P(x, x) \vee \nu_Q(x, x) = 0. \end{aligned}$$

Thus $Q \circ P(x, x) = (1, 0)$ for each $x \in X$. Hence $Q \circ P$ is an intuitionistic fuzzy reflexive relation on X. □

Proposition 2.14. Let X be a set and let $P, Q \in$ IFE(X). If $Q \circ P = P \circ Q$, then $P \circ Q \in$ IFE(X).

Proof. Let $x \in X$. Since P and Q are intutionistic fuzzy reflexive,

$$\mu_{P\circ Q}(x, x) = \bigvee_{y\in X} [\mu_Q(x, y) \wedge \mu_P(y, x)] \geq \mu_Q(x, x) \wedge \mu_P(x, x) = 1$$

and

$$\nu_{P\circ Q}(x, x) = \bigwedge_{y\in X} [\nu_Q(x, y) \vee \nu_P(y, x)] \leq \nu_Q(x, x) \vee \nu_P(x, x) = 0.$$

Thus $P \circ Q(x,x) = (1,0)$. So $P \circ Q$ is intutionistic fuzzy reflexive. Let $x, y \in S$. Then

$$\begin{aligned}
\mu_{P \circ Q}(x,z) &= \bigvee_{y \in X} [\mu_Q(x,y) \wedge \mu_P(y,z)] = \bigvee_{y \in X} [\mu_P(z,y) \wedge \mu_Q(y,x)] \\
&\quad \text{(Since } P \text{ and } Q \text{ are intutionistic fuzzy symmetric)} \\
&= \mu_{Q \circ P}(z,x) = \mu_{P \circ Q}(z,x) \text{ (Since } P \circ Q = Q \circ P)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{P \circ Q}(x,z) &= \bigwedge_{y \in X} [\nu_Q(x,y) \vee \nu_P(y,z)] = \bigwedge_{y \in X} [\nu_P(z,y) \vee \nu_Q(y,x)] \\
&= \nu_{Q \circ P}(z,x) = \nu_{P \circ Q}(z,x).
\end{aligned}$$

So $P \circ Q$ is intutionistic fuzzy symmetric. On the other hand,

$$\begin{aligned}
(P \circ Q) \circ (P \circ Q) &= (P \circ P) \circ (Q \circ Q) \text{ (By Proposition 2.5)} \\
&\subset P \circ Q. \\
&\quad \text{(Since } P \text{ and } Q \text{ are intutionistic fuzzy transitive)}
\end{aligned}$$

Hence $P \circ Q \in$ IFE(X). □

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows : for each $x \in X$,

$$Ra(x) = R(a,x).$$

Then clearly $Ra \in \text{IFS}(X)$. The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of* X *by* R and denoted by X/R.

Theorem 2.15. *Let* X *be a set and let* $R \in$ *IFE*(X)*. Then the following hold :*

(1) $Ra = Rb$ *if and only if* $R(a,b) = (1,0)$ *for any* $a, b \in X$.

(2) $R(a,b) = (0,1)$ *if and only if* $Ra \cap R_b = 0_\sim$ *for any* $a, b \in X$.

(3) $\bigcup_{a \in X} Ra = 1_{\sim}$.

(4) *There exists the surjection* $p : X \to X/R$ *(called the natural mapping) defined by* $p(x) = Rx$ *for each* $x \in X$.

Proof. (1) $(\Rightarrow)$: Suppose $Ra = Rb$. Since R is an intuitionistic fuzzy equivalence relation, $R(a,b) = Ra(b) = Rb(b) = R(b,b) = (1,0)$. Hence $R(a,b) = (1,0)$.
$(\Leftarrow)$: Suppose $R(a,b) = (1,0)$. Then $\mu_R(a,b) = 1$ and $\nu_R(a,b) = 0$. Let $x \in X$. Then

$$\begin{aligned} \mu_{Ra}(x) &= \mu_R(a,x) \\ &\geq \bigvee_{z \in X} [\mu_R(a,z) \wedge \mu_R(z,x)] \\ &\quad \text{(Since R is intutionistic fuzzy transitive)} \\ &\geq \mu_R(a,b) \wedge \mu_R(b,x) = 1 \wedge \mu_R(b,x) = \mu_R(b,x) = \mu_{Rb}(x) \end{aligned}$$

and

$$\begin{aligned} \nu_{Ra}(x) &= \nu_R(a,x) \leq \bigwedge_{z \in X} [\nu_R(a,z) \vee \nu_R(z,x)] \leq \nu_R(a,b) \vee \nu_R(b,x) \\ &= 0 \vee \nu_R(b,x) = \nu_R(b,x) = \nu_{Rb}(x). \end{aligned}$$

Thus $Ra \supset Rb$. By the similar arguments, we have $Ra \subset Rb$. Hence $Ra = Rb$.
The proofs of (2), (3) and (4) are easy. This completes the proof. □

Definition 2.16. Let R be a set and let $R \in \text{IFE}(X)$. For each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$, let

$$R^{(\lambda,\mu)} = \{(a,b) \in X \times X : \mu_R(a,b) \geq \lambda \textit{ and } \nu_R(a,b) \leq \mu\}.$$

This set is called the (λ, μ)-*level subset* of R.

It is clear that $R^{(\lambda,\mu)}$ is a relation on X.

Theorem 2.17. *Let* X *be a set and let* $R \in IFR(X)$. *Then* $R \in IFE(X)$ *if and only if* $R^{(\lambda,\mu)}$ *is an equivalence relation on* X *for each* $(\lambda, \mu) \in I \times I$ *with* $\lambda + \mu \leq 1$.

Proof. $(\Rightarrow)$: Suppose $R \in \mathrm{IFE}(X)$ and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Let $x \in X$. Since R is intuitionistic fuzzy reflexive,

$$\mu_R(x,x) = 1 \geq \lambda \text{ and } \nu_R(x,x) = 0 \leq \mu.$$

Thus $(x,x) \in R^{(\lambda,\mu)}$. So $R^{(\lambda,\mu)}$ is reflexive. It is clear that $R^{(\lambda,\mu)}$ is symmetric. Suppose $(x,y) \in R^{(\lambda,\mu)}$ and $(y,z) \in R^{(\lambda,\mu)}$. Then

$$\mu_R(x,y) \geq \lambda, \nu_R(x,y) \leq \mu \text{ and } \mu_R(y,z) \geq \lambda, \nu_R(y,z) \leq \mu.$$

Since R is intuitionistic fuzzy transitive,

$$\mu_R(x,z) \;\geq\; \bigvee_{t \in X} [\mu_R(x,t) \wedge \mu_R(t,z)] \geq \mu_R(x,y) \wedge \mu_R(y,z) \geq \lambda$$

and

$$\nu_R(x,z) \;\leq\; \bigwedge_{t \in X} [\nu_R(x,t) \vee \nu_R(t,z)] \leq \nu_R(x,y) \vee \nu_R(y,z) \leq \mu.$$

Thus $(x,z) \in R^{(\lambda,\mu)}$. So $R^{(\lambda,\mu)}$ is transitive. Hence $R^{(\lambda,\mu)}$ is an equivalence relation on X.

$(\Leftarrow)$: Suppose the necessary condition holds. Let $x \in X$. Since $R^{(1,0)}$ is reflexive, $(x,x) \in R^{(1,0)}$. Then $R(x,x) = (1,0)$. Thus R is intuitionistic fuzzy reflexive. Assume that there exist $x, y \in X$ such that $R(x,y) = (\lambda, \mu), R(y,x) = (s,t)$ and $\lambda < s, \mu > t$. Since $(y,x) \in R^{(s,t)}$ and $R^{(s,t)}$ is symmetric, $(x,y) \in R^{(s,t)}$. Then $\mu_R(x,y) \geq s$ and $\nu_R(x,y) \leq t$. Thus $\lambda \geq s$ and $\mu \leq t$. This is a contradiction. So $R(x,y) = R(y,x)$ for any $x, y \in X$, i.e., R is intuitionistic fuzzy symmetric. Now, for any $x, y, z \in X$, let $R(x,y) = (\lambda, \mu)$ and $R(y,z) = (s,t)$. It is clear that $(\lambda \wedge s, \mu \vee t) \in I \times I$ such that $(\lambda \wedge s) + (\mu \vee t) \leq 1$. Since $R^{(\lambda \wedge s, \mu \vee t)}$ is transitive, $(x,z) \in R^{(\lambda \wedge s, \mu \vee t)}$. Then

$$\mu_R(x,z) \geq \lambda \wedge s = \mu_R(x,y) \wedge \mu_R(y,z)$$

and

$$\nu_R(x,z) \leq \mu \vee t = \nu_R(x,y) \vee \nu_R(y,z).$$

Thus

$$\mu_R(x,z) \geq \bigvee_{t\in X} [\mu_R(x,t) \wedge \mu_R(t,z)] = \mu_{R\circ R}(x,z)$$

and

$$\nu_R(x,z) \leq \bigwedge_{t\in X} [\nu_R(x,t) \vee \nu_R(t,z)] = \nu_{R\circ R}(x,z).$$

So $R \circ R \subset R$, i.e., R is intuitionistic fuzzy transitive. Hence $R \in$ IFE(X). This completes the proof. □

Definition 2.18. Let X be a set, let $R \in$ IFR(X) and let $(\lambda,\mu) \in [0,1) \times (0,1]$ and $\lambda + \mu \leq 1$. We define a complex mapping $R_{(\lambda,\mu)}$: $X \times X \to I \times I$ as follows : for an $x, y \in X$,

$$R_{(\lambda,\mu)}(x,y) = \begin{cases} (1,0) & \text{if } \mu_R(x,y) > \lambda \text{and } \nu_R(x,y) < \mu, \\ (0,1) & \text{if } \mu_R(x,y) \leq \lambda \text{and } \nu_R(x,y) \geq \mu. \end{cases}$$

The following is the immediate result of Definition 2.18.

Proposition 2.19. Let $P, Q \in$ IFR(X). Then

(1) $P = Q$ if and only if $P_{(\lambda,\mu)} = Q_{(\lambda,\mu)}$ for each $(\lambda,\mu) \in [0,1) \times (0,1]$ with $\lambda + \mu \leq 1$.

(2) For each $(\lambda,\mu) \in [0,1) \times (0,1]$ with $\lambda + \mu \leq 1$,

$$(P \cap Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \cap Q_{(\lambda,\mu)}, (P \cup Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \cup Q_{(\lambda,\mu)},$$
$$(P \circ Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \circ Q_{(\lambda,\mu)}, (P \vee Q)_{(\lambda,\mu)} = P_{(\lambda,\mu)} \vee Q_{(\lambda,\mu)}.$$

3. An intuitionistic fuzzy equivalence relation generated by an intuitionistic fuzzy relation

Definition 3.1. Let X be a set, let $R \in$ IFR(X) and let $\{R_\alpha\}_{\alpha\in\Gamma}$ be the family of all the IFERs on X containing R. Then $\bigcap_{\alpha\in\Gamma} R_\alpha$ is called the IFER *generated* by R and denoted by R^e.

It is easily seen that R^e is the smallest intuitionistic fuzzy equivalence relation containing R.

Definition 3.2. Let X be a set and let $R \in \text{IFR}(X)$. Then the *intutionsitic fuzzy transitive closure* of R, denoted by R^∞, is defined as follows :

$$R^\infty = \bigcup_{n \in \mathbb{N}} R^n, \text{ where } R^n = R \circ R \circ \cdots \circ R,$$

in which R occurs n times.

The following is the immediate result of Definition 3.2.

Proposition 3.3. Let X be a set and let $R \in \text{IFR}(X)$. Then

(1) R^∞ is the smallest intuitionistic fuzzy transitive relation on X containing R.
(2) If there exists $n \in \mathbb{N}$ such that $R^{n+1} = R^n$, then $R^\infty = R \cup R^2 \cup \cdots \cup R^n$.

Example 3.4. Let $X = \{a, b, c\}$ and let $R = (\mu_R, \nu_R)$ be the IFR on X defined as follows :

R	a	b	c
a	(0.8,0.1)	(1,0)	(0.1,0.9)
b	(0,1)	(0.4,0.5)	(0,1)
c	(0.3,0.7)	(0,1)	(0.2,0.7)

Then

R^2	a	b	c
a	(0.8,0.1)	(0.8,0.1)	(0.1,0.9)
b	(0,1)	(0.4,0.5)	(0,1)
c	(0.3,0.7)	(0.3,0.7)	(0.2,0.7) ,

R^3	a	b	c
a	(0.8,0.1)	(0.8,0.1)	(0.1,0.9)
b	(0,1)	(0.4,0.5)	(0,1)
c	(0.3,0.7)	(0.3,0.7)	(0.2,0.7)

Thus $R^2 = R^3$. So $R^\infty = R \cup R^2$. Moreover $R^\infty \circ R^\infty \subset R^\infty$,

R^∞	a	b	c
a	(0.8,0.1)	(1,0)	(0.1,0.9)
b	(0,1)	(0.4,0.5)	(0,1)
c	(0.3,0.7)	(0.3,0.7)	(0.2,0.7) ,

$R^\infty \circ R^\infty$	a	b	c
a	(0.8,0.1)	(0.8,0.1)	(0.1,0.9)
b	(0,1)	(0.4,0.5)	(0,1)
c	(0.3,0.7)	(0.3,0.7)	(0.2,0.7)

Hence $R^\infty = R \cup R^2$ is intuitionistic fuzzy transitive. □

Proposition 3.5. If R is intuitionistic fuzzy symmetric, then so is R^∞.

Proof. For $n \geq 1$ and $x, y \in X$.

$$\begin{aligned}\mu_{R^n}(x,y) &= \bigvee_{z_1,\cdots,z_{n-1}} [\mu_R(x,z_1) \wedge \mu_R(z_1,z_2) \wedge \cdots \wedge \mu_R(z_{n-1},y)] \\ &= \bigvee_{z_{n-1},\cdots,z_1} [\mu_R(y,z_{n-1}) \wedge \cdots \wedge \mu_R(z_1,x)] = \mu_{R^n}(y,x)\end{aligned}$$

and

$$\begin{aligned}\nu_{R^n}(x,y) &= \bigwedge_{z_1,\cdots,z_{n-1}} [\nu_R(x,z_1) \vee \nu_R(z_1,z_2) \vee \cdots \vee \nu_R(z_{n-1},y)] \\ &= \bigwedge_{z_{n-1},\cdots,z_1} [\nu_R(y,z_{n-1}) \vee \cdots \vee \nu_R(z_1,x)] = \nu_{R^n}(y,x).\end{aligned}$$

Thus R^n is intuitionistic fuzzy symmetric for any $n \geq 1$. Hence R^∞ is intuitionistic fuzzy symmetric. □

Another proof : It is clear that $R^1 = R$ is intuitionistic fuzzy symmetric. Suppose R^k is intuitionistic fuzzy symmetric for $k > 1$. We show that R^{k+1} is intuitionistic fuzzy symmetric. Let $x, y \in X$. Then

$$\begin{aligned}\mu_{R^{k+1}}(x,y) &= \mu_{R \circ R^k}(x,y) \\ &= \bigvee_{z \in X} [\mu_{R^k}(x,z) \wedge \mu_R(z,y)] = \bigvee_{z \in X} [\mu_{R^k}(z,x) \wedge \mu_R(y,z)] \\ &= \bigvee_{z \in X} [\mu_{R^k}(y,z) \wedge \mu_R(z,x)] = \mu_{R^k \circ R}(y,x) = \mu_{R^{k+1}}(y,x)\end{aligned}$$

and

$$\begin{aligned}\nu_{R^{k+1}}(x,y) &= \nu_{R\circ R^k}(x,y)\\ &= \bigwedge_{z\in X}[\nu_{R^k}(x,z)\vee\nu_R(z,y)] = \bigwedge_{z\in X}[\nu_{R^k}(z,x)\vee\nu_R(y,z)]\\ &= \bigwedge_{z\in X}[\nu_{R^k}(y,z)\vee\nu_R(z,x)] = \nu_{R^k\circ R}(y,x) = \nu_{R^{k+1}}(y,x).\end{aligned}$$

So R^n is intuitionistic fuzzy symmetric for any $n \geq 1$. Hence R^∞ is intuitionistic fuzzy symmetric. □

Proposition 3.6. Let X be a set and let $P,\ Q \in \mathrm{IFR}(X)$. Then

(1) If $P \subset Q$, then $P^\infty \subset Q^\infty$.
(2) If $P \circ Q = Q \circ P$ and $P, Q \in \mathrm{IFE}(X)$, then $(P \circ Q)^\infty = P \circ Q$.

Proof.

(1) It is clear that $P^2 \subset Q^2$, by Proposition 2.4(2). Suppose $P^k \subset Q^k$ for any $k > 2$. Then, by Proposition 2.4(2), $P^{k+1} \subset Q^{k+1}$. Hence $P^\infty \subset Q^\infty$.
(2) Suppose $P \circ Q = Q \circ P$ and $P, Q \in \mathrm{IFE}(X)$. Then it is clear that $(P \circ Q)^1 = P \circ Q$.

Suppose $(P \circ Q)^k = P \circ Q$ for any $k \geq 2$. Then

$$\begin{aligned}(P\circ Q)^{k+1} &= (P\circ Q)^k\circ(P\circ Q) = (P\circ Q)\circ(P\circ Q)\\ &= (P\circ P)\circ(Q\circ Q) = P\circ Q.\end{aligned}$$

So $(P \circ Q)^n = P \circ Q$ for any $n \geq 1$. Hence $(P \circ Q)^\infty = P \circ Q$. □

Theorem 3.7. *If R is an IFR on a set X, then $R^e = [R \cup R^{-1} \cup \triangle]^\infty$.*

Proof. Let $Q = [R \cup R^{-1} \cup \triangle]^\infty$. Then clearly $R \subset Q$. By Proposition 3.3(1), Q is intutionistic fuzzy transitive. Let $x \in X$. Since $\triangle \subset Q$, $1 = \mu_\triangle(x,x) \leq \mu_Q(x,x)$ and $0 = \nu_\triangle(x,x) \geq \nu_Q(x,x)$. Thus $\mu_Q(x,x) = 1$ and $\nu_Q(x,x) = 0$. So Q is intutionistic fuzzy reflexive. It is clear that $R \cup R^{-1} \cup \triangle$ is intutionistic fuzzy symmetric. By Proposition 3.5, Q is intutionistic fuzzy symmetric. Hence $Q \in \mathrm{IFE}(X)$. Now $K \in$

IFE(X) such that $R \subset K$. Then $\triangle \subset K$ and $R^{-1} \subset K^{-1} = K$. Thus $R \cup R^{-1} \cup \triangle \subset K$. By Proposition 2.4(2), $[R \cup R^{-1} \cup \triangle]^n \subset K^n = K$ for any $n \geq 1$. So $Q \subset K$. Hence $R^e = Q = [R \cup R^{-1} \cup \triangle]^\infty$. This completes the proof. □

Proposition 3.8. Let X be a set and let $P, Q \in$ IFE(X). We define $P \vee Q$ as follows: $P \vee Q = (P \cup Q)^\infty$, i.e., $P \vee Q = \bigcup_{n \in \mathbb{N}} (P \cup Q)^n$. Then $P \vee Q \in$ IFE(X).

Proof. By Proposition 3.3, $P \vee Q$ is intuitionistic fuzzy transitive. Let $x \in X$. Since P and Q are intuitionistic fuzzy reflexive,

$$
\begin{aligned}
(P \vee Q)(x,x) &= \Big(\bigvee_{n\in\mathbb{N}} [\mu_P(x,x) \vee \mu_Q(x,x)]^n, \bigwedge_{n\in\mathbb{N}} [\nu_P(x,x) \wedge \nu_Q(x,x)]^n\Big) \\
&= \Big(\bigvee_{n\in\mathbb{N}} (1 \vee 1)^n, \bigwedge_{n\in\mathbb{N}} (0 \wedge 0)^n\Big) = (1,0).
\end{aligned}
$$

Thus $P \vee Q$ is intuitionistic fuzzy reflexive. Now let $x, y \in S$. Since P and Q are intuitionistic fuzzy symmetric,

$$
\begin{aligned}
(P \vee Q)(x,y) &= \Big(\bigvee_{n\in\mathbb{N}} [\mu_P(x,y) \vee \mu_Q(x,y)]^n, \bigwedge_{n\in\mathbb{N}} [\nu_P(x,y) \wedge \nu_Q(x,y)]^n\Big) \\
&= \Big(\bigvee_{n\in\mathbb{N}} [\mu_P(y,x) \vee \mu_Q(y,x)]^n, \bigwedge_{n\in\mathbb{N}} [\nu_P(y,x) \wedge \nu_Q(y,x)]^n\Big) \\
&= (P \vee Q)(y,x).
\end{aligned}
$$

Thus $P \vee Q$ is intutionistic fuzzy symmetric. Hence $P \vee Q \in$ IFE(X). □

The following result gives another description for $P \vee Q$ of two IFERs P and Q.

Theorem 3.9. *Let X be a set and let $P, Q \in$ IFE(X). If $P \circ Q \in$ IFE(X), then $P \circ Q = P \vee Q$, where $P \vee Q$ denotes the least upper bound for $\{P, Q\}$ with respect to the inclusion.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}\mu_{P\circ Q}(x,y) &= \bigvee_{z\in X}[\mu_Q(x,z)\wedge\mu_P(z,y)] \geq \mu_Q(x,y)\wedge\mu_P(y,y)\\ &= \mu_Q(x,y)\wedge 1 \text{ (Since } R \text{ is intuitionistic fuzzy reflexive)}\\ &= \mu_Q(x,y)\end{aligned}$$

and

$$\begin{aligned}\nu_{P\circ Q}(x,y) &= \bigwedge_{z\in X}[\nu_Q(x,z)\vee\nu_P(z,y)] \leq \nu_Q(x,y)\vee\nu_P(y,y)\\ &= \nu_Q(x,y)\vee 0 = \nu_Q(x,y).\end{aligned}$$

Thus $P\circ Q \supset Q$. Similarly, we have $P\circ Q\supset R$. So $P\circ Q$ is an upper bound for $\{P,Q\}$ with respect to "$\subset$".

Now let R be any intuitionistic fuzzy equivalence relation on X such that $R\supset P$ and $R\supset Q$. Let $x,y\in X$. Then

$$\begin{aligned}\mu_{P\circ Q}(x,y) &= \bigvee_{z\in X}[\mu_Q(x,z)\wedge\mu_P(z,y)] \leq \bigvee_{z\in X}[\mu_R(x,z)\wedge\mu_R(z,y)]\\ &= \mu_{R\circ R}(x,y) \leq \mu_R(x,y)\\ &\quad \text{(Since } R \text{ is intuitionistic fuzzy transitive)}\end{aligned}$$

and

$$\begin{aligned}\nu_{P\circ Q}(x,y) &= \bigwedge_{z\in X}[\nu_Q(x,z)\vee\nu_P(z,y)] \geq \bigwedge_{z\in X}[\nu_R(x,z)\vee\nu_R(z,y)]\\ &= \nu_{R\circ R}(x,y) \geq \nu_R(x,y).\end{aligned}$$

Thus $P\circ Q\subset R$. So $P\circ Q$ is the least upper bound for $\{P,Q\}$ with respect to "$\subset$". Hence $P\circ Q = P\vee Q$. □

Proposition 3.10. Let X be a set. If $P,Q\in$ IFE(X), then $P\vee Q = (P\circ Q)^\infty$.

Proof. Suppose $P,Q\in$ IFE(X). Then, by Theorem 3.7, $P\vee Q = (P\cup Q)^e = [(P\cup Q)\cup(P\cup Q)^{-1}\cup\triangle]^\infty$. Since $P,Q\in IFE(X)$, $(P\cup Q)\cup(P\cup Q)^{-1}\cup\triangle = P\cup Q$. Since $P\subset P\cup Q$ and $Q\subset P\cup Q$, by

Proposition 2.4(2) and (1), $P \circ Q \subset (P \cup Q) \circ (P \cup Q) = P \cup Q$. Thus, by Proposition 3.6(1), $(P \circ Q)^{\infty} \subset (P \cup Q)^{\infty}$. On the other had, since $P, Q \in$ IFE(X), $P \subset P \circ Q$ and $Q \subset P \circ Q$. Thus $P \cup Q \subset P \circ Q$. By Proposition 3.6(1), $(P \cup Q)^{\infty} \subset (P \circ Q)^{\infty}$. So $(P \circ Q)^{\infty} = (P \cup Q)^{\infty}$. Hence $P \vee Q = (P \cup Q)^{\infty} = (P \circ Q)^{\infty}$. □

The following is the immediate result of Proposition 3.10 and Proposition 3.6(2).

Corollary 3.11. Let X be a set. If $P, Q \in$ IFE(X) such that $P \circ Q = Q \circ P$, then $P \vee Q = P \circ Q$.

For a set X, it is clear that IFE(X) is a partially ordered set with respect to the inclusion relation " $\subset$ ". Moreover, for any $P, Q \in$ IFE(X), $P \cap Q$ is the greatest lower bound for P and Q in (IFE(X),$\subset$). Now, we define two binary operations $\vee$ and $\wedge$ on IFE(X) as follows : for any $P, Q \in$ IFE(X),

$$P \wedge Q = P \cap Q, \text{ and } P \vee Q = (P \cup Q)^{e}.$$

Then we obtain the following result from Proposition 2.10, Definition 2.12, Proposition 3.8 and Theorem 3.9.

Theorem 3.12. *Let X be a set. Then (IFE(X), $\vee, \wedge$) is a complete lattice with the least element $\triangle$ and the greatest element ∇.*

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References

[1] K.Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems, **20** (1986), 87-96.

[2] Baldev Banerjee and Dhiren Kr. Basnet, *Intuitionistic fuzzy subrings and ideals,* J.Fuzzy Math. **11(1)** (2003), 139-155.

[3] G.Birkhoff, Lattice Theory (A.M.S. Colloquium Pablication Vol XXV, 1963).

[4] R.Biswas, *Intuitionistic fuzzy subgroups,* Mathematical Forum x (1989), 37-46.

[5] H.Bustince and P.Burillo, *Structures on intuitionistic fuzzy relations,* Fuzzy Sets and Systems, **78** (1996), 293-303

[6] D.Çoker, *An introduction to intuitionistic fuzzy topological spaces,* Fuzzy Sets and Systems, **88**(1997), 81-89.

[7] D.Çoker and A.Haydar Es, *On fuzzy compactness in intuitionistic fuzzy topological spaces,* J. Fuzzy Math. **3** (1995), 899-909.

[8] G.Deschrijver and E.E.Kerre, *On the composition of intuitionistic fuzzy relations,* Fuzzy Sets and Systems, **136** (2003), 333-361.

[9] H.Gürçay, D.Çoker and A.Haydar Es, *On fuzzy continuity in intuitionistic fuzzy topological spaces,* J.Fuzzy Math. 5(1997), 365-378.

[10] K.Hur, S.Y.Jang and H.W.Kang, *Intuitionistic fuzzy subgroupoids,* International Journal of Fuzzy Logic and Intelligent Systems, **3(1)** (2003), 72-77.

[11] K.Hur, H.W.Kang and H.K.Song, *Intuitionistic fuzzy subgroups and subrings,* Honam Mathematical J.25(2) (2003), 19-41.

[12] S.J.Lee and E.P.Lee, *The category of intuitionistic fuzzy topological spaces,* Bull. Korean Math. Soc. **37(1)** (2000), 63-76.

[13] V.Murali, *Fuzzy equivalence relations,* Fuzzy Sets and Systems, **30** (1989), 155-163.

[14] W.C.Nemitz, *Fuzzy relations and fuzzy function,* Fuzzy Sets and Systems, **19** (1986), 177-191.

[15] L.A.Zadeh, *Fuzzy sets,* Inform. and Control, **8** (1965), 338-353.

[16] ________, *Similarity relations and fuzzy orderings,* Inform.Sci. **3** (1971), 177-200.

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