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NORMAL INTERPOLATION PROBLEMS IN CSL-ALGEBRA ALG$\mathcal{L}$

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Abstract. We investigate the equation $Ax = y$, where the vectors x and y are given and the operator A is normal and required to lie in CSL-algebra Alg$\mathcal{L}$. We desire a necessary and sufficient condition for the existence of a solution A.

1. Introduction

The equation $Ax = y$ in Hilbert space has been considered by a number of authors. Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. And n-vector interpolation problem asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[8]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[9]: his result was extended by Hopenwasser[4] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[10] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[5] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation for n-vectors.

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In [6], we studied the problem of finding A so that $Ax = y$ and A is required to lie in Alg$\mathcal{L}$ for a commutative subspace lattice $\mathcal{L}$.

THEOREM [6]. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ a commutative subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Then the following statements are equivalent.*
(1) There is an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\dfrac{\|\sum_{i=1}^{l}\alpha_i E_i y\|}{\|\sum_{i=1}^{l}\alpha_i E_i x\|} : l \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty.$

In this paper, we consider this problem of finding a normal operator A in Alg$\mathcal{L}$ that maps x to y : Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$. Given vectors x and y in $\mathcal{H}$, when does there exist a normal operator A in Alg$\mathcal{L}$ such that $Ax = y$?

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra and Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be vectors in a Hilbert space. Then $< x, y >$ means the inner product of vectors x and y. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ a commutative subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Then Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ the orthogonal complement of $\mathcal{M}$.

Let A be an operator acting on $\mathcal{H}$. A is *normal* if $AA^* = A^*A$.

THEOREM 1. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ a commutative subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Assume that*

$$\mathcal{M} = \left\{ \sum_{i=1}^{n} \alpha_i E_i x : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L} \right\} \text{ is dense in } \mathcal{H}.$$

Then the following statements are equivalent.

(1) There is an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is normal and every E in $\mathcal{L}$ reduces A.

(2) $\sup \left\{ \dfrac{\| \sum_{i=1}^{n} \alpha_i E_i y\|}{\| \sum_{i=1}^{n} \alpha_i E_i x\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L} \right\} < \infty$ *and*

there is a vector h in $\overline{\mathcal{M}}$ such that $< Ey, x > = < Ex, h >$ and $< Ey, y > = < Eh, h >$ for all E in $\mathcal{L}$.

Proof. If we assume that (1) holds, then by Theorem 1 [6],

$\sup \left\{ \dfrac{\| \sum_{i=1}^{n} \alpha_i E_i y\|}{\| \sum_{i=1}^{n} \alpha_i E_i x\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L} \right\} < \infty$. Let $A^*x = h$. Then $h \in \overline{\mathcal{M}}$ and $< Ex, h > = < Ex, A^*x > = < EAx, x > = < Ey, x >$ and $< Eh, h > = < Ex, AA^*x > = < Ex, A^*Ax > = < EAx, Ax > = < Ey, y >$ for all E in $\mathcal{L}$.

Conversely, under the given condition, there is an operator A in $\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A by Theorem 1 [6]. Since $< Ey, x > = < Ex, h >$,

$$\begin{aligned} < A(\sum_{i=1}^{n} \alpha_i E_i x), x > &= < \sum_{i=1}^{n} \alpha_i E_i y, x > \\ &= < \sum_{i=1}^{n} \alpha_i E_i x, h > . \end{aligned}$$

So $< Af, x >=< f, h >$ for all f in $\overline{\mathcal{M}} = \mathcal{H}$. Since $< Ag, x >= 0 =< g, h >$ for $g \in \overline{\mathcal{M}}^{\perp}$, $A^*x = h$. Since $< Ey, y >=< Eh, h >$ for all E in $\mathcal{L}$,

$$\begin{aligned} < A(\sum_{i=1}^{n} \alpha_i E_i x), y > &=< \sum_{i=1}^{n} \alpha_i E_i Ax, y > \\ &=< \sum_{i=1}^{n} \alpha_i E_i y, y > \\ &=< \sum_{i=1}^{n} \alpha_i E_i h, h > \\ &=< \sum_{i=1}^{n} \alpha_i E_i A^* x, h > \\ &=< \sum_{i=1}^{n} \alpha_i E_i x, Ah >, \end{aligned}$$

for all $n \in \mathbb{N}$, all $\alpha_i \in \mathbb{C}$ and all $E_i \in \mathcal{L}$. So $< Af, y >=< f, Ah >$ for all f in $\overline{\mathcal{M}} = \mathcal{H}$. Since $< Ag, y >=< g, Ah >$ and $< Ag, y >= 0 =< g, Ah >$ for $g \in \overline{\mathcal{M}}^{\perp}$, $A^*y = Ah$. Hence $AA^*x = A^*Ax$. Since $AE = EA$, $A^*E = EA^*$ for all E in $\mathcal{L}$. Since $AA^*x = A^*Ax$, $AA^*(\sum_{i=1}^{n} \alpha_i E_i x) = A^*A(\sum_{i=1}^{n} \alpha_i E_i x)$, for all $n \in \mathbb{N}$, all $\alpha_i \in \mathbb{C}$ and all $E_i \in \mathcal{L}$. So $AA^*f = A^*Af$ for all f in $\overline{\mathcal{M}} = \mathcal{H}$. Hence $AA^* = A^*A$. □

If we modify the proof of Theorem 1, we can prove the following theorems.

THEOREM 2. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ a commutative subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Let*

$\mathcal{M} = \left\{ \sum_{i=1}^{n} \alpha_i E_i x : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L} \right\}$. *Assume that Ey is in*

$\overline{\mathcal{M}}$ for all E in $\mathcal{L}$. Then the following statements are equivalent.

(1) There is an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is normal and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty$ *and there is a vector* h *in* $\overline{\mathcal{M}}$ *such that* $< Ey, x >=< Ex, h >$ *and* $< Ey, y >=< Eh, h >$ *for all* E *in* $\mathcal{L}$.

Proof. (1)$\Rightarrow$ (2). By Theorem 1 [6],

$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} < \infty$. Let $A^*x = h$.

Then $< Ey, y >=< Eh, h >$ and $< Ey, x >=< Ex, h >$ for all E in $\mathcal{L}$. Since $< g, h >=< g, A^*x >=< Ag, x >=< 0, x >= 0$ for all g in $\overline{\mathcal{M}}^{\perp}$, h is a vector in $\overline{\mathcal{M}}$.

(2)$\Rightarrow$ (1). Under the first condition of hypothesis, there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$ and every E in $\mathcal{L}$ reduces A by Theorem 1 [6]. Since $< Ey, x >=< Ex, h >$,

$$\begin{aligned} < A(\sum_{i=1}^{n}\alpha_i E_i x), x > &=< \sum_{i=1}^{n}\alpha_i E_i y, x > \\ &=< \sum_{i=1}^{n}\alpha_i E_i x, h > . \end{aligned}$$

Since $< g, h >= 0$ and $< Ag, x >= 0$ for all g in $\overline{\mathcal{M}}^{\perp}$, $A^*x = h$. Since $< Ey, y >=< Eh, h >$ for all E in $\mathcal{L}$,

$$\begin{aligned} < A(\sum_{i=1}^{n}\alpha_i E_i x), y > &=< \sum_{i=1}^{n}\alpha_i E_i Ax, y > \\ &=< \sum_{i=1}^{n}\alpha_i E_i y, y > \\ &=< \sum_{i=1}^{n}\alpha_i E_i h, h > \\ &=< \sum_{i=1}^{n}\alpha_i E_i A^*x, h > \\ &=< \sum_{i=1}^{n}\alpha_i E_i x, Ah > . \end{aligned}$$

Since $Ey \in \overline{\mathcal{M}}$ for all E in $\mathcal{L}$, $\sum_{i=1}^{n} \alpha_i E_i y \in \overline{\mathcal{M}} \cdots (i)$. Since $h \in \overline{\mathcal{M}}$, $Ah \in \overline{\mathcal{M}}$ by (i). Since $< g, Ah >= 0$, $< Ag, y >=< g, Ah >$ for all g in $\overline{\mathcal{M}}^{\perp}$. Hence $A^*y = Ah$. So $A^*Ax = AA^*x$. Hence $A^*Af = AA^*f$ for all f in $\overline{\mathcal{M}}$. Since $Ax = y$ and $Ey \in \overline{\mathcal{M}}$, $0 =< Ey, g >=< EAx, g >=< AEx, g >=< Ex, A^*g >$ for all E in $\mathcal{L}$ and all g in $\overline{\mathcal{M}}^{\perp}$. So $A^*g \in \overline{\mathcal{M}}^{\perp}$ and hence $AA^*g = 0$ for all g in $\overline{\mathcal{M}}^{\perp}$. So $AA^*g = 0 = A^*Ag$ for all g in $\overline{\mathcal{M}}^{\perp}$. Hence $AA^* = A^*A$. □

THEOREM 3. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be vectors in $\mathcal{H}$. Assume that*

$$\mathcal{N} = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\} \text{ is dense}$$

in $\mathcal{H}$. Then the following statements are equivalent.

(1) $\sup \left\{ \dfrac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\}$

$< \infty$ and there are vectors h_p's in $\overline{\mathcal{N}}$ such that $< Ey_p, y_q >=< Eh_p, h_q >$ and $< Ey_p, x_q >=< Ex_p, h_q >$ for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$.

(2) There is an operator A in Alg$\mathcal{L}$ such that $Ax_k = y_k$ $(k = 1, 2, \cdots, n)$, A is normal and every E in $\mathcal{L}$ reduces A.

Proof. (1)⇒ (2). Under the first condition of hypothesis, there is an operator A in Alg$\mathcal{L}$ such that $Ax_k = y_k$ $(k = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A by Theorem 2 [6]. Since $< Ey_p, x_q >=< Ex_p, h_q >$ for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$,

$$\begin{aligned} < A(\sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i), x_p > &=< \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i, x_p > \\ &=< \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i, h_p >, \end{aligned}$$

$m_i \in \mathbb{N}$, $l \leq n$, $\alpha_{k,i} \in \mathbb{C}$, $E_{k,i} \in \mathcal{L}$ and $p = 1, 2, \cdots, n$. So $< Af, x_p > = < f, h_p >$ for all f in $\overline{\mathcal{N}} = \mathcal{H}$. Hence $A^* x_p = h_p$ for all $p = 1, 2, \cdots, n$. Since $< Ey_p, y_q > = < Eh_p, h_q >$ for all E in $\mathcal{L}$ and $p, q = 1, 2, \cdots, n$,

$$\begin{aligned} < A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}x_i), y_p > &= < \sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}y_i, y_p > \\ &= < \sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}h_i, h_p > \\ &= < \sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}A^*x_i, h_p > \\ &= < \sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}x_i, Ah_p >, \end{aligned}$$

for all $m_i \in \mathbb{N}$, all $l \leq n$, $\alpha_{k,i} \in \mathbb{C}$ and all $E_{k,i} \in \mathcal{L}$. So $< Af, y_p > = < f, Ah_p >$ for all f in $\overline{\mathcal{N}} = \mathcal{H}$ and all $p = 1, 2, \cdots, n$. Thus $A^* y_p = Ah_p (p = 1, 2, \cdots, n)$. Hence $A^*Ax_p = AA^*x_p (p = 1, 2, \cdots, n)$. Since $AE = EA$, $A^*E = EA^*$ for all E in $\mathcal{L}$. Since $AA^*x_p = A^*Ax_p (p = 1, 2, \cdots, n)$, $AA^*(\sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}\, E_{k,i}x_i) = A^*A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}x_i)$, for all $m_i \in \mathbb{N}$, $l \leq n$, $\alpha_{k,i} \in \mathbb{C}$ and $E_{k,i} \in \mathcal{L}$. So $AA^*f = A^*Af$ for all f in $\mathcal{H}$. Hence $AA^* = A^*A$.

(2)$\Rightarrow$ (1). Under the conditions of hypothesis except that the operator A is normal,

$$\sup\left\{ \frac{\| \sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}y_i \|}{\| \sum_{k=1}^{m_i}\sum_{i=1}^{l} \alpha_{k,i}E_{k,i}x_i \|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\} < \infty$$

by Theorem 2 [6]. Let $A^*x_p = h_p (p = 1, 2, \cdots, n)$. Then $h_p \in \overline{\mathcal{N}} (p = 1, 2, \cdots, n)$ and

$$\begin{aligned} < Ex_p, h_q > &= < Ex_p, A^*x_q > \\ &= < AEx_p, x_q > \\ &= < EAx_p, x_q > \\ &= < Ey_p, x_q > \text{ and} \end{aligned}$$

$$\begin{aligned} < Eh_p, h_q > &=< EA^*x_p, A^*x_q > \\ &=< AA^*Ex_p, x_q > \\ &=< EAx_p, Ax_q > \\ &=< Ey_p, y_q > \end{aligned}$$

for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$. □

THEOREM 4. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be vectors in $\mathcal{H}$.*

Let $\mathcal{N} = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\}$.

Assume that $Ey_p \in \overline{\mathcal{N}}$ for all E in $\mathcal{L}$ and $p = 1, 2, \cdots, n$.

If $\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\}$

$< \infty$ *and if there are vectors h_p's in $\overline{\mathcal{N}}$ such that $< Ey_p, x_q >=< Ex_p, h_q >$ and $< Ey_p, y_q >= < Eh_p, h_q >$ for all E in $\mathcal{L}$ and all $p,\ q = 1, 2, \cdots, n$, then there is an operator A in Alg$\mathcal{L}$ such that $Ax_q = y_q$ $(q = 1, 2, \cdots, n)$, A is normal and every E in $\mathcal{L}$ reduces A.*

Proof. Under the first condition of hypothesis, there is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$ and every E in $\mathcal{L}$ reduces A by Theorem 2 [6]. Since $< Ey_p, x_q >=< Ex_p, h_q >$ for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$,

$$\begin{aligned} < A(\sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i), x_q > &=< \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i, x_q > \\ &=< \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i, h_q >, \end{aligned}$$

for all $m_i \in \mathbb{N}$, all $l \leq n$, all $\alpha_{k,i} \in \mathbb{C}$, all $E_{k,i} \in \mathcal{L}$ and $q = 1, 2, \cdots, n$. Let $f \in \overline{\mathcal{N}}$ and $\{f_p\}$ be a sequence in $\mathcal{N}$ such that $f_p \to f$. Then since

$< Af_p, x_q >=< f_p, h_q >$ for all $p = 1, 2, \cdots$, $< Af, x_q >=< f, h_q >$ for all $q = 1, 2, \cdots, n$. Since $< g, h_q >= 0$, $< Ag, x_q >= 0$ for all g in $\overline{\mathcal{N}}^{\perp}$, $A^*x_q = h_q$ for all $q = 1, 2, \cdots, n$. Since $< Ey_p, y_q >=< Eh_p, h_q >$ for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots, n$,

$$\begin{aligned} < A(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i), y_p > &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i, y_p > \\ &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}h_i, h_p > \\ &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}A^*x_i, h_p > \\ &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i, Ah_p > \end{aligned}$$

for all $p = 1, 2, \cdots, n$. So $< Af, y_p >=< f, Ah_p >$ for all f in $\overline{\mathcal{N}}$ and all $p = 1, 2, \cdots, n$. Since $Ey_p \in \overline{\mathcal{N}}$ $(p = 1, 2, \cdots, n)$, $\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i \in \overline{\mathcal{N}}$, for all $m_i \in \mathbb{N}$, all $l \leq n$, all $\alpha_{k,i} \in \mathbb{C}$ and all $E_{k,i} \in \mathcal{L} \cdots (ii)$. Since $< g, h_q >= 0$ for all g in $\overline{\mathcal{N}}^{\perp}$, $h_q \in \overline{\mathcal{N}}$ for all $q = 1, 2, \cdots, n$. Hence $Ah_q \in \overline{\mathcal{N}}$ by (ii). Since $< g, Ah_p >= 0$, $< Ag, y_p >=< g, Ah_p >$ for all g in $\overline{\mathcal{N}}^{\perp}$ and all $p = 1, 2, \cdots, n$. So $A^*y_p = Ah_p$ for all $p = 1, 2, \cdots, n$. Thus $A^*Ax_p = AA^*x_p$ for all $p = 1, 2, \cdots, n$ and hence $A^*Af = AA^*f$ for all f in $\overline{\mathcal{N}}$. Since $Ax_p = y_p$ and $Ey_p \in \overline{\mathcal{N}}$, $0 =< Ey_p, g >=< EAx_p, g >=< Ex_p, A^*g >$, $E \in \mathcal{L}$, $g \in \overline{\mathcal{N}}^{\perp}$ and $p = 1, 2, \cdots, n$. So $A^*g \in \overline{\mathcal{N}}^{\perp}$ and hence $AA^*g = 0$ for all g in $\overline{\mathcal{N}}^{\perp}$. Since $A^*Ag = 0$, $AA^*g = A^*Ag$ for all g in $\overline{\mathcal{N}}^{\perp}$. Hence $AA^* = A^*A$. □

THEOREM 5. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be vectors in $\mathcal{H}$.*

Let $\mathcal{N} = \left\{ \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\}.$

If there is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$, A is normal and every E in $\mathcal{L}$ reduces A, then

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\}$$

$< \infty$ and there are vectors h_p's in $\overline{\mathcal{N}}$ such that $< Ey_p, x_q >=< Ex_p, h_q >$ and $< Ey_p, y_q > = < Eh_p, h_q >$ for all E in $\mathcal{L}$ and all p, $q = 1, 2, \cdots, n$.

Proof. By Theorem 2[6], we can get the first part of result. Let $A^*x_q = h_q (q = 1, 2, \cdots, n)$. Then $h_q \in \overline{\mathcal{N}}$ and

$$\begin{aligned}
< Ex_p, h_q > &=< Ex_p, A^*x_q > \\
&=< AEx_p, x_q > \\
&=< Ey_p, x_q > \quad \text{and} \\
< Ey_p, y_q > &=< EAx_p, ax_q > \\
&=< EA^*x_p, A^*x_q > \\
&=< Eh_p, h_q > \quad (p, q = 1, 2, \cdots, n).
\end{aligned}$$

□

If we summarize Theorems 4 and 5, then we can get the following theorem.

THEOREM 6. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be vectors in $\mathcal{H}$.*

Let $\mathcal{N} = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i \in \mathbb{N}, l \leq n, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L}\right\}$.

Assume that $Ey_p \in \overline{\mathcal{N}}$ for all E in $\mathcal{L}$ and $p = 1, 2, \cdots, n$. Then the following statements are equivalent.

(1) There is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ $(p = 1, 2, \cdots, n)$, A is normal and every E in $\mathcal{L}$ reduces A.

$$(2)\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}: m_i\in\mathbb{N}, l\leq n, \alpha_{k,i}\in\mathbb{C} \text{ and } E_{k,i}\in\mathcal{L}\right\}$$

$<\infty$ and there are vectors h_p's in $\overline{\mathcal{N}}$ such that $< Ey_p, x_q >=< Ex_p, h_q >$ and $< Ey_p, y_q >=< Eh_p, h_q >$, $E \in \mathcal{L}$ and all $p, q = 1, 2, \cdots, n$.

If we modify the proofs of Theorems 2, 3, 4 and 5, we can prove the following theorems. So we will omit their proofs,

THEOREM 7. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. Assume that*

$$\mathcal{K}=\left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i, l\in\mathbb{N}, \alpha_{k,i}\in\mathbb{C} \text{ and } E_{k,i}\in\mathcal{L}\right\} \text{ is dense in}$$

$\mathcal{H}$. Then the following statements are equivalent.

(1) There is an operator A in Alg$\mathcal{L}$ such that $Ax_n = y_n$ ($n \in \mathbb{N}$), A is normal and every E in $\mathcal{L}$ reduces A.

$$(2)\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i, l\in\mathbb{N}, \alpha_{k,i}\in\mathbb{C} \text{ and } E_{k,i}\in\mathcal{L}\right\}$$

$<\infty$ and there is a sequence $\{h_n\}$ of vectors in $\overline{\mathcal{K}}$ such that $< Ey_p, y_q >=< Eh_p, h_q >$ and $< Ey_p, x_q >=< Ex_p, h_q >$ for all E in $\mathcal{L}$ and all $p, q = 1, 2, \cdots$.

THEOREM 8. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$.*

$$\text{Let } \mathcal{K}=\left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i, l\in\mathbb{N}, \alpha_{k,i}\in\mathbb{C} \text{ and } E_{k,i}\in\mathcal{L}\right\}.$$

Assume that $Ey_p \in \overline{\mathcal{K}}$, $E \in \mathcal{L}$ and all $p = 1, 2, \cdots$. Then the following statements are equivalent.

(1) There is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ $(p \in \mathbb{N})$, A is normal and every E in $\mathcal{L}$ reduces A.

(2) $\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i, l \in \mathbb{N}, \alpha_{k,i} \in \mathbb{C} \text{ and } E_{k,i} \in \mathcal{L} \right\}$

$< \infty$ and there is a sequence $\{h_n\}$ of vectors in $\overline{\mathcal{K}}$ such that $< Ey_p, y_q > = < Eh_p, h_q >$ and $< Ey_p, x_q > = < Ex_p, h_q >$ for all E in $\mathcal{L}$ and all $p, q \in \mathbb{N}$.

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