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CORNER SINGULARITY AT THE MULTIPLE JUNCTION OF THE ELECTRIC TRANSMISSION

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Abstract. We consider the several plane sector domains which are bonded together along common edges with vertex at the origin. Such domains appear in electric conducting problem with multi-layered heterogeneous media. Our aim is to give a structure theorem of the singularities of the electric field at the corner. Also, we provide a regularity theorem for the electric field.

1. Introduction

There have been many studies for the elliptic boundary value problems in non-smooth domains $\Omega \subset \mathbf{R}^n$ including Lipschitz domain. In this note, we are interested particularly in domain with corner points in the plane for electric field. Many results about cornered domains can be found in the works by Kondrat'ev[4] and Grisvard[3]. In the monography [3] by Grisvard he proved the existence of the solutions of boundary value problems for the Laplace operator in a plane domain with corner like a polygon. More precisely stating, under the Mellin transform in the polar coordinate for the polygonal domain, Poisson type problem for Laplace operator reduces to two point boundary value problem for each frequency. With the regularity condition on the function spaces, the solution of the boundary value problem can be decomposed into a regular part and a linear combination of the singular functions. Here, the regular part has the maximal regularity allowed by the right side of $\triangle u = f$. In addition, the nature of singularity does not depend on the forcing term of the differential equation but only on the domain and the differential operators.

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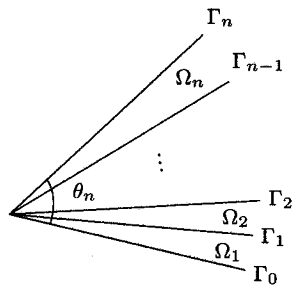
In particular, we consider the several plane sector domains which are bonded together along common edges with vertex at the origin. The problem arises in electric conducting problems with multi-layered heterogeneous media. In this note, our aim is to give a structure theorem of the singularities of the solution at the corner. So, we may use the method of Kondrat'ev like in [4] and [3] in a local sense.

On the other hand, due to the importance in engineering, there are several studies on transmission problem. In 1990, in the case of the system of linear elasticity for isotropic material, Reitich[6] studied two dimensional linear elasticity in a domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ with a jump in the elasticity coefficients across Γ where $\Omega_1 \subset \{(r,\theta) | r > 0, 0 < \theta < \omega_1\}$, $\Omega_2 \subset \{(r,\theta) | r > 0, -\omega_2 < \theta < 0\}$ satisfying $0 < \omega_1, \omega_2 < \pi$ and $\Gamma = \partial\Omega_1 \cap \partial\Omega_2 \subset \{\theta = 0\}$. In this case, it is observed the singular behavior of the solution near the corner point because of the lack of regularity and transmission condition. However, he only treated the case of $\omega_1 = \omega_2$ and it turns out a model case in our problem.

In section 2, we find a variational solution and *a priori* $W^{2,p}$ regularity estimate by forcing term. Then, we find singular expansion near corner point. The geometric picture of the domain is the following:



We consider a sequence of angles $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ and we let the domain $\Omega = \cup_i \Omega_i$ with $\Omega_i = \{(r,\theta) | r > 0, \theta_{i-1} < \theta < \theta_i\}$, $\Gamma_0 = \{\theta = 0\}, \Gamma_i = \{\theta = \theta_i\}$. The following function spaces are in use:

$$L^p(\Omega) = \left\{ u : ||u||_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$
$$W^{k,p}(\Omega) = \left\{ u : ||u||_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \le k} |\nabla^\alpha u|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

where α is multi-index and Einstein summation convention is used. When $p = 2$, we let $H^k(\Omega) = W^{k,2}(\Omega)$.

We consider the Laplace equation of Poisson type and the corresponding transmission problem: for $f^i \in L^p(\Omega_i), i = 1, \cdots, n$

$$(1.1) \qquad (T) \begin{cases} \triangle u^i = f^i & \text{in} \quad \Omega_i, \\ \kappa_i \dfrac{\partial u^i}{\partial \nu_i} = \kappa_{i+1} \dfrac{\partial u^{i+1}}{\partial \nu_i} & \text{on} \quad \Gamma_i, \\ u^i = u^{i+1} & \text{on} \quad \Gamma_i, \\ u^1 = 0 & \text{on} \quad \Gamma_0, \\ u^n = 0 & \text{on} \quad \Gamma_n. \end{cases}$$

We assume that κ_i is positive for the well-posedness and ν_i is the outward unit normal vector on Γ_i for Ω_i. To solve (T), we are to find *a priori* bounds for solutions in $W^{2,p}(\Omega)$ by the L^p norm of force f. We shall prove the inequality $||u||_{W^{2,p}(\Omega)} \leq c||f||_{L^p(\Omega)}$ for any solution $u \in W^{2,p}(\Omega)$, for some constant c depending on Ω, p. As the second step in solving (T), we reduce (T) to simple case. To do this process, we use the polar coordinate(see (2.2)) and the Fourier transform(see (2.3)). Then, we solve the two point boundary value problem for the case of homogeneous equation in each eigenvalue. Then, complying with the inhomogeneous forcing term, we get a characterization as the followings: If $u \in L^q$ is a solution of (1.1), then we have

$$u = \overline{u} + \Sigma c_m \phi_m, \quad \text{where} \quad \overline{u} \in W^{2,p}(\Omega).$$

Also, we find an explicit formula for eigenvalues and eigenfunction basis $\{\phi_m\}$ in section 3.

2. $W^{2,p}$ Estimate

The first step in solving (T) is the proof of the existence using the variational method in Hilbert space framework. We suppose $f \in L^2(\Omega)$ and let $u \in H_0^1(\cup_i \Gamma_i \cup \Omega)$ be the minimizer of

$$\sum_i \int_{\Omega_i} \frac{\kappa_i}{2} |\nabla u(x)|^2 + f(x)u(x)\, dx.$$

If we let $u^i = u\chi_{\Omega_i}$ and $f^i = f\chi_{\Omega_i}$ with the characteristic function χ_{Ω_i} for $i = 1, \cdots, n$, then u^i satisfy our transmission problem (T).

It is important to know $W^{2,p}$ regularity holds in cornered domain like free space when there is no singular zero of characteristic equation. The following theorem states H^2 estimate in two dimension.

THEOREM 2.1. *Suppose $f \in L^2(\Omega)\bigcap C^\infty(\Omega)$ and $u \in H^2(\cup_i\Gamma_i \cup \Omega)$ is a solution to (T). Then there is a constant c such that*

$$||u||_{H^2(\Omega)} \le c\left(||f||_{L^2(\Omega)} + ||u||^2_{1,2;\Omega}\right).$$

Proof. We assume f is smooth. Since $u^i = u^{i+1}$ on Γ_i, $\frac{\partial u^i}{\partial \tau} = \frac{\partial u^{i+1}}{\partial \tau}$ on Γ_i, where τ is the tangent vector along Γ_i. With suitable rotation, we assume Γ_i is the positive x-axis. Furthermore if we take a test function ϕ_x supported in $\Omega_i \cup \Omega_{i+1} \cup \Gamma_i$, we obtain

$$\int_{\Omega_i}(\kappa_i \nabla u \cdot \nabla \phi_x + f\phi_x) + \int_{\Omega_{i+1}}(\kappa_{i+1}\nabla u \cdot \nabla\phi_x + f\phi_x) = 0.$$

Integrating by parts we have

$$\int_{\Omega_i}(\kappa_i \nabla u_x \cdot \nabla \phi + f_x\phi) + \int_{\Omega_{i+1}}(\kappa_{i+1}\nabla u_x \cdot \nabla\phi + f_x\phi) = 0$$

and

$$\int_{\partial\Omega_i}\kappa_i\frac{\partial u_x^i}{\partial \nu^i}\phi\, d\sigma - \int_{\partial\Omega_{i+1}}\kappa_{i+1}\frac{\partial u_x^{i+1}}{\partial \nu^i}\phi\, d\sigma - \int_{\Omega_i}(\kappa_i\triangle u - f)_x\phi - \int_{\Omega_{i+1}}(\kappa_{i+1}\triangle u - f)_x\phi = 0.$$

From

$$\kappa_i\int_{\partial\Omega_i}\frac{\partial u_x^i}{\partial \nu^i}\phi\, d\sigma - \kappa_{i+1}\int_{\partial\Omega_{i+1}}\frac{\partial u_x^{i+1}}{\partial \nu^i}\phi\, d\sigma = \int_{\Gamma_i} D_y(\kappa_i u_x^i - \kappa_{i+1}u_x^{i+1})\phi = 0,$$

we get $\kappa_i u^i_{xy} = \kappa_{i+1}u^{i+1}_{xy}$ on Γ_i. Hence, from the integration by parts, we have

$$\kappa_i\int_{\Omega_i}u^i_{xx}u^i_{yy} - \kappa_i\int_{\Omega_i}u^i_{xy}u^i_{xy} + \kappa_{i+1}\int_{\Omega_{i+1}}u^{i+1}_{xx}u^{i+1}_{yy} - \kappa_{i+1}\int_{\Omega_{i+1}}u^{i+1}_{xy}u^{i+1}_{xy} = 0$$

and

$$\kappa_i\int_{\Omega_i}u_{xx}u_{yy} + \kappa_{i+1}\int_{\Omega_{i+1}}u_{xx}u_{yy} = \kappa_i\int_{\Omega_i}u^2_{xy} + \kappa_{i+1}\int_{\Omega_{i+1}}u^2_{xy}.$$

Then, it follows that

$$\kappa_i \int\limits_{\Omega_i} |\triangle u^i|^2 \kappa_{i+1} \int\limits_{\Omega_{i+1}} |\triangle u^{i+1}|^2$$

$$= \kappa_i \int\limits_{\Omega_i} ((u^i_{xx})^2 + (u^i_{yy})^2 + 2u^i_{xx}u^i_{yy})$$

$$+ \kappa_{i+1} \int\limits_{\Omega_{i+1}} ((u^{i+1}_{xx})^2 + (u^{i+1}_{yy})^2 + 2u^{i+1}_{xx}u^{i+1}_{yy})$$

$$= \kappa_i \int\limits_{\Omega_i} ((u^i_{xx})^2 + (u^i_{yy})^2 + 2(u^i_{xy})^2)$$

$$+ \kappa_{i+1} \int\limits_{\Omega_{i+1}} ((u^{i+1}_{xx})^2 + (u^{i+1}_{yy})^2 + 2(u^{i+1}_{xy})^2).$$

Therefore, we have

$$\kappa_i \int\limits_{\Omega_i} |\nabla^2 u|^2 + \kappa_{i+1} \int\limits_{\Omega_{i+1}} |\nabla^2 u|^2 = \kappa_i \int\limits_{\Omega_i} |\triangle u|^2 + \kappa_{i+1} \int\limits_{\Omega_{i+1}} |\triangle u|^2.$$

Considering all the interfaces Γ_i, we conclude our proof. □

From now on we consider the $W^{2,p}$ estimate. We shall now derive the inequality $||u||_{2,p;\Omega} \leq C\{||f||_{0,p;\Omega} + ||u||_{1,p;\Omega}\}$ for $1 < p < \infty$, $p \neq 2$. We consider some weighted spaces similar to those introduced by [3]. We denote by $\rho(x, y)$ the distance from the point (x, y) to the origin which is the corner point of Ω. We denote by $P^m_p(\Omega)$ the space of all function u defined in Ω such that $\rho^{|\alpha|-m} D^\alpha u \in L_p(\Omega)$ for all $|\alpha| \leq m$. Obviously we define a Banach space norm on $P^m_p(\Omega)$ by letting

$$||u||_{P^m_p(\Omega)} = \sum_{|\alpha|\leq m} ||\rho^{|\alpha|-m} D^\alpha u||_{0,p;\Omega}.$$

By using the above weighted space, we find *a priori* bound of the solution u in $P^m_p(\Omega)$ satisfying (T). By localization we assume u has a compact support and for convenience, we let v be a function satisfying $\triangle v = g$ for some g satisfying

$$||g||_{0,p,\Omega} \leq K(||f||_{0,p,\Omega} + ||u||_{1,p,\Omega}) \tag{2.1}$$

with the same boundary conditions and compact support. We set $w(t, \theta) = e^{-(2/q)t} u(e^{t+i\theta})$. Obviously, the polar coordinate is used and r is replaced by e^t. Then w is a solution of a boundary value problem in the

n-fold strip $B = \mathbf{R} \times \cup_{i=0}^{n-1}(\theta_i, \theta_{i+1})$ and under this transformation the equation is

$$(2.2) \qquad (A) \begin{cases} w_{tt} + w_{\theta\theta} + \frac{4}{q} w_t + \frac{4}{q^2} w = k & \text{in} \quad B, \\ w^1(t, \theta_0) = 0, \\ w^n(t, \theta_n) = 0, \\ w^i(t, \theta_i) = w^{i+1}(t, \theta_i), \\ \kappa_i \frac{\partial w^i}{\partial \theta}(t, \theta_i) = \kappa_{i+1} \frac{\partial w^{i+1}}{\partial \theta}(t, \theta_i), \end{cases}$$

where $k = k(t, \theta) = e^{-(2/q)t}\{e^{2t} g(e^{t+i\theta})\}$. After performing the Fourier transform with respect to t on (A) we get

$$(2.3) \qquad (\hat{A}) \begin{cases} \hat{w}_{\theta\theta} + (b + ia\xi - \xi^2)\hat{w} = \hat{k}, \\ \hat{w}^1(\xi, \theta_0) = 0, \\ \hat{w}^n(\xi, \theta_n) = 0, \\ \hat{w}^i(\xi, \theta_i) = \hat{w}^{i+1}(\xi, \theta_i), \\ \kappa_i \frac{\partial \hat{w}^i}{\partial \theta}(\xi, \theta_i) = \kappa_{i+1} \frac{\partial \hat{w}^{i+1}}{\partial \theta}(\xi, \theta_i), \end{cases}$$

where $a = 4/q$, $b = 4/q^2$.
A fundamental solution to $(\hat{A})$ is the couple of functions $(h_1(\theta), h_2(\theta)) = (\sin \rho\theta, \cos \rho\theta)$ where $\rho = (b + ia\xi - \xi^2)^{1/2}$ for $\xi \in \mathbf{R}$. The problem $(\hat{A})$ is well-defined if and only if the homogeneous solution is trivial. Thus we are to consider the homogeneous solution. We let the homogeneous solution to $(\hat{A})$ be

$$w^i = c_{i1} h_1 + c_{i2} h_2 \quad \text{for} \quad \theta_i < \theta < \theta_{i+1}.$$

The boundary condition is fulfilled if and only if for $i = 1, ..., n$,
(2.4)

$$\begin{cases} c_{11} \sin(\rho\theta_0) + c_{12} \cos(\rho\theta_0) = 0, \\ c_{n1} \sin(\rho\theta_n) + c_{n2} \cos(\rho\theta_n) = 0, \\ c_{i1} \sin(\rho\theta_i) + c_{i2} \cos(\rho\theta_i) = c_{(i+1)1} \sin(\rho\theta_i) + c_{(i+1)2} \cos(\rho\theta_i), \\ c_{i1} \cos(\rho\theta_i) - c_{i2} \sin(\rho\theta_i) = \alpha_i c_{(i+1)1} \cos(\rho\theta_i) - \alpha_i c_{(i+1)2} \sin(\rho\theta_i), \end{cases}$$

where $\alpha_i = \frac{\kappa_{i+1}}{\kappa_i}$. Since this is a linear system in (c_{i1}, c_{i2}), we obtain a matrix form such that

$$\begin{pmatrix} \sin\rho\theta_0 & \cos\rho\theta_0 & 0 & 0 & 0 & \dots & 0 \\ \sin\rho\theta_1 & \cos\rho\theta_1 & -\sin\rho\theta_1 & -\cos\rho\theta_1 & 0 & \dots & 0 \\ \cos\rho\theta_1 & -\sin\rho\theta_1 & -\alpha_1\cos\rho\theta_1 & \alpha_1\sin\rho\theta_1 & 0 & \dots & 0 \\ 0 & 0 & \sin\rho\theta_2 & \cos\rho\theta_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & \sin(\rho\theta_n) & \cos(\rho\theta_n) \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \\ \vdots \\ c_{n1} \\ c_{n2} \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Thus if the coefficient matrix is nonsingular, there is not nontrivial solution. Hence we are interested in the determinant of the above coefficient matrix $M_{2(n+1)}$ for $n \geq 1$ and we get a nontrivial solution if the determinant doesn't vanish.

We let D_n be the determinant of the matrix $M_{2(n+1)}$. Note that we have $D_1(\rho\theta_0, \rho\theta_1) = -\sin(\rho(\theta_1 - \theta_0))$. Then, with D_1, we get the recurrence relation for the determinant D_n as follows:

$$\begin{aligned} &D_n(\rho\theta_0, \cdots, \rho\theta_n) \\ (2.5) \qquad &= -\alpha_{n-1}\cos(\rho(\theta_n - \theta_{n-1}))D_{n-1}(\rho\theta_0, \cdots, \rho\theta_{n-1}) \\ &\quad - \sin(\rho(\theta_n - \theta_{n-1}))D_{n-1}\left(\rho\theta_0, \cdots, \rho\theta_{n-2}, \rho\theta_{n-1} + \frac{\pi}{2}\right) \end{aligned}$$

for $n \geq 2$. If we define the determinant vector H_n in phase shift by

$$H_n = \left(D_n(\rho\theta_0, \cdots, \rho\theta_n), D_n(\rho\theta_0, \cdots, \rho\left(\theta_n + \frac{\pi}{2}\right)\right)^T,$$

we obtain a recurrence vector relation

$$H_n = \begin{pmatrix} -\alpha_{n-1}\cos(\rho(\theta_n - \theta_{n-1})), & -\sin(\rho(\theta_n - \theta_{n-1})) \\ \alpha_{n-1}\sin(\rho(\theta_n - \theta_{n-1})), & -\cos(\rho(\theta_n - \theta_{n-1})) \end{pmatrix} H_{n-1}$$

with $H_1 = (-\sin(\rho(\theta_1-\theta_0)), -\cos(\rho(\theta_1-\theta_0)))^T$. We know that

$$\lim_{|\xi|\to\infty} \frac{\sin(\rho\theta)}{e^{|\xi|\theta}} = -\frac{i}{2} \quad \text{and} \quad \lim_{|\xi|\to\infty} \frac{\cos(\rho\theta)}{e^{|\xi|\theta}} = \frac{1}{2}.$$

Thus, with straightforward computation, we find that D_n behaves like $i\left(\frac{-1}{2}\right)^n e^{|\xi|\theta_n}(\alpha_1+1)\cdots(\alpha_{n-1}+1)$ as $|\xi|$ goes to infinity. Hence there is no zero of D_n near infinity. Since D_n is analytic as a function of ξ, each zero is isolated and there are only a finite number of zeros of D_n on the real line.

REMARK. When there are only two layers, namely $n=2$, we get the determinant is equal to the following

$$\sin(\rho(\theta_2-\theta_1))\cos(\rho(\theta_1-\theta_0)) + \alpha_1\cos(\rho(\theta_2-\theta_1))\sin(\rho(\theta_1-\theta_0)),$$

where $\rho = (4/p^2 + i(4/p)\xi - \xi^2)^{1/2}$ for $\xi \in \mathbf{R}$. So the problem $(\hat{A})$ is well-posed unless the above determinant vanishes.

To handle the inhomogeneous term in $(\hat{A})$, we let $\hat{v}^i$ be the solution to

$$(2.6) \qquad (\hat{B})\begin{cases} \hat{v}^i_{\theta\theta} + (b + ia\xi - \xi^2)\hat{v}^i = \hat{f}, \\ \hat{v}^i(\xi,\theta_i) = 0, \\ \hat{v}^i(\xi,\theta_{i-1}) = 0, \end{cases}$$

in $\theta_{i-1} < \theta < \theta_i$. From the Green function expression, we get

$$\int_{-\infty}^{\infty} |v^i(t,\theta)|^p dt \le c\int_{\theta_{i-1}}^{\theta_i} |M(\theta,z)|^p \Big(\int_{-\infty}^{\infty} |f(t,z)|^p dt\Big)^{1/p} dz,$$

where M is computed from the Green function as Lemma 4.2.1.3 in [3]. If we let $\hat{e} = \hat{w} - \hat{v}$, then $\hat{e}$ is a solution to

$$(2.7) \qquad (\hat{B}^*)\begin{cases} \hat{e}_{\theta\theta} + \rho^2\hat{e} = 0, \\ \hat{e}^1(\xi,\theta_0) = 0, \\ \hat{e}^n(\xi,\theta_n) = 0, \\ \hat{e}^i(\xi,\theta_i) = \hat{e}^{i+1}(\xi,\theta_i), \\ \frac{\partial\hat{e}^i}{\partial\theta}(\xi,\theta_i) + \frac{\partial\hat{v}^i}{\partial\theta}(\xi,\theta_i) = \alpha_i\frac{\partial\hat{e}^{i+1}}{\partial\theta}(\xi,\theta_i) + \alpha_i\frac{\partial\hat{v}^{i+1}}{\partial\theta}(\xi,\theta_i). \end{cases}$$

REMARK. When there are only two layers, namely, n=2, we find the solution $\hat{e}$ such that

$$(2.8) \quad \hat{e} = \begin{cases} \hat{e}^1 = A\sin\rho\theta & \text{for } \; 0<\theta<\theta_1, \\ \hat{e}^2 = \frac{A\sin\rho\theta_1}{\sin\rho(\theta_1-\theta_2)}\sin\rho(\theta-\theta_2) & \text{for } \; \theta_1<\theta<\theta_2, \end{cases}$$

where $A = \{\alpha_1 \frac{\partial v^2}{\partial \theta}(\theta_1) - \frac{\partial v^1}{\partial \theta}(\theta_1)\}/\{\rho \cos \rho\theta_1 - \rho\alpha_1 \frac{\sin \rho\theta_1}{\sin \rho(\theta_1-\theta_2)} \cos \rho(\theta_1 - \theta_2)\}$.

THEOREM 2.2. *Suppose D_n doesn't vanish for all $\xi \in \mathbf{R}$ with $\rho = (b + ia\xi - \xi^2)^{1/2}$, then the solution $\hat{e}$ in the cone Ω satisfies*

$$||e||_{W^{2,p}(\Omega)} \leq c||v||_{W^{2,p}(\Omega)} \leq c||f||_{L^p(\Omega)},$$

where $e = \mathcal{F}^{-1}(\hat{e})$. Here, $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Proof. We refer [3](p.191) for the related proof. □

Therefore with the $W^{2,p}$ estimates of v and e, we obtain $W^{2,p}$ estimate of w and consequently P_p^2 estimate of u.

THEOREM 2.3. *Suppose there is no real root of D_n which is defined by (2.5). Then, there exists a constant c such that*

$$||u||_{P_p^2(\Omega)} \leq c\{||f||_{0,p,\Omega} + ||u||_{1,p,\Omega}\}$$

for all $u \in P_p^2(\Omega)$ which is solution to (T) under the well-posed condition of (T).

3. Singular expansion

We are interested in the structure of u satisfying (1.1), which is L^p class. In particular, the behavior of u near the corner point is our main concern. Using separation of variables in polar coordinates we are able to derive precise expansions of the solution near the corner. Suppose $L\phi = (\sum_i \kappa_i \chi_{\Omega_i} \phi')'$, where χ_A is the characteristic function supported on A. Then, L is self-adjoint and positive definite operator on $H_0^1(\Omega)$ with the norm $||u||_L = \sqrt{\sum_i \kappa_i ||\nabla u||_{L^2}^2}$. Hence, all the eigenvalues are positive real numbers and there is a nontrivial solution ϕ_m for the eigenvalue λ_m^2 to

$$(3.1) \qquad (E)\begin{cases} (\kappa_i(\phi_m^i)')' + \lambda_m^2 \phi_m^i = 0 \quad \text{in} \quad \Omega_i, \\ \phi_m^1(\theta_0) = \phi_m^n(\theta_n) = 0, \\ \phi_m^i(\theta_i) = \phi_m^{i+1}(\theta_i), \\ \frac{\partial \phi_m^i}{\partial \theta}(\theta_i) = \alpha_i \frac{\partial \phi_m^{i+1}}{\partial \theta}(\theta_i). \end{cases}$$

We can compute the eigenvalues λ_m as in section 2. We let $D_n(\lambda, \theta_0, \cdots, \lambda_n))$ be the characteristic function corresponding to $\{\theta_0, \cdots, \theta_n\}$. We have $D_1(\lambda, \theta_0, \lambda_1) = -\sin(\lambda\frac{\theta_1-\theta_0}{\sqrt{\kappa_1}})$. Then we get the recurrence relation for the characteristic function D_n as follows:

$$
\begin{aligned}
&D_n(\lambda, \theta_0, \cdots, \theta_n) \\
&= -\sqrt{\frac{\kappa_n}{\kappa_{n-1}}}\cos\left(\lambda\frac{\theta_n-\theta_{n-1}}{\sqrt{\kappa_n}}\right) D_{n-1}(\lambda, \theta_0, \cdots, \theta_{n-1}) \\
&\quad -\sin\left(\lambda\frac{\theta_n-\theta_{n-1}}{\sqrt{\kappa_n}}\right) D_{n-1}\left(\lambda, \theta_0, \cdots, \theta_{n-2}, \theta_{n-1}+\frac{\pi}{2}\right)
\end{aligned}
$$

for $n \geq 2$. If we define the determinant vector H_n in phase shift by

$$H_n = \left(D_n(\lambda, \theta_0, \cdots, \theta_n), D_n\left(\lambda, \theta_0, \cdots, \theta_n + \frac{\pi}{2}\right)\right)^T,$$

we obtain a recurrence vector relation

$$H_n = \begin{pmatrix} -\sqrt{\frac{\kappa_n}{\kappa_{n-1}}}\cos(\lambda\frac{\theta_n-\theta_{n-1}}{\sqrt{\kappa_n}}), & -\sin(\lambda\frac{\theta_n-\theta_{n-1}}{\sqrt{\kappa_n}}) \\ \sqrt{\frac{\kappa_n}{\kappa_{n-1}}}\sin(\lambda\frac{\theta_n-\theta_{n-1}}{\sqrt{\kappa_n}}), & -\cos(\lambda\frac{\theta_n-\theta_{n-1}}{\sqrt{\kappa_n}}) \end{pmatrix} H_{n-1}$$

with $H_1 = \left(-\sin(\lambda\frac{\theta_1-\theta_0}{\sqrt{\kappa_1}}), -\cos(\lambda\frac{\theta_1-\theta_0}{\sqrt{\kappa_1}})\right)$.

REMARK. When there are only two layers, namely $n = 2$, we get the characteristic function is equal to the following:

$$\sin\left(\lambda\frac{\theta_2-\theta_1}{\sqrt{\kappa_2}}\right)\cos\left(\lambda\frac{\theta_1-\theta_0}{\sqrt{\kappa_1}}\right) + \sqrt{\frac{\kappa_2}{\kappa_1}}\cos\left(\lambda\frac{\theta_2-\theta_1}{\sqrt{\kappa_2}}\right)\sin\left(\lambda\frac{\theta_1-\theta_0}{\sqrt{\kappa_1}}\right)$$

For the given eigenvalue λ_m, the eigenfunction ϕ_m is
(3.2)

$$\phi_m(\theta) = \begin{cases} \phi_m^1(\theta) = A\sin(\lambda_m\frac{\theta}{\sqrt{\kappa_1}}) & \text{for } 0 < \theta < \theta_1, \\ \phi_m^2(\theta) = \frac{A\sin(\lambda_m\frac{\theta_1}{\sqrt{\kappa_1}})}{\sin(\lambda_m\frac{\theta_2-\theta_1}{\sqrt{\kappa_2}})}\sin(\lambda_m\frac{\theta_2-\theta}{\kappa_2}) & \text{for } \theta_1 < \theta < \theta_2 \end{cases}$$

for a constant A.

In the first layer, $\phi_m^1(\theta) = c\sin(\lambda_m\frac{\theta}{\sqrt{\kappa_1}})$ for a constant c and ϕ_m^i can be considered as a solution to initial value problem instead of two point boundary value problem in each layer. Therefore, from the uniqueness of the initial value problem, we find any two solutions are linearly dependent.

We let $J = \{\phi \in H^1(\theta_0, \theta_n) : \phi|_{[\theta_i,\theta_{i+1}]} \equiv \phi^i(\theta) \in H^2(\theta_i, \theta_{i+1}),\ \phi^1(0) = 0,\ \phi^n(\theta_n) = 0,\ \phi^i(\theta_i) = \phi^{i+1}(\theta_i),\ \kappa_i\frac{\partial\phi^i}{\partial\theta}(\theta_i) = \kappa_{i+1}\frac{\partial\phi^{i+1}}{\partial\theta}(\theta_i)\}$. We define the inner product J by

$$< \phi, \psi >= \sum_{i=1}^{n} \frac{\kappa_i}{2} \int_{\theta_{i-1}}^{\theta_i} \frac{\partial \phi}{\partial \theta} \frac{\partial \psi}{\partial \theta} d\theta.$$

Then, with normalization, we have that $\{\phi_m\}$ is an orthonormal basis of J.

PROPOSITION 3.1. *Suppose that* $v \in C^\infty((0, R) \times \cup_{i=1}^n (\theta_{i-1}, \theta_i))$ *is a solution to* $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$ *and satisfies the transmission condition. We let* $d_m = \max_i \frac{\lambda_m}{\sqrt{\kappa_i}}$. *If* $v \in L^p(\cup_{i=1}^n \Omega_i)$, *we have*

$$\begin{aligned} v(re^{i\theta}) = & \sum_{m \geq 1} \alpha_m r^{\frac{\lambda_m}{\sqrt{\kappa_i}}} \phi_m(\theta) \\ & + \sum_{0 < d_m < 2/p} \beta_m r^{-\frac{\lambda_m}{\sqrt{\kappa_i}}} \phi_m(\theta) \quad \text{for} \quad \theta_{i-1} < \theta < \theta_i. \end{aligned} \tag{3.3}$$

Proof. We note 0 can not be an eigenvalue since $\kappa_i > 0$. Since the sequence ϕ_m, $m = 1, 2, ...$ is a basis of J, we have that for a fixed r

$$v(re^{i\theta}) = \sum_{m \geq 1} v_m(r) \phi_m(\theta), \text{ where } v_m(r) =< v(re^{i\theta}), \phi_m(\theta) > .$$

Since v is smooth, we have for $\theta_{i-1} < \theta < \theta_i$

$$v_m'' + \frac{1}{r} v_m' - \frac{\lambda_m^2}{\kappa_i r^2} v_m(r) = 0, \quad 0 < r < R$$

and accordingly we obtain $v_m(r) = \alpha_m r^{\frac{\lambda_m}{\sqrt{\kappa_i}}} + \beta_m r^{-\frac{\lambda_m}{\sqrt{\kappa_i}}}$. Therefore

$$|v_m(r)|^p \leq c \sum_i \int_{\theta_{i-1}}^{\theta_i} |v(re^{i\theta})|^p d\theta$$

and

$$\int_0^R |v_m(r)|^p r dr \leq c \int_0^R \int_{\theta_0}^{\theta_n} |v(re^{i\theta})|^p r d\theta dr.$$

This implies $\beta_m = 0$ when $d_m \geq 2/p$. □

REMARK. Roughly speaking, we compute λ_m from the characteristic equation and then find the range of λ_m such that the second term in the right side of (3.3) does not belong to $W^{2,p}$. A singular mode corresponding to such a λ can not appear in decomposition because of regularity.

From the above result, we get the main theorem.

THEOREM 3.2. *Assume Ω is a n-fold plane sector with a corner point at the origin like a figure. Let $u \in L^p$ be a solution of* (1.1). *Then we have*

$$u = \overline{u} + \sum_{m \geq 1, 0 < d_m < 2/p} c_m \phi_m, \quad \text{where} \quad \overline{u} \in W^{2,p}(\Omega),$$

where c_m is a constant and ϕ_m is given in the above proposition 3.1.

References

[1] M. Costabel and M. Dauge, *Singularities of electromagnetic Fields in Polyhedral Domains*, Arch. Ration. Mech. Anal. **151** (2000), 221–276.

[2] M. Dauge, *Elliptic boundary value problems on corner domains*, Lecture Notes in Math. no. 1341, 1988.

[3] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985.

[4] V. A. Kondrat'ev, *Boundary problem for elliptic equations in domains with conical or angular points*, Trudy Moscovkogo Mat. Obschetsva and Transactions Moscow. Mat. Soc. 227–313, **16** (1967), 209–292.

[5] Y. Lungan, *Interface Problem for elliptic Differential Equations*, Chinese Ann. Math. **18B** (1997), no. 2, 53–76.

[6] Fernando Reitich, *Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions*, Numer. Math. **59** (1991), 179–216.

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