J. Korean Math. Soc. **42** (2005), No. 6, pp. 1265–1278

LIMIT BEHAVIORS FOR THE INCREMENTS OF A d–DIMENSIONAL MULTI-PARAMETER GAUSSIAN PROCESS

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Abstract. In this paper, we establish limit theorems containing both the moduli of continuity and the large incremental results for finite dimensional Gaussian processes with N parameters, via estimating upper bounds of large deviation probabilities on suprema of the Gaussian processes.

1. Introduction and results

The limit theory on the increments of Wiener processes, partial sum processes, empirical processes and etc. is integrated in Csörgő and Révész [9] and Lin and Lu[19].

Since then, many various limit theories for fractional Brownian motions, renewal processes, Gaussian processes and related stochastic processes have been developed in [1, 2, 3, 4, 5, 10, 11, 14, 17, 18, 24, 25, 27, 28] and etc.

In this paper, we establish limit theorems containing both the moduli of continuity and the large incremental results for finite dimensional Gaussian processes with N parameters under mild conditions. Throughout the paper, we always assume the following conditions: Let $\{X_j(\mathbf{t}), \mathbf{t} \in [0,\infty)^N\}$, $j = 1, 2, \cdots, d$, be real-valued continuous and centered Gaussian processes with $X_j(\mathbf{0}) = 0$ and $E\{X_j(\mathbf{t}) - X_j(\mathbf{s})\}^2 =$

Received September 18, 2004.
2000 Mathematics Subject Classification: 60F15, 60G15.
Key words and phrases: Gaussian process, quasi-increasing, regularly varying function, large deviation probability.
The second author was supported by NSFC(10071072), NSFZP(199016) and KRF-2002-042-D00008, and the other authors were supported by KOSEF(R05-2003-000-11438-0).

$\sigma_j^2(\|\mathbf{t}-\mathbf{s}\|)$, where $\sigma_j(h)$ are positive and nondecreasing continuous functions of $h > 0$ and $\|\cdot\|$ is the usual Euclidean norm. Put $\sigma(d,h) = \max_{1\le j\le d}\sigma_j(h)$ and we assume that, for some $\alpha > 0$, $\sigma(d,h)/h^\alpha$ is *quasi-increasing*, that is, there is a constant $c > 0$ such that $\sigma(d,s)/s^\alpha \le c\,\sigma(d,t)/t^\alpha$ for $0 < s < t < \infty$.

Let $\{X^d(\mathbf{t}) = (X_1(\mathbf{t}),\cdots,X_d(\mathbf{t})), \mathbf{t} \in [0,\infty)^N\}$ be a d-dimensional Gaussian process with norm $\|\cdot\|$ and N parameters $t_1,\cdots,t_N \in [0,\infty)$, where $\mathbf{t} = (t_1,\cdots,t_N)$. Denote:

$$\mathbf{0} = (0,\cdots,0) \text{ and } \mathbf{1} = (1,\cdots,1) \text{ in } [0,\infty)^N,$$
$$\mathbf{t} \le \mathbf{s} \text{ if } t_i \le s_i \text{ for all integers } 1 \le i \le N,$$
$$\mathbf{t} \pm \mathbf{s} = (t_1 \pm s_1,\cdots,t_N \pm s_N), \quad \mathbf{ts} = (t_1s_1,\cdots,t_Ns_N),$$
$$a\mathbf{t} = (at_1,\cdots,at_N) \quad \text{for } a \in (-\infty,\infty),$$
$$\mathbf{a}(T) = (a_1(T),\cdots,a_N(T)), \quad \mathbf{b}(T) = (b_1(T),\cdots,b_N(T)),$$
$$\beta_1(T) = \Big\{2\big(\log(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|)^N + \log|\log\sigma(d,\|\mathbf{a}(T)\|)|\big)\Big\}^{1/2},$$
$$\beta_2(T) = \Big\{2N\log\big(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|\big)\Big\}^{1/2},$$

where $a_i(T)$ and $b_i(T)$, $i = 1,2,\cdots,N$ are positive continuous functions of $T > 0$, and $\log x = \ln(\max\{x,1\})$.

The following results generalize some main theorems for one dimensional Gaussian processes with one parameter in [1, 5, 8, 9, 21, 27, 28].

The main results are as follows:

THEOREM 1.1. *Assume that*

(i) $$\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} + \|\mathbf{a}(T)\| \to \infty \quad as \quad T \to \infty.$$

Then we have

$$\limsup_{T\to\infty}\ \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\,\beta_1(T)} \le 1 \quad \text{a.s.} \tag{1.1}$$

The condition (i) implies that $\mathbf{a}(T)$ and $\mathbf{b}(T)$ may be many diverse functions. However, in order to obtain the opposite inequality of (1.1), the conditions on $\mathbf{a}(T)$, $\mathbf{b}(T)$ and $\sigma(d,\cdot)$ are a little bit restricted as in the following Theorem 1.2.

A positive function $\sigma(h)$, $h > 0$, is said to be *regularly varying* with exponent $\alpha > 0$ at $b \ge 0$ if $\lim_{h\to b}\{\sigma(xh)/\sigma(h)\} = x^\alpha$, $x > 0$.

THEOREM 1.2. *Assume that $\sigma(d,h)$ is a regularly varying function with exponent $\alpha\,(0<\alpha<1)$ at 0 or ∞ and that there exist positive constants $c_1$and c_2 such that*

$$\text{(ii)}\qquad \left|\frac{d\sigma^2(d,h)}{dh}\right| \le c_1\,\frac{\sigma^2(d,h)}{h} \quad \text{and} \quad \left|\frac{d^2\sigma^2(d,h)}{dh^2}\right| \le c_2\,\frac{\sigma^2(d,h)}{h^2}.$$

Suppose that

$$\text{(iii)}\qquad \lim_{T\to\infty} \frac{\log\left(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|\right)}{\log|\log\|\mathbf{b}(T)\||} = \infty.$$

Then we have

$$\text{(1.2)}\qquad \liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\beta_2(T)} \ge 1 \quad \text{a.s.}$$

The class of variance functions σ^2 satisfying the condition (ii) contains all concave functions with $0<\alpha\le 1/2$ (e.g. $\sigma^2(d,h)=\sqrt{h}$) and convex functions with $1/2<\alpha<1$. We recall that the correlation function on increments of a stochastic process with stationary increments is nonpositive if and only if its variance function is nearly concave (cf. see (2.8) of this paper, (3.10) and (4.2) in Csáki et al.[5] and (2.7) in Lin and Qin[20]), and vice versa.

The condition (iii) guarantees that the class of vector functions $\mathbf{a}(T)$ and $\mathbf{b}(T)$ satisfying (iii) contains many various functions such that they can go to zero, constants or infinity as T tends to infinity.

By combining Theorems 1.1 and 1.2, we obtain the following limit theorem containing both the modulus of continuity and the large incremental result:

COROLLARY 1.1. *Under the assumptions of Theorem 1.2, we have*

$$\begin{aligned} &\lim_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\,\beta_2(T)} \\ &= \lim_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\,\beta_2(T)} \\ &= 1 \quad \text{a.s.} \end{aligned}$$

The structures of main theorems above and the techniques for their proofs which have been studied from one-dimensional Gaussian processes with one-parameter can be applied to develop the limit theories on increments of finite dimensional multi-parameter random fields with respect to the following stochastic processes: Ornstein-Uhlenbeck process (e.g. Csáki et al.[5]), renewal process (Steinebach[25]), lag sum process (Choi and Hwang[3]), local-time process (Csörgő et al.[6]), partial sum process (Szyszkowicz[26], Steinebach[24], Deheuvels and Steinebach[13], Csörgő et al.[7]), self-normalized partial sum process (Csörgő et al.[12], Shao[22]) and etc.

EXAMPLE 1.1. (large incremental result) Let $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$, $j = 1, 2, \cdots, d$, be N-parameter fractional Brownian motions of orders α_j with $0 < \alpha_j < 1$, that is, let $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$, $j = 1, 2, \cdots, d$, be Gaussian random fields with $X_j(\mathbf{0}) = 0$ and $\sigma_j(h) = h^{\alpha_j}$, $h > 0$. When $\alpha_j = 1/2$, then $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$ are standard Wiener random fields. For convenience, put

$$\mathbf{b}(T) = (e^T, \sqrt{2}e^T, \cdots, \sqrt{N}e^T) \quad \text{and} \quad \mathbf{a}(T) = Te^{-T}\mathbf{b}(T).$$

Then $\sigma_j(h)$, $\mathbf{a}(T)$ and $\mathbf{b}(T)$ satisfy all conditions of Corollary 1.1 with

$$\begin{aligned} &\|\mathbf{b}(T)\| = \sqrt{N(N+1)/2}\, e^T =: b_N\, e^T, \\ &\beta_2(T) \approx \sqrt{2NT} \quad \text{and} \quad \sigma(d, \|\mathbf{a}(T)\|) = (b_N T)^\alpha \end{aligned}$$

for sufficiently large T, where $\alpha = \max_{1 \le j \le d} \alpha_j$. Thus we have, by Corollary 1.1,

$$\begin{aligned} &\lim_{T\to\infty} \sup_{\|\mathbf{t}\| \le b_N e^T} \sup_{\|\mathbf{s}\| \le b_N T} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{T^{(2\alpha+1)/2}} \\ = &\lim_{T\to\infty} \sup_{\|\mathbf{t}\| \le b_N e^T} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{T^{(2\alpha+1)/2}} \\ = &\sqrt{2N}(b_N)^\alpha \qquad \text{a.s.} \end{aligned}$$

EXAMPLE 1.2. (modulus of continuity) Let $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$, $j = 1, 2, \cdots, d$, be as in Example 1.1. Put

$$\mathbf{b}(T) = (e^{-T}, T^{-5}\log T, T^{-1}) \quad \text{and} \quad \mathbf{a}(T) = T^{-1}\,\mathbf{b}(T).$$

Then we get

$$\|\mathbf{b}(T)\| < 3T^{-1}(\log T) \quad \text{and} \quad \beta_2(T) = \sqrt{2N\log T}$$

for sufficiently large T. Thus, by Corollary 1.1, we have

$$\begin{aligned}
&\lim_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\sqrt{\log T}} \\
&= \lim_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T))-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\sqrt{\log T}} \\
&= \sqrt{2N} \quad \text{a.s.}
\end{aligned}$$

2. Proofs

Theorem 1.1 is proved by using the following lemma (cf. Lin and Choi[18]):

LEMMA 2.1. *For any $\varepsilon > 0$, there exists a positive constant C_ε depending only on ε such that*

$$\begin{aligned}
&P\left\{ \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)} \ge x \right\} \\
&\le C_\varepsilon \left(\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} \right)^N \Phi_d\Big(\frac{2}{2+\varepsilon}x\Big), \quad x > 1,
\end{aligned}$$

where $\Phi_d(x) = P\{\|N^d(0,1)\| \ge x\}$ and $N^d(0,1)$ is a d-dimensional standardized normal random vector.

It is well-known that

$$\Phi_d(x) \le c\,x^{d-2}e^{-x^2/2}, \quad x > 1$$

for some $c > 0$ (cf. Lemma 1 in Kôno[14]).

Proof of Theorem 1.1. Let $\theta = 1+\varepsilon$ for any given $\varepsilon > 0$. Define

$$\begin{aligned}
A_k &= \{T : \theta^k \le \sigma(d,\|\mathbf{a}(T)\|) \le \theta^{k+1}\}, \quad -\infty < k < \infty, \\
A_{k,j} &= \left\{ T : \theta^j \le \frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} \le \theta^{j+1},\ T \in A_k \right\}, \quad 0 \le j < \infty, \\
\|\mathbf{a}_{T_{k,j}}\| &= \sup\{\|\mathbf{a}(T)\| : T \in A_{k,j}\}, \\
\|\mathbf{b}_{T_{k,j}}\| &= \sup\{\|\mathbf{b}(T)\| : T \in A_{k,j}\}.
\end{aligned}$$

By the condition (i), we have

(2.1)

$$\begin{aligned}
&\limsup_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\,\beta_1(T)}\\
&\le \limsup_{|k|+l\to\infty} \sup_{j\ge l\ge 0} \sup_{T\in A_{k,j}} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\,\beta_1(T)}\\
&\le \limsup_{|k|+l\to\infty} \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\theta^k\,G(k,j)},
\end{aligned}$$

where $G(k,j)=\big\{2\big(\log\theta^{Nj}+\log\log\theta^{|k|}\big)\big\}^{1/2}$.
We will show that

$$\begin{aligned}
&\limsup_{|k|+l\to\infty} \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\theta^k\,G(k,j)}\\
&\le \theta \limsup_{|k|+l\to\infty} \sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}_{T_{k,j}}\|)\,G(k,j)} \qquad (2.2)\\
&\le \theta^2 \quad \text{a.s.}
\end{aligned}$$

By Lemma 2.1, there exists $C_\varepsilon>0$ such that

$$\begin{aligned}
&P\Big\{\sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}_{T_{k,j}}\|)G(k,j)}\ge\theta\Big\}\\
&\le C_\varepsilon \sum_{j\ge l}\Big(\frac{\|\mathbf{b}_{T_{k,j}}\|}{\|\mathbf{a}_{T_{k,j}}\|}\Big)^N \exp\Big(-\frac{2(1+\varepsilon)}{2+\varepsilon}\big(\log\theta^{Nj}+\log\log\theta^{|k|}\big)\Big) \qquad (2.3)\\
&\le C_\varepsilon \sum_{j\ge l} \theta^{-\varepsilon' Nj}|k\vee 1|^{-1-\varepsilon'}\\
&\le C_\varepsilon\,|k\vee 1|^{-1-\varepsilon'}\,\theta^{-\varepsilon' Nl}
\end{aligned}$$

for sufficiently large $|k|+l$, where $\varepsilon'=\varepsilon/(2+\varepsilon)$ and $k\vee 1=\max\{k,\,1\}$.
Hence we have

$$\sum_{l=0}^{\infty}\sum_{|k|=1}^{\infty} P\Big\{\sup_{j\ge l} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}_{T_{k,j}}\|)G(k,j)}\ge\theta\Big\}<\infty,$$

and the inequality (2.2) follows from the Borel-Cantelli lemma. Combining (2.2) with (2.1) yields (1.1) by the arbitrariness of θ. This completes the proof. □

The following Lemmas 2.2–2.5 are needed to prove Theorem 1.2 :

LEMMA 2.2. *Assume that the condition* (ii) *of Theorem 1.2 is satisfied. Let* $\mathbf{a} > \mathbf{0}$ *and* $\mathbf{b} > \mathbf{1}$ *be N-dimensional vectors. Then there exists a positive constant c such that*

$$\left| \int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|} d\sigma^2(d,x) - \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}\|} d\sigma^2(d,x) \right| \le c\,\frac{\sigma^2(d, \|\mathbf{a}\|\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2}.$$

Proof. We have

$$\begin{aligned}
&\left| \int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|} d\sigma^2(d,x) - \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}\|} d\sigma^2(d,x) \right| \\
&= \left| \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \left(\frac{d\sigma^2(d, x+\|\mathbf{a}\|\,\|\mathbf{b}\|-\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|)}{dx} \right.\right. \\
&\qquad \left.\left. - \frac{d\sigma^2(d,x)}{dx} \right) dx + \int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \left(\frac{d\sigma^2(d,x)}{dx} \right) dx \right| \\
&\le \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \int_x^{x+\|\mathbf{a}\|\,\|\mathbf{b}\|-\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|} \left| \frac{d^2\sigma^2(d,y)}{dy^2} \right| dy dx \\
&\quad + \int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \left| \frac{d\sigma^2(d,x)}{dx} \right| dx \\
&=: I_1 + I_2, \quad \text{say.}
\end{aligned}$$

Thus,

$$\begin{aligned}
&I_1 \\
&\le \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \int_x^{x+\|\mathbf{a}\|\,\|\mathbf{b}\|-\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|} \left(c_2 \frac{\sigma^2(d,y)}{y^2} \right) dy dx \\
&\le c_2 \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \left(\frac{\sigma^2(d, x+\|\mathbf{a}\|\,\|\mathbf{b}\|-\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|)}{x^2} \right) \\
&\qquad\qquad \times \big(\|\mathbf{a}\|\,\|\mathbf{b}\| - \|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|\big) dx \\
&\le c_2 \frac{\sigma^2(d, \|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{a}\|^2\,\|\mathbf{b}-\mathbf{1}\|^2} \big(\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\| - \|\mathbf{a}\|\,\|\mathbf{b}\|\big)\big(\|\mathbf{a}\|\,\|\mathbf{b}\| - \|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|\big) \\
&\le c\,\frac{\sigma^2(d, \|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2},
\end{aligned}$$

where $c > 0$ is a constant, and

$$
\begin{aligned}
I_2 &\leq \int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|+\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|-\|\mathbf{a}\|\,\|\mathbf{b}\|} \Big(c_1 \frac{\sigma^2(d,x)}{x}\Big) dx \\
&\leq c_1 \frac{\sigma^2(d, \|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \\
&\qquad \times \big(\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\| - \|\mathbf{a}\|\,\|\mathbf{b}\| - (\|\mathbf{a}\|\,\|\mathbf{b}\| - \|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|)\big) \\
&\leq c_1 \frac{\sigma^2(d, \|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{a}\|\,\|\mathbf{b}\|} \\
&\qquad \times \Big(\frac{\|\mathbf{a}\|^2\,\|\mathbf{b}+\mathbf{1}\|^2 - \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2\,\|\mathbf{b}-\mathbf{1}\|^2)}{2\|\mathbf{a}\|\,\|\mathbf{b}\|}\Big) \\
&\leq c\, \frac{\sigma^2(d, \|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2}. \qquad \square
\end{aligned}
$$

LEMMA 2.3. (Slepian[23]) *Suppose that* $\{V_i,\ i = 1, \cdots, n\}$ *and* $\{W_i,\ i = 1, \cdots, n\}$ *are jointly standardized normal random variables with* $Cov(V_i, V_j) \leq Cov(W_i, W_j)$, $i \neq j$. *Then, for any real* u_i $(i = 1, \cdots, n)$, *we have*

$$P\{V_i \leq u_i,\ i = 1, \cdots, n\} \leq P\{W_i \leq u_i,\ i = 1, \cdots, n\}.$$

LEMMA 2.4. (Leadbetter et al.[15], Li and Shao[16]) *Let* $\mathbb{N} = (n_1, \cdots, n_N)$ *be a* N*-dimensional vector, where* $n_1, \cdots, n_N = 1, 2, \cdots, L$. *Suppose that* $\{Y(\mathbb{N})\}$ *is a sequence of* N*-parameter standard normal random variables with* $\Lambda(\mathbb{N}, \mathbb{N}') := \mathrm{Cov}\,(Y(\mathbb{N}), Y(\mathbb{N}'))$ *such that*

$$\delta := \max_{\mathbb{N} \neq \mathbb{N}'} |\Lambda(\mathbb{N}, \mathbb{N}')| < 1.$$

Let $\{l_{\mathbb{N}} = (l_{n_1}, \cdots, l_{n_N})\}$ *be a subsequence of* $\{\mathbb{N}\}$. *Denote* $\mathbf{m} = (m_1, \cdots, m_N)$ *with* $m_i \leq L$, $1 \leq i \leq N$. *Then, for any real number* u, *we have*

$$
\begin{aligned}
& P\Big\{\max_{\mathbf{1} \leq \mathbb{N} \leq \mathbf{m}} Y(l_{\mathbb{N}}) \leq u\Big\} \\
& \leq \big\{\Phi(u)\big\}^{(\prod_{i=1}^{N} m_i)} \\
& \quad + \sum_{\substack{\mathbb{N} \neq \mathbb{N}' \\ \mathbf{1} \leq \mathbb{N}, \mathbb{N}' \leq \mathbf{m}}} c\,|\lambda(\mathbb{N}, \mathbb{N}')| \exp\Big(-\frac{u^2}{1 + |\lambda(\mathbb{N}, \mathbb{N}')|}\Big)
\end{aligned} \tag{2.4}
$$

for some $c > 0$, *where* $\lambda(\mathbb{N}, \mathbb{N}') = \Lambda(l_{\mathbb{N}}, l_{\mathbb{N}'})$ *and* $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$

Estimating an upper bound for the second term of the right hand side of (2.4), we obtain the following lemma, whose proof is similar to that of Lemma 7 in Choi and Kôno[4].

LEMMA 2.5. *Let* $Y(\mathbb{N})$, δ *and* $\lambda(\mathbb{N}, \mathbb{N}')$ *be as in* Lemma 2.4. *Further, assume that the inequality*

$$|\lambda(\mathbb{N}, \mathbb{N}')| < \|\mathbb{N} - \mathbb{N}'\|^{-\nu}$$

holds for some $\nu > 0$. *Set* $u = \left\{(2-\eta)\log\left(\prod_{i=1}^{N} m_i\right)\right\}^{1/2}$, *where* $0 < \eta < (1-\delta)\nu/(1+\nu+\delta)$. *Then we have*

$$\sum := \sum_{\substack{\mathbb{N}\neq\mathbb{N}' \\ \mathbf{1}\leq\mathbb{N},\mathbb{N}'\leq\mathbf{m}}} |\lambda(\mathbb{N}, \mathbb{N}')| \exp\left(-\frac{u^2}{1+|\lambda(\mathbb{N}, \mathbb{N}')|}\right) \leq c\left(\prod_{i=1}^{N} m_i\right)^{-\delta_0},$$

where $\delta_0 = \{\nu(1-\delta) - \eta(1+\delta+\nu)\}/\{(1+\nu)(1+\delta)\} > 0$ *and* c *is a positive constant independent of* $\mathbb{N}$ *and* u.

Proof of Theorem 1.2. Let $1 < \theta < e$. For integers $j_1, \cdots, j_N$ and k, denote $\mathbf{j} = (j_1, \cdots, j_N)$, $j = \frac{1}{N}\sum_{i=1}^{N} j_i$, $\Theta^{a\mathbf{j}} = (\theta^{aj_1}, \cdots, \theta^{aj_N})$ for $-\infty < a < \infty$ and

$$B_{k,\mathbf{j}} = \{T : \theta^{k-1} \leq \|\mathbf{b}(T)\| \leq \theta^{k},\ \ \theta^{j_i-1} \leq a_i(T) \leq \theta^{j_i},\ 1 \leq i \leq N\}.$$

Note that $\|\mathbf{a}(T)\| \geq \theta^{j-1}$ for $T \in B_{k,\mathbf{j}}$. First, assume that $\|\mathbf{b}(T)\| \to 0$ (or ∞) as $T \to \infty$. By the condition (iii), there exists $\gamma > 0$ such that

$$p := k - j > \gamma\,(\log\log\theta^{|k|})/(\log\theta)^2 =: K$$

for sufficiently large $|k|$. Put $m_i = [\theta^{k-j_i-1}/(\sqrt{N}M)]$, $1 \leq i \leq N$, where $M > 0$ is large enough and $[\cdot]$ denotes the integer part. We can write

(2.5)

$$\begin{aligned}
&\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\beta_2(T)} \\
&\geq \liminf_{|k|\to\infty} \inf_{p>K} \sup_{\|\mathbf{t}\|\leq\theta^{k-1}} \frac{\|X^d(\mathbf{t}+\Theta^{\mathbf{j}}) - X^d(\mathbf{t})\|}{\sigma(d, \|\Theta^{\mathbf{j}}\|)\{2\log(\prod_{i=1}^{N} m_i)\}^{1/2}} \\
&\quad - \limsup_{|k|\to\infty} \sup_{p>K} \sup_{\|\mathbf{t}\|\leq\theta^{k}} \sup_{\Theta^{\mathbf{j}-1}\leq\mathbf{s}\leq\Theta^{\mathbf{j}}} \frac{\|X^d(\mathbf{t}+\Theta^{\mathbf{j}}) - X^d(\mathbf{t}+\mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j}-\mathbf{1}}\|)\{2\log(\prod_{i=1}^{N} m_i)\}^{1/2}} \\
&=: Q_1 - Q_2.
\end{aligned}$$

We claim that

$$Q_1 \geq 1 \quad \text{a.s.} \tag{2.6}$$

By the definition of $\sigma(d,h)$, there exists an integer i_0 $(1 \leq i_0 \leq d)$ such that $\sigma_{i_0}(\|\Theta^{\mathbf{j}}\|) = \sigma(d, \|\Theta^{\mathbf{j}}\|)$, where $i_0 = i_0(\mathbf{j})$ is a function of $\mathbf{j}$. Put $\mathbf{m} = (m_1, \cdots, m_N)$. Then

$$\begin{aligned} Q_1 &\geq \liminf_{|k|\to\infty} \inf_{p>K} \sup_{\|\mathbf{t}\|\leq \theta^{k-1}} \frac{X_{i_0}(\mathbf{t}+\Theta^{\mathbf{j}}) - X_{i_0}(\mathbf{t})}{\sigma_{i_0}(\|\Theta^{\mathbf{j}}\|)\{2\log(\prod_{i=1}^N m_i)\}^{1/2}} \\ &\geq \liminf_{|k|\to\infty} \inf_{p>K} \max_{\mathbf{1}\leq \boldsymbol{l}\leq \mathbf{m}} \frac{X_{i_0}((M\boldsymbol{l}+\mathbf{1})\Theta^{\mathbf{j}}) - X_{i_0}(M\boldsymbol{l}\Theta^{\mathbf{j}})}{\sigma_{i_0}(\|\Theta^{\mathbf{j}}\|)\{2\log(\prod_{i=1}^N m_i)\}^{1/2}}. \end{aligned} \tag{2.7}$$

Let

$$Z_{\mathbf{j}}(\boldsymbol{l}) = \frac{X_{i_0}((M\boldsymbol{l}+\mathbf{1})\Theta^{\mathbf{j}}) - X_{i_0}(M\boldsymbol{l}\Theta^{\mathbf{j}})}{\sigma_{i_0}(\|\Theta^{\mathbf{j}}\|)}, \quad \mathbf{1} \leq \boldsymbol{l} \leq \mathbf{m}.$$

Using the elementary relation $ab = (a^2+b^2-(a-b)^2)/2$, then it follows that, for all $\boldsymbol{l}$ and $\boldsymbol{l}'$ with $\boldsymbol{l} > \boldsymbol{l}'$,

$$\begin{aligned} &\lambda_{\mathbf{j}}(\boldsymbol{l},\boldsymbol{l}') := \operatorname{Cov}\big(Z_{\mathbf{j}}(\boldsymbol{l}), Z_{\mathbf{j}}(\boldsymbol{l}')\big) \\ &\quad = \frac{1}{2\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\|)}\Big\{\sigma_{i_0}^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\Theta^{\mathbf{j}} + \Theta^{\mathbf{j}}\|) - \sigma_{i_0}^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\Theta^{\mathbf{j}}\|) \\ &\qquad - \big(\sigma_{i_0}^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\Theta^{\mathbf{j}}\|) - \sigma_{i_0}^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\Theta^{\mathbf{j}} - \Theta^{\mathbf{j}}\|)\big)\Big\}. \end{aligned} \tag{2.8}$$

If the right hand side of (2.8) is less than or equal to zero, then it follows from Lemma 2.3 that, for any $0 < \varepsilon < 1$,

$$\begin{aligned} &P\Bigg\{\inf_{p>K}\max_{\mathbf{1}\leq\boldsymbol{l}\leq\mathbf{m}} \frac{Z_{\mathbf{j}}(\boldsymbol{l})}{\sqrt{2\log(\Pi_{i=1}^N m_i)}} \leq \sqrt{1-\varepsilon}\Bigg\} \\ &\quad \leq \sum_{p>K}\Big\{\Phi\Big(\sqrt{(2-2\varepsilon)\log(\Pi_{i=1}^N m_i)}\Big)\Big\}^{\Pi_{i=1}^N m_i} \end{aligned} \tag{2.9}$$

On the other hand, if the right hand side of (2.8) is positive, that is, $\sigma_{i_0}^2$ is a nearly convex function, then it follows from the regular variation of $\sigma_{i_0}^2$ and Lemma 2.2 with $\mathbf{a} = \Theta^{\mathbf{j}}$ and $\mathbf{b} = M(\boldsymbol{l}-\boldsymbol{l}')$ that

$$
\begin{aligned}
&|\lambda_{\mathbf{j}}(\boldsymbol{l},\boldsymbol{l}')| \\
&\le \frac{1}{\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\|)}\Big|\int_{\|\Theta^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')\|}^{\|\Theta^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|} d\sigma_{i_0}^2(x) - \int_{\|\Theta^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')-\mathbf{1}\|}^{\|\Theta^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')\|} d\sigma_{i_0}^2(x)\Big| \\
&\le c\,\frac{\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|)}{\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\|)\,\|M(\boldsymbol{l}-\boldsymbol{l}')-\mathbf{1}\|^2} \\
&\le c\,\frac{\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|^2}{\|M(\boldsymbol{l}-\boldsymbol{l}')-\mathbf{1}\|^2}\,\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|^{2\alpha-2} \\
&< \xi\,\|\boldsymbol{l}-\boldsymbol{l}'\|^{-\nu}
\end{aligned}
$$

for sufficiently small $\xi > 0$, where $\nu = 1-\alpha > 0$. Let us apply Lemmas 2.4 and 2.5 for

$$
\begin{aligned}
Y(l_l) &= Z_{\mathbf{j}}(\boldsymbol{l}), \quad \mathbf{1} \le \boldsymbol{l} \le \mathbf{m}, \\
|\lambda(\boldsymbol{l},\boldsymbol{l}')| &= |\lambda_{\mathbf{j}}(\boldsymbol{l},\boldsymbol{l}')| < \xi\,\|\boldsymbol{l}-\boldsymbol{l}'\|^{-\nu}, \\
u &= \{(2-\eta)\log(\Pi_{i=1}^N m_i)\}^{1/2}, \quad \eta = 2\varepsilon.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&P\Big\{\inf_{p>K}\max_{\mathbf{1}\le\boldsymbol{l}\le\mathbf{m}} \frac{Z_{\mathbf{j}}(\boldsymbol{l})}{\sqrt{2\log(\Pi_{i=1}^N m_i)}} \le \sqrt{1-\varepsilon}\Big\} \\
&\le \sum_{p>K}\Big\{\big(\Phi(u)\big)^{(\Pi_{i=1}^N m_i)} + c\Big(\prod_{i=1}^N m_i\Big)^{-\delta_0}\Big\} \\
&\le \sum_{p>K}\Big\{\exp\big(-c\,\theta^{\varepsilon Np}\big) + c\,\big(\theta^{Np}\big)^{-\delta_0}\Big\} \\
&\le c\sum_{p>K}\theta^{-N\delta_0 p} \le c\,\theta^{-N\delta_0\gamma(\log_\theta \log\theta^{|k|})/\log\theta} \\
&\le c\,|k|^{-N\delta_0\gamma/\log\theta}
\end{aligned}
\tag{2.10}
$$

for sufficiently large $|k|$. Note that the right hand side of (2.9) is less than or equal to that of (2.10). Taking $\theta > 1$ such that $\log\theta < N\delta_0\gamma$, then the Borel-Cantelli lemma implies (2.6) via (2.7).

Now we turn to show that

$$
Q_2 \le 2c\,\varepsilon^{\alpha/2} \quad \text{a.s.} \tag{2.11}
$$

for any small $\varepsilon > 0$, where $c > 0$ is a constant. Since $\sigma(d,h)$ is regularly varying, we have

$$\frac{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j-1}}\|)}{\sigma(d, \|\Theta^{\mathbf{j-1}}\|)} \le c\varepsilon^{\alpha/2}.$$

Therefore, (2.11) is proved if we show that

(2.12)

$$\limsup_{|k|\to\infty} \sup_{p>K} \sup_{\|\mathbf{t}\|\le\theta^k} \sup_{\Theta^{\mathbf{j-1}}\le\mathbf{s}\le\Theta^{\mathbf{j}}} \frac{\|X^d(\mathbf{t}+\Theta^{\mathbf{j}}) - X^d(\mathbf{t}+\mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j-1}}\|)\sqrt{2\log(\Pi_{i=1}^N m_i)}} \le 2 \quad \text{a.s.}$$

Applying the same way as the proof of Lemma 2.1, then it follows that, for sufficiently large $|k|$,

$$\begin{aligned} &P\left\{ \sup_{\|\mathbf{t}\|\le\theta^k} \sup_{\Theta^{\mathbf{j-1}}\le\mathbf{s}\le\Theta^{\mathbf{j}}} \frac{\|X^d(\mathbf{t}+\Theta^{\mathbf{j}}) - X^d(\mathbf{t}+\mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j-1}}\|)\sqrt{2\log(\Pi_{i=1}^N m_i)}} \ge 2+\varepsilon \right\} \\ &\le c\,\frac{\theta^{Nk}}{\|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j-1}}\|^N} \exp\left(-\frac{4(2+\varepsilon)^2}{(2+\varepsilon)^2} \log\theta^{Np}\right) \\ &\le c\,\theta^{-3Np}. \end{aligned}$$

Since

$$\sum_{|k|=1}^{\infty} \sum_{p>K} \theta^{-3Np} \le c \sum_{|k|=1}^{\infty} |k|^{-\gamma/\log\theta} < \infty,$$

we obtain (2.12) and hence (1.2) holds true by (2.11), (2.6), and (2.5).

Next, assume that the vector function $\mathbf{b}(T)$ is constant such that $b_i(T) = \theta^c\,(i = 1, \cdots, N)$ for a constant c $(-\infty < c < \infty)$. Put $m_i = [\theta^{c-j_i}/M]$, $1 \le i \le N$, in $B_{c,\mathbf{j}}$. By (iii), we can write

$$\begin{aligned} &\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\beta_2(T)} \\ &\ge \liminf_{j\to-\infty} \sup_{\|\mathbf{t}\|\le\sqrt{N}\theta^c} \frac{\|X^d(\mathbf{t}+\Theta^{\mathbf{j}}) - X^d(\mathbf{t})\|}{\sigma(d, \|\Theta^{\mathbf{j}}\|)\{2\log(\prod_{i=1}^N m_i)\}^{1/2}} \\ &\quad - \limsup_{j\to-\infty} \sup_{\|\mathbf{t}\|\le\sqrt{N}\theta^c} \sup_{\Theta^{\mathbf{j-1}}\le\mathbf{s}\le\Theta^{\mathbf{j}}} \frac{\|X^d(\mathbf{t}+\Theta^{\mathbf{j}}) - X^d(\mathbf{t}+\mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j-1}}\|)\{2\log(\prod_{i=1}^N m_i)\}^{1/2}} \\ &=: Q_1' - Q_2', \quad \text{say.} \end{aligned}$$

According to the same lines as in (2.6)–(2.12), we can easily prove that (1.2) holds true as well. □

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