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PARTIAL DIFFERENTIAL EQUATIONS AND SCALAR CURVATURES ON SPACE-TIMES

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Abstract. In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct Lorentzian metrics on $M = [a, b) \times_f N$ with specific scalar curvatures

1. Introduction

In recent studies ([10, 11]), M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metrics with some prescribed scalar curvature functions. In this paper, we study also the existence and nonexistence of Lorentzian warped metric with prescribed scalar curvature functions on some Lorentzian warped product manifolds.

By the results of Kazdan and Warner ([7, 8, 9]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three catagories:

(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.

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(B) A Smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.

(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [3, 10, 11], the authors considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [4], the authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature.

In [5], the author considered the existence of a warping function on a Lorentzian warped product manifold $M = [a, \infty) \times_f N$. Similarly in [6], the authors also considered the existence of a warping funtion on $M = (-\infty, \infty) \times_f N$. In this paper, when N is a compact Riemannian manifold, we consider the null future completeness of Lorentzian metrics on $M = [a, b) \times_f N$ with specific scalar curvatures, where a and b are positive constants. When the base manifold is a finite interval, the results about the existence of a warping function are different from those of [5] and [6], in which cases the base manifolds are infinite intervals. It is shown that if the fiber manifold N belongs to (A), (B) or (C), then M admits a Lorentzian metric with some prescribed scalar curvature outside a compact set.

2. Fiber manifold in class (A) or (B)

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, b) \to R^+$ be a smooth function, where a and b are positive numbers.

The Lorentzian warped product of N and $[a,b)$ with warping function f is defined to be the product manifold $([a,b)\times_f N, g')$ with

$$g' = -dt^2 + f^2(t)g \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N,g). Then the scalar curvature $R(t,x)$ of g' is given by the equation

$$R(t,x) = \frac{1}{f^2(t)}\{R(g)(x) + 2nf(t)f''(t) + n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a,b)$ and $x \in N$. (For details, cf. [5] or [6])

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad b > t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u''(t) - R(t,x)u(t) + R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary. For $n \geq 3$, let $M = [a,b)\times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Trivially, since the base manifold is a finite interval, then for any warping function the resulting warped product metric is a future timelike geodesically incomplete one ([1], [2]). But for the null geodesical completeness we have the following proposition.

Prposition 2.1 ([12]) *All null geodesics are future complete on* $[a,b)\times_{f(t)} N$ *if and only if* $\int_{t_0}^{b} f(t)dt = +\infty$ *for some* $t_0 \in [a,b)$.

If N admits a Riemannian metric of negative or zero scalar curvature, i.e., N is in class (A) or (B), then we let $u(t) = (b-t)^{\alpha}$ in (2.3), where $\alpha \in (0,1)$ is a constant, and we have

$$R(t,x) \leq -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{(b-t)^2} < 0, \quad b > t > a.$$

Theorem 2.2 *For $n \geq 3$, let $M = [a,b) \times_f N$ be the Lorentzian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (A) or (B), then on M there exists a Lorentzian metric of negative scalar curvature outside a compact set.*

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = \frac{1}{2}$. And when $u = (b-t)^{\frac{1}{2}}$ and N admits a Riemannian metric of zero scalar curvature, we have

$$R = -\frac{4n}{n+1}\frac{1}{4}\frac{1}{(b-t)^2}, \quad b > t > a.$$

If $R(t,x)$ is the function of only t-variable, then we have the following proposition whose proof is smilar to that of Lemma 1.8 in [11].

Proposition 2.3 *If $R(g) = 0$, then there is no positive solution to equation (2.3) with*

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{(b-t)^2} \quad \text{for} \quad b > t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{(b-t)^2} \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Equations (2.3) gives $(b-t)^2 u'' + \frac{c}{4}u \leq 0$. Let $u(t) = (b-t)^{\alpha}v(t), \quad t \geq t_0$, where $\alpha > 0$ is a constant and $v(t) > 0$ is a smooth function. Then we have

$$u'' = \alpha(\alpha-1)(b-t)^{\alpha-2}v(t) - 2\alpha(b-t)^{\alpha-1}v'(t) + (b-t)^{\alpha}v''(t).$$

And we obtain

$$(2.4)\quad (b-t)^{\alpha}v(t)[\alpha(\alpha-1)+\frac{c}{4}] - 2\alpha(b-t)^{\alpha+1}v'(t) + (b-t)^{\alpha+2}v''(t) \le 0.$$

Let δ be a positive constant such that $\delta^2 = \frac{c-1}{4}$. Then we have

$$\alpha(\alpha-1)+\frac{c}{4} = (\alpha-\frac{1}{2})^2 + \frac{c-1}{4} \ge \delta^2.$$

Here δ is a constant independent on α. Equation (2.4) gives

$$(2.5)\qquad -2\alpha(b-t)v'(t) + (b-t)^2 v''(t) \le -\delta^2 v(t).$$

Let $\beta = 2\alpha$ and we choose $\alpha > 0$ such that $\beta > 1$, that is, $\alpha > \frac{1}{2}$. Then (2.5) becomes

$$((b-t)^{\beta}v'(t))' \le -\frac{\delta^2 v(t)}{(b-t)^{2-\beta}}.$$

Upon integration we have

$$(2.6)\quad (b-t)^{\beta}v'(t) - (b-\tau)^{\beta}v'(\tau) \le -\int_{\tau}^{t} \frac{\delta^2 v(s)}{(b-s)^{2-\beta}}ds, \qquad b > t > \tau > t_0.$$

Here we have two following cases:

i) If $v'(\tau) \le 0$ for some $\tau > t_0$, then (2.6) implies that $(b-t)^{\beta}v'(t) \le -C$ for some positive constant C. We have

$$v(t) \le v(\tau) - \int_{\tau}^{t} \frac{C}{(b-s)^{\beta}}ds = v(\tau) + \frac{c(b-s)^{1-\beta}}{1-\beta}\Big|_{\tau}^{t} \to -\infty,$$

as $t \to b$. Hence $v(t) < 0$ for some t, contradicting that $v(t) > 0$ for all $t \in [a, b)$

ii) We have $v'(t) > 0$ for all $t > t_0$. Equation (2.6) implies that

$$(b-\tau)^{\beta}v'(\tau) - \int_{\tau}^{t} \frac{\delta^2 v(s)}{(b-s)^{2-\beta}} ds \geq 0$$

for all $t > \tau > t_0$. As $v'(t) > 0$ for all $t > t_0$, we have

$$(b-\tau)^{\beta}v'(\tau) \geq v(\tau) \int_{\tau}^{t} \frac{\delta^2}{(b-s)^{2-\beta}} ds = v(\tau)[\frac{1}{(b-s)^{1-\beta}} \frac{\delta^2}{1-\beta}]|_{\tau}^{t}$$

Letting $t \to b$, we have $(b-\tau)^{\beta}v'(\tau) \geq \frac{v(\tau)}{(b-\tau)^{1-\beta}} \frac{\delta^2}{\beta-1}$. Or after changing the parameter, we have $\frac{v'(t)}{v(t)} \geq \frac{1}{b-t} \frac{\delta^2}{\beta-1}, \quad t > t_0$.

Choosing $\alpha > \frac{1}{2}$ close to $\frac{1}{2}$ so that $\beta > 1$ is close to 1 and using the fact that δ is independent on α or β, we have $\frac{v'(t)}{v(t)} \geq \frac{N}{b-t}$ for a big integer $N > 2$. This gives $v(t) \geq C(b-t)^{-N}, \quad t > t_0$, where C is a positive constant. (2.6) implies that

$$(b-t)^{\beta}v'(t) \leq (b-\tau)^{\beta}v'(\tau) - \int_{\tau}^{t} \frac{C\delta^2 (b-s)^{-N}}{(b-s)^{2-\beta}} ds \to -\infty \quad \text{as} \quad t \to b.$$

Thus $v'(t) < 0$ for t sufficiently close to b, which is also a contradiction. Hence there is no solution to equation (2.3).

Theorem 2.4 *Suppose that* $R(g) = 0$ *and* $R(t,x) = R(t) \in C^{\infty}([a,b))$. *Assume that for* $t > t_0 \geq a$, *there exist an upper solution* $u_+(t)$ *and a lower solution* $u_-(t)$ *such that* $0 < u_-(t) \leq u_+(t)$. *Then there exists a solution* $u(t)$ *of equation (2.3) such that for* $t > t_0 \quad 0 < u_-(t) \leq u(t) \leq u_+(t)$.

Proof. We need only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a,b) \quad \tilde{u}_-(t) \leq \tilde{u}_+(t)$. Since $R(t) \in C^{\infty}([a,b))$, there exists a positive constant d such that $|R(t)| \leq \frac{4n}{n+1}d^2$ for $t \in [a,t_0]$.

Since $\frac{4n}{n+1}u_+''(t) - R(t)u_+(t) \leq \frac{4n}{n+1}(u_+''(t) + d^2 u_+(t))$, if we devide the given interval $[a,t_0]$ into small intervals $\{I_i\}_{i=1}^{n}$, then for each interval I_i

we have an upper solution $u_+^i(t)$ by parallel transporting $\cos dt$ such that $0 < c_0 \le u_+^i(t) \le 1$. That is to say, for each interval I_i, $\frac{4n}{n+1} u_+^i(t)'' - R(t)u_+^i(t) \le \frac{4n}{n+1}(u_+^i(t)'' + d^2 u_+^i(t)) = 0$, which means that $u_+^i(t)$ is an upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_+^i(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution such that $c_0 \le \tilde{u}_+(t) \le 1$ for all $t \in [a, t_0]$. Put $\tilde{u}_-(t) = c_0 e^{-\alpha t}$ for $t \in [a, t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a, t_0]$, $\frac{4n}{n+1} u_-''(t) - R(t)u_-(t) \ge \frac{4n}{n+1}(u_-''(t) - d^2 u_-(t)) = \frac{4n}{n+1} c_0 e^{-\alpha t}(\alpha^2 - d^2) \ge 0$ for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a, b)$ $0 < \tilde{u}_-(t) \le \tilde{u}_+(t)$.

Theorem 2.5 *Suppose that $R(g) = 0$. Assume that $R(t, x) = R(t) \in C^\infty([a, b))$ is a function such that*

$$-\frac{4n}{n+1}\frac{c}{4}\frac{1}{(b-t)^2} < R(t) \le \frac{4n}{n+1}\frac{d}{(b-t)^\alpha} \quad \text{for} \quad \mathrm{t} > \mathrm{t}_0,$$

where $t_0 > a, 0 < c < 1, d > 0$ and $\alpha > 0$ are constants. Then equation (2.3) has a positive solution on $[a, b)$ and the resulting warped product metric is a null geodesically incomplete one.

Proof. Since $R(g) = 0$, put $u_+(t) = (b-t)^{\frac{1}{2}}$.
Then $u_+''(t) = \frac{-1}{4}(b-t)^{\frac{1}{2}-2}$. Hence

$$\begin{aligned}
&\frac{4n}{n+1} u_+''(t) - R(t)u_+(t) \\
&= \frac{4n}{n+1}\frac{-1}{4}(b-t)^{\frac{1}{2}-2} - R(t)(b-t)^{\frac{1}{2}} \\
&= \frac{4n}{n+1}(b-t)^{\frac{1}{2}}[\frac{-1}{4}(b-t)^{-2} - \frac{n+1}{4n}R(t)] \\
&\le \frac{4n}{n+1}(b-t)^{\frac{1}{2}-2}[-\frac{1}{4} + \frac{c}{4}] \\
&\le 0
\end{aligned}$$

Therefore $u_+(t)$ is our upper solution. And put $u_-(t) = e^{-\beta(b-t)^{-k}}$, where β is a positive constant and will be determined later and k is also a positive constant such that $k \geq \frac{\alpha-2}{2}$.
Then $u''_-(t) = e^{-\beta(b-t)^{-k}}\,[\beta^2 k^2 (b-t)^{-2k-2} - \beta k (k+1)(b-t)^{-k-2}]$.
Hence

$$\begin{aligned}
&\frac{4n}{n+1}u''_-(t) - R(t)\,u_-(t) \\
&= \frac{4n}{n+1}\,[\beta^2k^2(b-t)^{-2k-2} - \beta k(k+1)(b-t)^{-k-2}]\,e^{-\beta(b-t)^{-k}} \\
&\qquad - R(t)\,e^{-\beta(b-t)^{-k}} \\
&\geq \frac{4n}{n+1}\,e^{-\beta(b-t)^{-k}}[\beta^2k^2(b-t)^{-2k-2} - \beta k(k+1)(b-t)^{-k-2} \\
&\qquad - d^2(b-t)^{-\alpha}] \\
&\geq 0
\end{aligned}$$

for large β and as $t \to b$. Since $t > t_0 > a$, we can take β large so that $u_-(t)$ is a lower solution and $0 < u_-(t) < u_+(t)$. By Theorem 2.5, equation (2.3) has a positive solution $u(t)$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$. And trivially Proposition 2.1 implies that the resulting warped product metric is a null geodesically incomplete one.

Example 2.6 We consider the Lorentzian warped product manifold with $R(g) = 0$ and $R(t,x) = R(t) = \frac{4n}{n+1}\frac{C}{(b-t)^2}$. If we use the technique of Cauchy-Euler equation, then we have the following solutions of equation (2.3):

$u_1(t) = (b-t)^{\frac{1-\sqrt{1+4C}}{2}}$ and $u_2(t) = (b-t)^{\frac{1+\sqrt{1+4C}}{2}}$.

If $C \geq \frac{(n+1)(n+3)}{4}$, then the warped product manifold using the warping function $f(t) = u_1(t)^{\frac{2}{n+1}}$ is null future geodesically complete, but the warped product manifold using the warping function $f(t) = u_2(t)^{\frac{2}{n+1}}$ is null future geodesically incomplete. In the incomplete case, we get the solution $u_2(t)$ by the result of Theorem 2.5, but in the complete case, we could not have gotten the proper upper solution and the proper lower solution.

3. Fiber manifold in class (C)

In this section, we assume that the fiber manifold N of $M = [a,b) \times_f N$ belongs to class (C), where a, b are positive numbers. In this case, N admits a Riemannian metric of positive scalar curvature. If we let $u(t) = (b-t)^\alpha$, where $\alpha \in (0,1)$ is a constant, then we have

$$R(t,x) \geq -\frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{(b-t)^2} \geq -\frac{4n}{n+1}\frac{1}{4}\frac{1}{(b-t)^2}, \quad a < t < b.$$

Since $R(g) \geq 0$, if $R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{(b-t)^2}$ for $b > t \geq t_0$, then we can induce that $(b-t)^2 u'' + \frac{c}{4}u \leq 0$. So by a similar argument as in Proposition 2.3, we have the following:

Theorem 3.1 *If $R(g)$ is positive, then there is no positive solution to equation (2.3) with*

$$R(t) \leq -\frac{4n}{n+1}\frac{c}{4}\frac{1}{(b-t)^2} \quad \text{for} \quad b > t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

If N belongs to (C), then any smooth function on N is the scalar curvature of some Riemannian metric. So we can take a Riemannian metic g on N with scalar curvature $R(g) = \frac{4n}{n+1}k^2$, where k is a positive constant. Then equation (2.3) becomes

$$\text{(3.1)} \qquad \frac{4n}{n+1}u''(t) + \frac{4n}{n+1}k^2 u(t)^{1-\frac{4}{n+1}} - R(t,x)u(t) = 0.$$

Theorem 3.2 *Suppose that* $R(g) = \frac{4n}{n+1}k^2$ *and* $R(t,x) = R(t) \in C^\infty([a,b))$*. Assume that for* $t > t_0$*, there exist an upper solution* $u_+(t)$ *and a lower solution* $u_-(t)$ *of equation (3.1) such that* $0 < u_-(t) \le u_+(t)$*. Then there exists a solution* $u(t)$ *of equation (3.1) such that for* $t > t_0 \quad 0 < u_-(t) \le u(t) \le u_+(t)$.

Proof. The proof is similar to that of Theorem 2.3. Hence we also have only to show that there exist an upper solution $\tilde{u}_+(t)$ and a lower solution $\tilde{u}_-(t)$ such that for all $t \in [a,b) \quad \tilde{u}_-(t) \le \tilde{u}_+(t)$. Since $R(t) \in C^\infty([a,b))$, there exists a positive constant d such that $|R(t)| \le \frac{4n}{n+1}d^2$ for $t \in [a,t_0]$. We assume that $u_+(t) \ge 1$ for $t \in [a,t_0]$. Then we have

$$\begin{aligned}
&\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
\le\ &\frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t) + \frac{4n}{n+1}d^2 u_+(t) \\
=\ &\frac{4n}{n+1}[u_+''(t) + (k^2+d^2)u_+(t)].
\end{aligned}$$

And if we devide the given interval $[a,t_0]$ into small intervals $\{I_i\}_{i=1}^n$, then for each interval I_i we have an upper solution $u_+^i(t)$ by parallel transporting $c_1 cos(\sqrt{k^2+d^2}t)$ such that $u_+^i(t) \ge 1$ for some constant c_1. That is to say, for each interval I_i, $\frac{4n}{n+1}u_+^i(t)'' + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+^i(t) \le \frac{4n}{n+1}(u_+^i(t)'' + (k^2+d^2)u_+^i(t)) = 0$, which means that $u_+^i(t)$ is an upper solution for each interval I_i. Then put $\tilde{u}_+(t) = u_+^i(t)$ for $t \in I_i$ and $\tilde{u}_+(t) = u_+(t)$ for $t > t_0$, which is our desired (weak) upper solution on $[a,b)$ such that $\tilde{u}_+(t) \ge 1$ for all $t \in [a,t_0]$.

Put $\tilde{u}_-(t) = e^{-\alpha t}$ for $t \in [a,t_0]$ and some large positive α, which will be determined later, and $\tilde{u}_-(t) = u_-(t)$ for $t > t_0$. Then, for $t \in [a,t_0]$, $\frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} + R(t)u_-(t) \ge \frac{4n}{n+1}(u_-''(t) - d^2 u_-(t)) = \frac{4n}{n+1}e^{-\alpha t}(\alpha^2 - d^2) \ge 0$ for large α. Thus $\tilde{u}_-(t)$ is our desired (weak) lower solution such that for all $t \in [a,b) \quad 0 < \tilde{u}_-(t) \le \tilde{u}_+(t)$.

If $R(t,x)$ is the function of only t- variable, then we have the following theorem.

Theorem 3.3 *Assume that* $R(t,x) = R(t) \in C^\infty([a,b))$ *is a positive function such that*

$$\frac{4n}{n+1}\frac{C_1}{(b-t)^\alpha} \geq R(t) \geq -\frac{4n}{n+1}\frac{C_2}{(b-t)^\beta},$$

for $b > t \geq t_0 > a$ *where* α *is arbitrary and* $\beta < 2$ *and* C_1 *and* C_2 *are positive constants. Then equation (3.1) has a positive solution on* $[a,b)$.

Proof. We let $u_-(t) = (b-t)^m$ where m is a positive number. If we take m large enough so that $m\frac{4}{n+1} > \alpha$, then we have

$$\begin{aligned}
& \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - R(t)u_-(t) \\
\geq\ & \frac{4n}{n+1}u_-''(t) + \frac{4n}{n+1}k^2 u_-(t)^{1-\frac{4}{n+1}} - \frac{4n}{n+1}\frac{C_1}{(b-t)^\alpha}u_-(t) \\
=\ & \frac{4n}{n+1}(b-t)^m[\frac{m(m-1)}{(b-t)^2} + \frac{k^2}{(b-t)^{m\frac{4}{n+1}}} - \frac{C_1}{(b-t)^\alpha}] \\
\geq\ & 0, \quad b > t \geq t_0 > a,
\end{aligned}$$

which is possible for large fixed m.

And put $u_+(t) = e^{(b-t)^\delta}$, where δ is a positive constant such that $0 < \delta < \min\{1, 2-\beta\}$. Then $u_+''(t) = e^{(b-t)^\delta}[\delta^2(b-t)^{2\delta-2} + \delta(\delta-1)(b-t)^{\delta-2}]$. Hence

$$
\begin{aligned}
& \frac{4n}{n+1}u_+''(t) + \frac{4n}{n+1}k^2 u_+(t)^{1-\frac{4}{n+1}} - R(t)u_+(t) \\
= \; & \frac{4n}{n+1}[\delta^2(b-t)^{2\delta-2} + \delta(\delta-1)(b-t)^{\delta-2}]e^{(b-t)^\delta} \\
& + \frac{4n}{n+1}k^2 e^{(b-t)^\delta(1-\frac{4}{n+1})} - R(t)e^{(b-t)^\delta} \\
\leq \; & \frac{4n}{n+1}e^{(b-t)^\delta}[\delta^2(b-t)^{2\delta-2} + \delta(\delta-1)(b-t)^{\delta-2} \\
& + k^2 e^{(b-t)^\delta(-\frac{4}{n+1})} + C_2(b-t)^{-\beta}] \leq 0
\end{aligned}
$$

as $t \to b$. For sufficiently close to b, we can take the upper solution $u_+(t)$ so that $0 < u_-(t) < u_+(t)$. So by the upper and lower solution method, we obtain a positive solution.

Remark The above resulting warped product metric is a null geodesically incomplete one because $\int_{t_0}^b f(t)dt \leq \int_{t_0}^b u_+(t)^{\frac{2}{n+1}}dt = \int_{t_0}^b e^{(b-t)^\delta \frac{2}{n+1}}dt < \infty$.

References

[1] J.K. Beem and P.E. Ehrlich, Global Lorentzian Geometry, Pure and Applied Mathematics, **67**, Dekker, New York, 1981.

[2] J.K. Beem, P.E. Ehrlich and Th.G. Powell, *Warped product manifolds in relativity,* Selected Studies (Th.M. Rassias, G.M. Rassias, eds.), North-Holland, 1982, 41-56.

[3] J. Bland and M. Kalka, *Negative scalar curvature metrics on non-compact manifolds,* Trans. A.M.S. **316** (1989), 433-446.

[4] P.E. Ehrlich, Yoon-Tae Jung and Seon-Bu Kim, *Constant Scalar Curvaturers on Warped Product Manifolds,* Tsukuba J. Math. **20(1)** (1996), 239-256.

[5] Y.T. Jung, *Partial Differential Equations on Semi-Riemannian Manifolds,* J.M.A.A, **241** (2000), 238-253

[6] Y.T. Jung and Y.J. Kim, *Nonlinear Partial Differential Equations on Semi-Riemannian Manifolds,* Bull.Korean Math.Soc. **37** (2000), 317-336.

[7] J.L. Kazdan and F.W. Warner, *Scalar curvature and conformal deformation of Riemannian structure,* J.Diff.Geo. **10** (1975), 113-134.

[8] J.L. Kazdan and F.W. Warner, *Existence and conformal deformation of metrics with prescribed Guassian and scalar curvature,* Ann. of Math. **101** (1975), 317-331.

[9] J.L. Kazdan and F.W. Warner, *Curvature functions for compact 2 - manifolds,* Ann. of Math. **99** (1974), 14-74.

[10] M.C. Leung, *Conformal scalar curvature equations on complete manifolds,* Comm. in P.D.E. **20** (1995), 367-417.

[11] M.C. Leung, *Conformal deformation of warped products and scalar curvature functions on open manifolds,* Bulletin des Sciences Mathematiques. **122** (1998), 369-398.

[12] T.G. Powell, Lorentzian manifolds with non-smooth metrics and warped products, Ph.D thesis, Univ. of Missouri-Columbia, 1982.

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