Honam Mathematical J. **27**(2005), No. 2, pp. 233–241

LOCALIZATION PROPERTY AND FRAMES

YOUNG-HWA HA AND JU-YEON RYU

Abstract. A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is said to be exponentially localized with respect to a Riesz basis $\{g_i\}_{i=1}^{\infty}$ for H if there exist positive constants $r < 1$ and C such that for all $i, j \in N$, $|\langle f_i, g_j\rangle| \le Cr^{|i-j|}$ and $|\langle f_i, \tilde{g}_j\rangle| \le Cr^{|i-j|}$ where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual basis of $\{g_i\}_{i=1}^{\infty}$. It can be shown that such sequence is always a Bessel sequence. We present an additional condition which guarantees that $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

1. Introduction

The main feature of a basis $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is that every $f \in H$ can be represented as a convergent series in terms of the elements f_i : $f = \sum_{i=1}^{\infty} c_i f_i$. The coefficients c_i are unique. A frame is also a sequence of elements $\{f_i\}_{i=1}^{\infty}$ in H, which allows every $f \in H$ to be written as a series like a basis. But, the corresponding coefficients are not necessarily unique. So a frame is more flexible than orthonormal basis, and it play an important role in wavelet theory.

Let H denote a separable Hilbert space. A family of elements $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called a **Bessel sequence** if there exists a constant $B < \infty$ such that for every $f \in H$,

$$\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

Received May 31, 2005. Revised June 11, 2005.

2000 Mathematics Subject Classification : 42C15, 46E99, 46B15.

Key words and phrases : frame, Bessel sequence, Riesz basis, exponentially localized.

A Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is called a **frame** if it satisfies an additional condition: there exists a constant $A > 0$ such that for every $f \in H$,

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2.$$

The numbers A, B are called **frame bounds**. If $\{f_i\}_{i=1}^{\infty}$ is a frame, then it is well-known that for every $f \in H$, $f = \sum_{i=1}^{\infty} c_i f_i$ for some coefficients $\{c_i\}_{i=1}^{\infty} \in l^2(N)$.

A family of elements $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called a **Riesz basis** if there exists a bounded invertible operator T on H such that for every i, $f_i = Te_i$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for H. There are well-known characterizations for Riesz bases(see [8]). For instance, it is an ω-independent frame.

It is a classical result that a sufficiently small perturbation of an orthonormal basis gives a Riesz basis(see [8]). It is also well-known that if $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds A, B and a sequence $\{f_i\}_{i=1}^{\infty}$ satisfies a Paley-Wiener condition: there exist nonnegative constants λ, μ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and $\|\sum_{i=1}^{n} c_i(f_i - g_i)\| \leq \lambda \|\sum_{i=1}^{n} c_i f_i\| + \mu\left(\sum_{i=1}^{n} |c_i|^2\right)^{1/2}$ for all $c_1, c_2, \cdots, c_n (n = 1, 2, \cdots)$, then $\{f_i\}_{i=1}^{\infty}$ is a frame for H(see [2]). And we know that if $\{g_i\}_{i=1}^{\infty}$ is a frame and a sequence $\{f_i\}_{i=1}^{\infty}$ is quadratically close to $\{g_i\}_{i=1}^{\infty}$, $\sum_{i=1}^{\infty} \|f_i - g_i\|^2 < \infty$, then $\{f_i\}_{i=1}^{\infty}$ is a frame sequence(see [3]). Paley-Wiener condition and quadratically closeness can be considered as perturbations. Localization of a sequence can also be considered as perturbations.

Definition 1.1. A sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space H is called **exponentially localized** with respect to a Riesz basis $\{g_i\}_{i=1}^{\infty}$ for H if for some positive constants $r < 1$ and C

$$|\langle f_i, g_j \rangle| \leq C r^{|i-j|} \tag{1.1}$$

and

$$|\langle f_i, \tilde{g}_j \rangle| \leq C r^{|i-j|} \tag{1.2}$$

for all positive integers i, j, where $\{\tilde{g}_i\}_{i=1}^{\infty}$ is the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$.

Note that if $\{g_i\}_{i=1}^{\infty}$ is an orthonomal basis for H then the condition (1.1)and (1.2) are identical because $\{g_i\}_{i=1}^{\infty} = \{\tilde{g}_i\}_{i=1}^{\infty}$. K. Gröchenig first defined a concept of localization of frame(see [6]). His definition and our definition are identical for the case of frames.

2. main results

We first state some of important lemmas for Bessel condition.

Lemma 2.1. *For any positive constant $r < 1$, there exists a constant $C > 0$ such that*

$$\sum_{l=1}^{\infty} r^{|l-k|} r^{|l-j|} \leq C r^{\frac{1}{2}|k-j|}$$

for all positive integers k, j.

Proof. Fix k, j and let $I_1 = \{l \in N : |l-j| \leq \frac{1}{2}|k-j|\}$ and $I_2 = N - I_1$.

If $l \in I_1$, then $|l-k| \geq \frac{1}{2}|k-j|$. Hence, we have

$$\begin{aligned}
\sum_{l=1}^{\infty} r^{|l-k|} r^{|l-j|} &= \sum_{l \in I_1} r^{|l-k|} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} r^{|l-j|} \\
&\leq \sum_{l \in I_1} r^{\frac{1}{2}|k-j|} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} r^{\frac{1}{2}|k-j|} \\
&= r^{\frac{1}{2}|k-j|} \Big(\sum_{l \in I_1} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} \Big) \\
&= C r^{\frac{1}{2}|k-j|}
\end{aligned}$$

where $C = \sum_{l \in I_1} r^{|l-j|} + \sum_{l \in I_2} r^{|l-k|} < \infty$ since $r < 1$. □

The following lemma gives a sufficient condition for the Gram matrix associated with $\{f_i\}_{i=1}^{\infty}$ to be a Bessel sequence(see [1]).

Lemma 2.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H, and suppose that there exists a constant $B > 0$ such that for every $j \in N$*

$$\sum_{k=1}^{\infty} |\langle f_j, f_k\rangle| \leq B.$$

Then, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence with bound B.

Proposition 2.3. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H and $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis for H. If $\{f_i\}_{i=1}^{\infty}$ is exponentially localized with respect to $\{g_i\}_{i=1}^{\infty}$; that is, for some positive constants $r < 1$ and C, $|\langle f_i, g_j\rangle| \leq Cr^{|i-j|}$ and $|\langle f_i, \tilde{g}_j\rangle| \leq Cr^{|i-j|}$ for all positive integers i, j, then $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence.*

PROOF. By Lemma 2.2, it suffices to show that there exists a constant $B > 0$ such that for every $j \in N$

$$\sum_{i=1}^{\infty} |\langle f_j, f_i\rangle| \leq B.$$

Since $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis, we can write $f_j = \sum_{l=1}^{\infty}\langle f_j, g_l\rangle \tilde{g}_l$ for every $j \in N$. Thus, by Lemma 2.1, for any $j \in N$,

$$\begin{aligned}
\sum_{i=1}^{\infty} |\langle f_j, f_i\rangle| &= \sum_{i=1}^{\infty} \Big|\sum_{l=1}^{\infty}\langle f_j, g_l\rangle\langle \tilde{g}_l, f_i\rangle\Big| \\
&\leq \sum_{i=1}^{\infty}\sum_{l=1}^{\infty} |\langle f_j, g_l\rangle||\langle f_i, \tilde{g}_l\rangle| \\
&\leq C^2 \sum_{i=1}^{\infty}\sum_{l=1}^{\infty} r^{|j-l|} r^{|i-l|} \\
&\leq D \sum_{i=1}^{\infty} r^{\frac{1}{2}|i-j|}.
\end{aligned}$$

Since $\sum_{i=1}^{\infty} r^{\frac{1}{2}|i-j|} < \infty$, setting $B = D\sum_{i=1}^{\infty} r^{\frac{1}{2}|i-j|}$, we have $\sum_{i=1}^{\infty} |\langle f_j, f_i\rangle| \leq B$. □

Corollary 2.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H and $\{e_i\}_{i=1}^{\infty}$ be an orthonomal basis for H. If $\{f_i\}_{i=1}^{\infty}$ is exponentially localized with respect to $\{e_i\}_{i=1}^{\infty}$, then $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence.*

To prove Theorem 2.7, we need the well-known fact that a diagonally dominant matrix is invertible(see [7] for proof).

Lemma 2.5. *Suppose that a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ defines a bounded self-adjoint operator on $l^2(N)$ and that A satisfies the condition of diagonal dominance for every $i \in N$; that is, there exists a positive constant δ such that*

$$|a_{ii}| - \sum_{j:j\neq i} |a_{ij}| \geq \delta$$

for every $i \in N$. Then A is invertible on $l^2(N)$.

If $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, we can define a bounded operators T, usually called a *pre-frame operator* associated to $\{f_i\}_{i=1}^{\infty}$:

$$T : l^2(N) \to H, \qquad T\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i.$$

Then its adjoint $T^* : H \to l^2(N)$ is defined by $T^*f = \{\langle f, f_i\rangle\}_{i=1}^{\infty}$ for every $f \in H$. By composing T and T^*, we obtain the *frame operator* S:

$$S : H \to H, \qquad Sf = TT^*f = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i.$$

If $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, the series defining S converges unconditionally for all $f \in H$, and S is a bounded self-adjoint operator on H.

Theorem 2.6. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in a Hilbert space H and $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis for H. If the matrix $\{\langle Sg_i, g_j\rangle\}_{i,j=1}^{\infty}$ is invertible on $l^2(N)$, then $\{f_i\}_{i=1}^{\infty}$ is a frame for H.*

Proof. Let $\{\tilde{g}_i\}_{i=1}^{\infty}$ be the dual Riesz basis of $\{g_i\}_{i=1}^{\infty}$. Since $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, we need only to show that $\{f_i\}_{i=1}^{\infty}$ has a lower frame bound. Since $\{\langle Sg_i, g_j\rangle\}_{i,j=1}^{\infty}$ is the matrix representation of S with respect to the bases $\{g_i\}_{i=1}^{\infty}$ and $\{\tilde{g}_i\}_{i=1}^{\infty}$, and it is invertible by the hypothesis, S is invertible. So, for any $f \in H$,

$$\begin{aligned} ||f||^4 &= \langle f, f\rangle^2 = \langle SS^{-1}f, f\rangle^2 = |\sum_{i=1}^{\infty}\langle S^{-1}f, f_i\rangle\langle f_i, f\rangle|^2 \\ &\leq \sum_{i=1}^{\infty}|\langle S^{-1}f, f_i\rangle|^2\sum_{i=1}^{\infty}|\langle f, f_i\rangle|^2 = \langle S^{-1}f, f\rangle\sum_{i=1}^{\infty}|\langle f, f_i\rangle|^2 \\ &\leq ||f||^2||S^{-1}||\sum_{i=1}^{\infty}|\langle f, f_i\rangle|^2. \end{aligned}$$

Since S^{-1} is bounded, we now have

$$\frac{1}{||S^{-1}||}||f||^2 \leq \sum_{i=1}^{\infty}|\langle f, f_i\rangle|^2,$$

for every $f \in H$. Therefore, $\{f_i\}_{i=1}^{\infty}$ is a frame for H. □

Theorem 2.7. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space H and $\{g_i\}_{i=1}^{\infty}$ be a Riesz basis for H. Suppose that $\{f_i\}_{i=1}^{\infty}$ is exponentially localized with respect to $\{g_i\}_{i=1}^{\infty}$; that is, for some positive constants $r < 1$ and C_1,*

$$|\langle f_i, g_j\rangle| \leq C_1 r^{|i-j|}$$

and

$$|\langle f_i, \tilde{g}_j\rangle| \leq C_1 r^{|i-j|}$$

for all positive integers i, j.

If there exists a positive constant C_2 such that

$$\sum_{i=1}^{\infty}|\langle f_i, g_j\rangle|^2 \geq C_2^2$$

and

$$\sqrt{2}C_2 > \frac{1+r}{1-r}C_1,$$

then $\{f_i\}_{i=1}^{\infty}$ is a frame for H.

PROOF. By Proposition 2.3, $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence. So it suffices to show that $\{f_i\}_{i=1}^{\infty}$ has a lower frame bound. Fix i and consider

$$|\langle Sg_i, g_i\rangle| - \sum_{j:j\neq i} |\langle Sg_i, g_j\rangle|.$$

Since $\langle Sg_i, g_j\rangle = \sum_{k=1}^{\infty}\langle g_i, f_k\rangle\langle f_k, g_j\rangle$ and $\langle Sg_i, g_i\rangle = \sum_{k=1}^{\infty}|\langle g_i, f_k\rangle|^2$, we have

$$\begin{aligned}\sum_{j:j\neq i} |\langle Sg_i, g_j\rangle| &= \sum_{j=1}^{\infty}|\langle Sg_i, g_j\rangle| - |\langle Sg_i, g_i\rangle| \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| - \sum_{k=1}^{\infty}|\langle g_i, f_k\rangle|^2 \\ &\leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| - C_2^2.\end{aligned}$$

Now since $\{f_i\}_{i=1}^{\infty}$ is exponentially localized with respect to $\{g_i\}_{i=1}^{\infty}$,

$$\begin{aligned}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle| &\leq C_1\sum_{k=1}^{\infty}r^{|i-k|} = C_1\Big(\sum_{k=1}^{i-1}r^k + \sum_{k=0}^{\infty}r^k\Big) \\ &\leq C_1\Big(\frac{r}{1-r} + \frac{1}{1-r}\Big) = C_1\Big(\frac{1+r}{1-r}\Big),\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{j=1}^{\infty}|\langle f_k, g_j\rangle| &\leq C_1\sum_{j=1}^{\infty}r^{|j-k|} \\ &\leq C_1\Big(\frac{1+r}{1-r}\Big).\end{aligned}$$

¿From these inequalities, we obtain

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\langle g_i, f_k\rangle||\langle f_k, g_j\rangle| \leq C_1^2\Big(\frac{1+r}{1-r}\Big)^2.....$$

Hence,

$$\sum_{j:j\neq i} |\langle Sg_i, g_j\rangle| \leq C_1^2\Big(\frac{1+r}{1-r}\Big)^2 - C_2^2.$$

Finally,

$$|\langle Sg_i, g_i\rangle| - \sum_{j:j\neq i} |\langle Sg_i, g_j\rangle| \;\geq\; 2C_2^2 - C_1^2\Big(\frac{1+r}{1-r}\Big)^2 > 0,$$

where the constants r, C_1, C_2 are independent of i. Therefore, $\{f_i\}_{i=1}^{\infty}$ is a frame for H. □

As an application of our main theorem, Theorem 2.7, we show an example.

Example 2.8. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for H. Let $\{g_i\}_{i=1}^{\infty} = \{e_i\}_{i=1}^{\infty}$ and

$$\begin{aligned} f_1 &= e_1, \\ f_i &= e_i + \frac{1}{\sqrt{i+98}} e_{i-1}, \qquad i \geq 2. \end{aligned}$$

Then, for every i,

$$\begin{aligned} |\langle f_i, g_i\rangle| &= 1, \\ |\langle f_{i+1}, g_i\rangle| &= \frac{1}{\sqrt{i+99}} \leq \frac{1}{10}, \\ |\langle f_j, g_i\rangle| &= 0, \qquad j \neq i, i+1, \end{aligned}$$

and

$$\sum_{j=1}^{\infty} |\langle f_j, g_i\rangle|^2 = 1 + \Big(\frac{1}{\sqrt{i+99}}\Big)^2 = 1 + \frac{1}{i+99} \geq 1.$$

Let $C_1 = 1, C_2 = 1, r = \frac{1}{10}$. Then,

$$\sqrt{2}C_2 = \sqrt{2} > \frac{11}{9} = \frac{1+\frac{1}{10}}{1-\frac{1}{10}} = \Big(\frac{1+r}{1-r}\Big)C_1.$$

Hence, $\{f_i\}_{i=1}^{\infty}$ is a frame for H. In fact, $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis, since it is ω-independent. But, $\{f_i\}_{i=1}^{\infty}$ is not quadratically close to $\{g_i\}_{i=1}^{\infty}$ because $\sum_{i=1}^{\infty} ||f_i - g_i||^2 = \sum_{i=1}^{\infty} \frac{1}{i+98} = \infty$. So a quadratically closeness is not applicable to this example.

References

[1] O. Christensen, An introduction to frames and and Riesz basis, Birkhäuser, Boston, 2003.

[2] O. Christensen, *A Paley-Wiener theorem for frames,* Proc. Amer. Math. Soc. **123** (1995), 2199-2201.

[3] O. Christensen, *Frame perturbations,* Proc. Amer. Math. Soc. **123** (1995), 1217-1220.

[4] P. G. Casazza, O. Christensen, *Approximation of the inverse frame opertor and applications to Gabor frames,* Jounal of Approximation Thoery **103** (2000), 338-356.

[5] I. Daubechies, Ten Lectures on Wavelets, SIAM Conference Series in Applied Mathematiccs, SIAM, Boston, 1992.

[6] K. Gröchenig, *Localization of frames, Banach frames, and the invertiblility of the frame operator,* J. Fourier Anal. Appl. **10** (2004), 105-132.

[7] K. Gröchenig, Foundations of time-frequency analysis, Birkhäuser, Boston, 2001.

[8] R. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, NewYork, 1980.

Young-Hwa Ha
Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail : yhha@ajou.ac.kr

Ju-Yeon Ryu
Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail : juyeonryu@ajou.ac.kr