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# THE PROPERTIES OF THE TRANSVERSAL KILLING SPINOR AND TRANSVERSAL TWISTOR SPINOR FOR RIEMANNIAN FOLIATIONS

Seoung Dal Jung and Yeong Bong Moon

Abstract. We study the properties of the transversal Killing and twistor spinors for a Riemannian foliation with a transverse spin structure. And we investigate the relations between them. As an application, we give a new lower bound for the eigenvalues of the basic Dirac operator by using the transversal twistor operator.

## 1. Introduction

Twistor spinors were introduced by R. Penrose in General Relativity ([19]). In [15], A. Lichnerowicz introduced the twistor operator acting on the spinors, which is a conformally invariant, and proved that the twistor spinors are zeroes of the twistor operator. Further, it is remarkable that the twistor spinors corrrespond to parallel sections in a certain bundle (see [2], [5]). It is well known ([5], [15]) that given a twistor spinor $\Psi$ there are two interesting conformal scalar invariants $C(\Psi)$, $Q(\Psi)$ which are constant. Moreover, it was proved that a non-vanishing twistor spinor $\Psi$ is conformally equivalent to a real Killing spinor if and only if $C(\Psi)\neq 0$ and $Q(\Psi)=0$. Similarly, we define two transversally conformal invariants $C'(\Psi), Q'(\Psi)$ for a certain Riemannian foliation (see section 5).

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$. In [9], the first author introduced the transversal Killing spinor $\Psi$ given by the equation

$$\nabla_X\Psi + f\pi(X)\cdot\Psi = 0, \quad \forall\, X\in\Gamma TM, \tag{1.1}$$

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where $f$ is a basic function and $\pi : TM \to Q$ is a natural projection(see (2.1)). And any eigenvalue $\lambda$ of the basic Dirac operator $D_b$ satisfies the inequality

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M(\sigma^\nabla + |\kappa|^2), \tag{1.2}$$

where $q = \mathrm{codim}\mathcal{F}$, $\sigma^\nabla$ is the transversal scalar curvature and $\kappa$ is the mean curvature form of $\mathcal{F}$. And in the limiting case, $M$ admits a transversal Killing spinor.

In this paper, we study the properties of transversal Killing spinors and transversal twistor spinors. Moreover, we investigate the relations between them in terms of $C'(\Psi), Q'(\Psi)$.

The paper is organized as follows. In Section 2, we review the known facts on a foliated Riemannian manifold. In Section 3, we introduce the transversal twistor (resp. W-twistor) spinor defined by the transversal twistor(resp. W-twistor) equation

(1.3)

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D_{tr}\Psi = 0(\text{resp. } \nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0),\ \forall X \in \Gamma TM.$$

Moreover, we prove that the transversal W-twistor spinors correspond to parallel basic sections in a certain foliated bundle(cf. [10]). In Section 4, we study the transversal Killing spinor. In Section 5, we define two transversally conformal invariants $C'(\Psi)$ and $Q'(\Psi)$, which are similar to ones on [5]. By these invariants, we investigate the transversally conformal relation between transversal twistor spinors and transversal Killing spinors. In Section 6, we estimate the eigenvalue of the basic Dirac operator, which is sharper than (1.2).

## 2. Preliminaries and known facts

In this section, we review the basic properties of the Riemannian foliation ([13], [20]). Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. We recall the exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 \tag{2.1}$$

determined by the tangent bundle $L$ and the normal bundle $Q = TM/L$ of $\mathcal{F}$. For a distinguished chart $\mathcal{U} \subset M$ the leaves of $\mathcal{F}$ in $\mathcal{U}$ are given as the fibers of a Riemannian submersion $f : \mathcal{U} \to \mathcal{V} \subset N$ onto an open subset $\mathcal{V}$ of a model Riemannian manifold $N$. For overlapping charts

$U_\alpha \cap U_\beta$, the corresponding local transition functions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on $N$ are isometries. Further, we denote by $\nabla$ the transversal Levi-Civita connection of the normal bundle $Q$ of $\mathcal{F}$. It is defined by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L, \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases} \tag{2.2}$$

where $s \in \Gamma Q$, and $Y_s \in \Gamma L^\perp$ corresponding to $s$ under the canonical isomorphism $L^\perp \cong Q$. The connection $\nabla$ is metrical with respect to $g_M$ and torsion free. It corresponds to the Riemannian connection of the model space $N$. The curvature $R^\nabla$ of $\nabla$ is defined by

$$R^\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \quad \forall\ X,\ Y \in \Gamma TM.$$

Since $i(X)R^\nabla = 0$ for any $X \in \Gamma L$ ([13]), we can define the transversal Ricci curvature $\rho^\nabla : \Gamma Q \to \Gamma Q$ and the transversal scalar curvature $\sigma^\nabla$ of $\mathcal{F}$ by

$$\rho^\nabla(X) = \sum_a R^\nabla(X, E_a)E_a, \quad \sigma^\nabla = \sum_a g_Q(\rho^\nabla(E_a), E_a),$$

for $X \in \Gamma Q$, where $\{E_a\}_{a=1,\cdots,q}$ is a local orthonormal basic frame for $Q$. $\mathcal{F}$ is said to be transversally *Einsteinian* if the model space $N$ is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot id \tag{2.3}$$

with constant transversal scalar curvature $\sigma^\nabla$. The *mean curvature vector field* of $\mathcal{F}$ is defined by

$$\kappa^\sharp = \sum_i \pi(\nabla^M_{E_i} E_i), \tag{2.4}$$

where $\{E_i\}_{i=1,\cdots,p}$ is a local orthonormal frame for $L$. Its dual form $\kappa$, the *mean curvature form* for $L$, is then given by

$$\kappa(X) = g_Q(\kappa^\sharp, X), \quad \forall\ X \in \Gamma Q. \tag{2.5}$$

Let $\Omega^r_B(\mathcal{F})$ be the space of all *basic r-forms*, i.e.,

$$\Omega^r_B(\mathcal{F}) = \{\psi \in \Omega^r(M) |\ i(X)\phi = 0,\ \theta(X)\phi = 0,\ \forall\ X \in \Gamma L\},$$

where $\theta$ is the transverse Lie derivative operator. $\mathcal{F}$ is said to be *minimal* (resp. *isoparametric*) if $\kappa = 0$(resp. $\kappa \in \Omega^1_B(\mathcal{F})$). It is well known([20]) that on a compact manifold $\kappa$ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric. The cohomology

$$H_B(M/\mathcal{F}) = H(\Omega^*_B(\mathcal{F}), d_B) \tag{2.6}$$

is called the *basic cohomology* of $\mathcal{F}$. Note that $\Omega_B^*(\mathcal{F})$ is a transversal Clifford algebra with the Clifford multiplication defined as follows: if $\phi \in \Omega_B^1(\mathcal{F})$ and $\psi \in \Omega_B^r(\mathcal{F})$, then

$$\phi \cdot \psi = \phi \wedge \psi - i(v)\psi, \tag{2.7}$$

where $v$ is the $g_Q$-dual vector to $\phi$. Let $\delta_B$ be the formal adjoint operator on $\Omega_B^*(\mathcal{F})$ of $d_B$. Then it is written as ([1], [9])

$$d_B = \sum_a \theta_a \wedge \nabla_{E_a}, \quad \delta_B = -\sum_a i(E_a)\nabla_{E_a} + i(\kappa_B^\sharp), \tag{2.8}$$

where $\kappa_B^\sharp$ is the $g_Q$-dual to the basic component $\kappa_B$ of $\kappa$ ([1]) and $\theta_a$ is the $g_Q$-dual 1-form to $E_a$. The *basic Laplacian* acting on $\Omega_B^*(\mathcal{F})$ is defined by ([18])

$$\Delta_B = d_B\delta_B + \delta_B d_B. \tag{2.9}$$

Note that $\Delta_B$ corresponds to the ordinary Laplacian of $N$.

## 3. Transversal twistor spinors

In this section, we improve some facts in [10] for the transversal twistor spinor. Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$(see [9] for definition). Let $S(\mathcal{F})$ be a foliated spinor bundle of $\mathcal{F}$ and $< \cdot, \cdot >$ a hermitian scalar product on $S(\mathcal{F})$. It was considered ([9], [14]) the curvature transform $R^S$ given by

$$R^S(X,Y)\Psi = \frac{1}{4}\sum_{a,b} g_Q(R^\nabla(X,Y)E_a, E_b)E_a \cdot E_b \cdot \Psi \tag{3.1}$$

for $X, Y \in \Gamma TM$ and $\Psi \in \Gamma S(\mathcal{F})$, where $X \cdot \Psi$ denotes the Clifford multiplication of the vector $X \in Q$ by $\Psi$. Then it holds ([9])

$$\sum_{a<b} E_a \cdot E_b \cdot R^S(E_a, E_b)\Psi = \frac{1}{4}\sigma^\nabla \Psi, \tag{3.2}$$

$$\sum_a E_a \cdot R^S(X, E_a)\Psi = -\frac{1}{2}\rho^\nabla(\pi(X)) \cdot \Psi \tag{3.3}$$

for $X \in \Gamma TM$. Let $m : Q \otimes S(\mathcal{F}) \to S(\mathcal{F})$ be the Clifford multiplication. Then Ker $m$ is a subbundle of $Q \otimes S(\mathcal{F})$ and there exists a projection

$p : Q \otimes S(\mathcal{F}) \to \mathrm{Ker}\ m$ given by the formula

$$p(X \otimes \Psi) = X \otimes \Psi + \frac{1}{q}\sum_{a=1}^{q} E_a \otimes E_a \cdot X \cdot \Psi. \tag{3.4}$$

There are two operators on $\Gamma S(\mathcal{F})$, the *transversal Dirac operator* $D'_{tr}$ and the *transversal twistor operator* $P'_{tr}$ of $\mathcal{F}$, which are defined by

$$D'_{tr} = m \circ \hat{\pi} \circ \nabla^S, \quad P'_{tr} = p \circ \hat{\pi} \circ \nabla^S,$$

respectively, where $\hat{\pi} : \Gamma(T^*M \otimes S(\mathcal{F})) \to \Gamma(Q^* \otimes S(\mathcal{F})) \cong \Gamma(Q \otimes S(\mathcal{F}))$ is the projection and $\nabla^S$ is the spinor derivation on $S(\mathcal{F})$ induced by (2.2). If it does not cause any confusion, we will henceforward use $\nabla = \nabla^S$. They are locally given by

$$D'_{tr}\Psi = \sum_a E_a \cdot \nabla_{E_a}\Psi, \quad P'_{tr}\Psi = \sum_a E_a \otimes P'_{E_a}\Psi, \tag{3.5}$$

respectively, where $P'_X\Psi = \nabla_X\Psi + \frac{1}{q}\pi(X) \cdot D'_{tr}\Psi$ for any $X \in \Gamma TM$. It was shown ([3], [6]) that the formal adjoint $D'^*_{tr}$ is given by $D'^*_{tr} = D'_{tr} - \kappa\cdot$ and so

$$D_{tr} = D'_{tr} - \frac{1}{2}\kappa\cdot \tag{3.6}$$

is a symmetric, transversally elliptic differential operator. It is well-known ([6], [9], [12]) that on a compact Riemannian manifold $(M, g_M, \mathcal{F})$ with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$ such that $\delta\kappa = 0$

$$D^2_{tr}\Psi = \nabla^*_{tr}\nabla_{tr}\Psi + \frac{1}{4}(\sigma^\nabla + |\kappa|^2)\Psi, \tag{3.7}$$

where $\nabla^*_{tr}\nabla_{tr}\Psi = -\sum_a \nabla^2_{E_a,E_a}\Psi + \nabla_{\kappa^\sharp}\Psi$ and $\nabla^2_{V,W} = \nabla_V\nabla_W - \nabla_{\nabla_V W}$ for any $V,\ W \in \Gamma TM$. From (3.6), we have

$$D^2_{tr}\Psi = D'^2_{tr}\Psi - \frac{1}{2}\{\kappa \cdot D'_{tr}\Psi + D'_{tr}(\kappa \cdot \Psi)\} - \frac{1}{4}|\kappa|^2\Psi. \tag{3.8}$$

A direct calculation with (2.7) and (2.8) yields

$$D'_{tr}(\kappa \cdot \Psi) + \kappa \cdot D'_{tr}\Psi = (d_B\kappa + \delta_B\kappa - |\kappa|^2)\Psi - 2\nabla_{\kappa^\sharp}\Psi. \tag{3.9}$$

Hence we have from (3.8) and (3.9) the following proposition.

PROPOSITION 3.1. *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$ such that $\delta\kappa = 0$. Then it holds*

$$D^2_{tr}\Psi = D'^2_{tr}\Psi + \frac{1}{4}|\kappa|^2\Psi + \nabla_{\kappa^\sharp}\Psi. \tag{3.10}$$

Similarly, we put

$$P_{tr}\Psi = \sum_a E_a \otimes P_{E_a}\Psi, \tag{3.11}$$

where $P_X\Psi = \nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D_{tr}\Psi$ for any $X \in \Gamma TM$. It is obvious that for any vector field $X$

$$P_X\Psi = P'_X\Psi - \frac{1}{2q}\pi(X)\cdot\kappa\cdot\Psi. \tag{3.12}$$

We define the subspace $\Gamma_B S(\mathcal{F})$ of all basic sections of $S(\mathcal{F})$ by

$$\Gamma_B S(\mathcal{F}) = \{\Psi \in \Gamma S(\mathcal{F}) |\ \nabla_X\Psi = 0, \quad \forall\, X \in \Gamma L\}. \tag{3.13}$$

Then $D_b = D_{tr}|_{\Gamma_B S(\mathcal{F})}$ preserves the basic sections if the foliation $\mathcal{F}$ is isoparametric, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$. In this case, $D_b$ is called the *basic Dirac operator* for $\mathcal{F}$. A spinor field of the kernel of $P_{tr}$(resp. kernel of $P'_{tr}$) is called a *transversal twistor*(resp. *W-twistor*) *spinor*, if it satisfies the *transversal twistor*(resp. *W-twistor*) *equation*

(3.14)

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D_{tr}\Psi = 0 \text{ (resp. } \nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0), \ X \in \Gamma TM.$$

It is trivial that $\mathrm{Ker}P'_{tr} \subset \Gamma_B S(\mathcal{F})$.

THEOREM 3.2. *If $M$ admits a non-vanishing transversal twistor spinor $\Psi$, then $\mathcal{F}$ is minimal.*

*Proof.* Let $(0 \neq)\Psi \in \mathrm{Ker}P_{tr}$ be a transversal twistor spinor. Then we have

$$\begin{aligned} 0 = \sum_a E_a \cdot P_{E_a}\Psi &= \sum_a E_a\cdot\nabla_{E_a}\Psi + \frac{1}{q}\sum_a E_a\cdot E_a\cdot D_{tr}\Psi \\ &= D_{tr}\Psi + \frac{1}{2}\kappa\cdot\Psi - D_{tr}\Psi \\ &= \frac{1}{2}\kappa\cdot\Psi, \end{aligned}$$

which implies that $\kappa = 0$. Therefore $\mathcal{F}$ is minimal. □

REMARK. Theorem 3.2 says that there does not exist a non-trivial solution of (3.14) if $\mathcal{F}$ is not minimal. So in this case it maybe helpful to consider the operator $P'_{tr}$. From (3.12) and Theorem 3.2, any transversal twistor spinor is a transversal W-twistor spinor. But the converse is not true in general.

PROPOSITION 3.3. *A spinor field $\Psi \in \mathrm{Ker}P'_{tr}$ is a transversal W-twistor spinor if and only if for any $X,\ Y \in \Gamma TM$*

$$\pi(X)\cdot\nabla_Y\Psi + \pi(Y)\cdot\nabla_X\Psi = \frac{2}{q}g_Q(\pi(X),\pi(Y))D'_{tr}\Psi. \tag{3.15}$$

*Proof.* It is easy to verify the sufficiency. Conversely, (3.15) with $Y = E_a$ yields

$$\sum_a E_a\cdot\pi(X)\cdot\nabla_{E_a}\Psi + \sum_a E_a\cdot E_a\cdot\nabla_X\Psi = \frac{2}{q}\sum_a g_Q(\pi(X),E_a)E_a\cdot D'_{tr}\Psi,$$

which follows from (3.5) that

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi = 0.$$

This means that $\Psi$ is a transversal W-twistor spinor. □

PROPOSITION 3.4. *Under the same condition as in Proposition 3.1, every transversal W-twistor spinor $\Psi \in \mathrm{Ker}P'_{tr}$ satisfies*

$$D'^2_{tr}\Psi = \frac{q}{4(q-1)}\sigma^\nabla\Psi, \tag{3.16}$$

(3.17)

$$\nabla_X D'_{tr}\Psi = \frac{q}{2(q-2)}\left\{\frac{\sigma^\nabla}{2(q-1)}\pi(X) - \rho^\nabla(\pi(X))\right\}\Psi,\ \forall X \in \Gamma TM.$$

*Proof.* Let $x \in M$ and choose an orthonormal basic frame $\{E_a\}$ with the property that $(\nabla E_a)_x = 0$ for all $a$. From (3.14), we have at $x$ that for any transversal W-twistor spinor $\Psi$

$$\sum_a \nabla_{E_a}\nabla_{E_a}\Psi + \frac{1}{q}D'^2_{tr}\Psi = 0. \tag{3.18}$$

On the other hand, from (3.7) and (3.10), we have

$$D'^2_{tr}\Psi = -\sum_a \nabla_{E_a}\nabla_{E_a}\Psi + \frac{1}{4}\sigma^\nabla\Psi. \tag{3.19}$$

Hence (3.16) is obtained by (3.18) and (3.19).

Next, let $X \in \Gamma TM$ be a local vector field arising from a vector in $T_xM$ by parallel displacement along transversal geodesics. Then we have from (3.14)

$$R^S(X,E_a)\Psi = \frac{1}{q}\left\{\pi(X)\cdot\nabla_{E_a}D'_{tr}\Psi - E_a\cdot\nabla_X D'_{tr}\Psi\right\}. \tag{3.20}$$

It follows from (3.3) that

$$\begin{aligned}\rho^{\nabla}(\pi(X))\cdot\Psi &= -2\sum_a E_a\cdot R^S(X,E_a)\Psi\\ &= -\frac{2}{q}\sum_a E_a\cdot\{\pi(X)\cdot\nabla_{E_a}D'_{tr}\Psi - E_a\cdot\nabla_X D'_{tr}\Psi\}\\ &= -\frac{2}{q}\{(q-2)\nabla_X D'_{tr}\Psi - \pi(X)\cdot D'^2_{tr}\Psi\},\end{aligned}$$

which, combined with (3.16), follows (3.17). □

Let us define the bundle map $K : TM \to Q$ by

$$K(X) = \frac{1}{q-2}\left\{\frac{\sigma^{\nabla}}{2(q-1)}\pi(X) - \rho^{\nabla}(\pi(X))\right\} \tag{3.21}$$

for $X \in \Gamma TM$. We consider the bundle $E = S(\mathcal{F}) \oplus S(\mathcal{F})$ and the covariant derivative $\nabla^E$ in $E$ defined by

$$\nabla^E_X\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot\Phi\\ \nabla_X\Phi - \frac{q}{2}K(X)\cdot\Psi\end{pmatrix}. \tag{3.22}$$

Then we have the following proposition.

PROPOSITION 3.5. *Under the same condition as in Proposition 3.1, every transversal W-twistor spinor $\Psi$ satisfies*

$$\nabla^E\begin{pmatrix}\Psi\\ D'_{tr}\Psi\end{pmatrix} \equiv 0.$$

*Conversely, if* $\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} \in \Gamma_B E$ *is* $\nabla^E$*-parallel, then* $\Psi$ *is a transversal W-twistor spinor and* $\Phi = D'_{tr}\Psi$.

*Proof.* Let $\Psi \in \mathrm{Ker}P'_{tr}$ be a transversal W-twistor spinor. Then the definition together with (3.17) gives rise to

$$\nabla^E_X\begin{pmatrix}\Psi\\ D'_{tr}\Psi\end{pmatrix} = \begin{pmatrix}\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot D'_{tr}\Psi\\ \nabla_X D'_{tr}\Psi - \frac{q}{2}K(X)\cdot\Psi\end{pmatrix} = 0.$$

Conversely, let $\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} \in \Gamma_B E$ be a $\nabla^E$-parallel section. Then we have

$$\nabla_X\Psi + \frac{1}{q}\pi(X)\cdot\Phi = 0, \quad \text{for } X \in TM$$

and thus

$$\sum_a E_a\cdot\nabla_{E_a}\Psi + \sum_a \frac{1}{q}E_a\cdot E_a\cdot\Phi = 0.$$

Hence $D'_{tr}\Psi = \Phi$. This means that $\Psi$ is a solution of the transversal W-twistor equation. □

On the other hand, we have from (2.9) that for any transversal W-twistor spinor $\Psi$

$$\Delta_B|\Psi|^2 = < D_b^2\Psi, \Psi > + < \Psi, D_b^2\Psi > -\frac{1}{2q}(q\sigma^\nabla + (q+1)|\kappa|^2)|\Psi|^2$$
$$-\frac{2}{q}|D_b\Psi|^2 - \frac{1}{q}\{< \kappa\cdot\Psi, D_b\Psi > + < D_b\Psi, \kappa\cdot\Psi >\}.$$

From (3.10) and (3.16), we have the following proposition.

PROPOSITION 3.6. *Under the same condition as in Proposition 3.1, every transversal W-twistor spinor $\Psi$ satisfies*

$$\frac{q}{2}\Delta_B|\Psi|^2 = \frac{q}{4(q-1)}\sigma^\nabla|\Psi|^2 + \frac{1}{4}|\kappa|^2|\Psi|^2 - |D_b\Psi|^2. \tag{3.23}$$

REMARK. On a complete Riemannian manifold with an isoparametric foliation $\mathcal{F}$, if all leaves are compact, then $\kappa$ is closed([11]). Hence Proposition 3.1, 3.4, 3.5, and 3.6 are true on a complete Riemannian manifold with an isoparametric foliation of $\delta\kappa = 0$.

## 4. Transversal Killing spinor

Throughout this section $(M, g_M, \mathcal{F})$ is considered as a connected Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$. We recall([9]) that $\Psi \in \Gamma_B S(\mathcal{F})$ is a *transversal Killing spinor* if it satisfies

$$\nabla^f_X\Psi \equiv \nabla_X\Psi + f\pi(X)\cdot\Psi = 0 \tag{4.1}$$

for any $X \in \Gamma TM$ and a basic function $f(\neq 0)$.

LEMMA 4.1. *If $\Psi$ is a transversal Killing spinor, then the associated vector field $X_\Psi$ defined by*

$$X_\Psi = i\sum_a < \Psi, E_a\cdot\Psi > E_a$$

*is transversal Killing, i.e., $\theta(X_\Psi)g_Q = 0$.*

*Proof.* The definition together with (3.14) implies for $Y, Z \in \Gamma Q$

$$\begin{aligned}\nabla_Y X_\Psi &= i\sum_a Y < \Psi, E_a \cdot \Psi > E_a \\ &= -if\sum_a \{< Y \cdot \Psi, E_a \cdot \Psi > + < \Psi, E_a \cdot Y \cdot \Psi >\}E_a,\end{aligned}$$

so that

$$g_Q(\nabla_Y X_\Psi, Z) = -if\{< Y \cdot \Psi, Z \cdot \Psi > + < \Psi, Z \cdot Y \cdot \Psi >\}.$$

It follows that

$$(\theta(X_\Psi)g_Q)(Y, Z) = g_Q(\nabla_Y X_\Psi, Z) + g_Q(Y, \nabla_Z X_\Psi) = 0,$$

which means that $X_\Psi$ is transversal Killing. □

LEMMA 4.2. *If $\Psi$ is a transversal Killing spinor, then $|\Psi|^2$ is constant.*

*Proof.* Let $\Psi$ be a transversal Killing spinor such that $\nabla_X \Psi = -f\pi(X) \cdot \Psi$ for any $X \in \Gamma TM$. Then (4.1) implies

$$\begin{aligned}X|\Psi|^2 &= < \nabla_X \Psi, \Psi > + < \Psi, \nabla_X \Psi > \\ &= -f\{< \pi(X) \cdot \Psi, \Psi > + < \Psi, \pi(X) \cdot \Psi >\} \\ &= 0.\end{aligned}$$

Therefore $|\Psi|^2$ is constant. □

THEOREM 4.3. [9] *If $M$ admits a non-vanishing transversal Killing spinor $\Psi$ with $\nabla^f_{tr}\Psi = 0$, then*

*(1) $f$ is constant and $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$.*

*(2) $\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature $\sigma^\nabla > 0$.*

THEOREM 4.4. *If $\Psi$ is a transversal Killing spinor with $\nabla^f_{tr}\Psi = 0$, then*

$$|D_{tr}\Psi|^2 = \frac{1}{4}\left(\frac{q}{q-1}\sigma^\nabla + |\kappa|^2\right)|\Psi|^2 \tag{4.2}$$

$$\mathrm{Re} < D_{tr}\Psi, \kappa \cdot \Psi > = -\frac{1}{2}|\kappa|^2|\Psi|^2. \tag{4.3}$$

*Proof.* Let $\Psi$ be a transversal Killing spinor with $\nabla^f_X \Psi = 0$. From (4.1), we have

$$D_{tr}\Psi = fq\Psi - \frac{1}{2}\kappa \cdot \Psi, \tag{4.4}$$

where $f^2 = \frac{\sigma^\nabla}{4q(q-1)}$ is constant. It follows that

$$< D_{tr}\Psi, D_{tr}\Psi >= < fq\Psi - \frac{1}{2}\kappa \cdot \Psi, fq\Psi - \frac{1}{2}\kappa \cdot \Psi >$$
$$= (f^2q^2 + \frac{1}{4}|\kappa|^2) < \Psi, \Psi >,$$

which prove (4.2). Since $< X \cdot \Psi, \Psi >$ is pure imaginary, the equation (4.3) follows from (4.4). □

From (4.4), we have the following corollary.

COROLLARY 4.5. *On the minimal foliation $\mathcal{F}$, every transversal Killing spinor is an eigenspinor of $D_b$.*

Now we recall the generalized Myers' theorem.

THEOREM 4.6. [8] *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ and complete bundle-like metric $g_M$. If there is a positive lower bound of the transversal Ricci curvature, then the leaf spact $M/\mathcal{F}$ of $\mathcal{F}$ is compact, and the basic cohomology $H^1(M/\mathcal{F}) = 0$.*

Summing up Theorem 4.3 and Theorem 4.6, we have the following theorem.

COROLLARY 4.7. *Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and complete bundle-like metric $g_M$. If $M$ admits a transversal Killing spinor, then the leaf space $M/\mathcal{F}$ of $\mathcal{F}$ is compact and $H^1(M/\mathcal{F}) = 0$.*

## 5. The conformal relation between transversal twistor and transversal Killing spinors

Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$. Now, we consider the transversally conformal change $\bar{g}_Q = e^{2u}g_Q$ of $g_Q$ for any real basic function $u$ on $M$. Let $\bar{S}(\mathcal{F})$ be the foliated spinor bundle associated with $\bar{g}_Q$. If $< , >_{g_Q}$ and $< , >_{\bar{g}_Q}$ denote the natural Hermitian metrics on $S(\mathcal{F})$ and $\bar{S}(\mathcal{F})$ respectively, then for any $\Phi,\ \Psi \in \Gamma S(\mathcal{F})$ we have ([12])

(5.1) $$< \Phi, \Psi >_{g_Q} = < \bar{\Phi}, \bar{\Psi} >_{\bar{g}_Q},$$

and the corresponding Clifford multiplication in $\bar{S}(\mathcal{F})$ is given by

(5.2) $$\bar{X} \mathbin{\bar{\cdot}} \bar{\Psi} = \overline{X \cdot \Psi}, \quad \forall\, X \in \Gamma Q.$$

Let $\bar\nabla$(resp. $\bar D_{tr}$) be the transversal Levi-Civita connection(resp. transver sal Dirac operator) corresponding to $\bar g_Q$. Then we have the following proposition.

PROPOSITION 5.1. [12] *For any $X, Y \in \Gamma TM$ and $\Psi \in \Gamma S(\mathcal{F})$*

(1) $\bar\nabla_X\pi(Y) = \nabla_X\pi(Y) + X(u)\pi(Y) + Y(u)\pi(X) - g_Q(\pi(X), \pi(Y))\,\mathrm{grad}_\nabla(u)$,

(2) $e^{2u}\rho^{\bar\nabla}(X) = \rho^\nabla(X) + (2-q)\nabla_X\mathrm{grad}_\nabla(u) + (2-q)|\mathrm{grad}_\nabla(u)|^2X + (q-2)X(u)\mathrm{grad}_\nabla(u) + \{\Delta_B u - \kappa^\sharp(u)\}X.$

(3) $e^{2u}\sigma^{\bar\nabla} = \sigma^\nabla + (q-1)(2-q)|\mathrm{grad}_\nabla(u)|^2 + 2(q-1)\{\Delta_B u - \kappa^\sharp(u)\}.$

(4) $\bar\nabla_X\bar\Psi = \overline{\nabla_X\Psi} - \frac{1}{2}\overline{\pi(X)\cdot\mathrm{grad}_\nabla(u)\cdot\Psi} - \frac{1}{2}g_Q(\mathrm{grad}_\nabla(u), \pi(X))\bar\Psi.$

(5) $\bar D_{tr}(e^{-\frac{q-1}{2}u}\bar\Psi) = e^{-\frac{q+1}{2}u}\overline{D_{tr}\Psi}.$

Let $\{\bar E_a\}$ be a local orthonormal basic frame associated with $\bar g_Q$. Then $\bar D_{tr}$ is locally expressed by

$$\bar D_{tr}\bar\Psi = \bar D'_{tr}\bar\Psi - \frac{1}{2}\kappa_{\bar g}\,\bar\cdot\,\bar\Psi \tag{5.3}$$

for $\bar\Psi \in \Gamma\bar S(\mathcal{F})$, where $\bar D'_{tr}\bar\Psi = \sum_a \bar E_a\,\bar\cdot\,\bar\nabla_{\bar E_a}\bar\Psi$ and $\kappa_{\bar g}$ is the mean curvature form associated with $\bar g_Q$, which satisfies $\kappa_{\bar g} = e^{-2u}\kappa$. It follows from (5.3) that

$$\bar D_{tr}\bar\Psi = e^{-u}\{\overline{D_{tr}\Psi} + \frac{q-1}{2}\overline{\mathrm{grad}_\nabla(u)\cdot\Psi}\}. \tag{5.4}$$

Since $D_{tr}(f\Psi) = \mathrm{grad}_\nabla(f)\cdot\Psi + fD_{tr}\Psi$ for a function $f$, we have

$$\bar D_{tr}(f\bar\Psi) = e^{-u}\overline{\mathrm{grad}_\nabla(f)\cdot\Psi} + f\bar D_{tr}\bar\Psi, \tag{5.5}$$

so that

$$\bar D'_{tr}(e^{-\frac{q-1}{2}u}\bar\Psi) = e^{-\frac{q+1}{2}u}\overline{D'_{tr}\Psi}. \tag{5.6}$$

Therefore we conclude that the dimensions of the kernel of $D_{tr}$ and $D'_{tr}$ are transversally conformal invariants.

Let $\bar P'_{tr}$ be the transversal W-twistor operator of $\bar g_M = g_L \oplus \bar g_Q$, where $\bar g_Q = e^{2u}g_Q$ for a basic function $u$. A similar way shows the following proposition.

PROPOSITION 5.2. *For any spinor field $\Psi \in \Gamma_B S(\mathcal{F})$, we have*

$$\bar P'_{tr}(e^{\frac{u}{2}}\bar\Psi) = e^{-\frac{u}{2}}\overline{P'_{tr}\Psi}.$$

*In particular, $\Psi \in \Gamma_B S(\mathcal{F})$ is a transversal W-twistor spinor on $(M, g_M)$ if and only if $e^{\frac{u}{2}}\bar\Psi \in \Gamma_B\bar S(\mathcal{F})$ is a transversal W-twistor spinor on $(M, \bar g_M)$.*

On the vector space $\mathrm{Ker}P'_{tr}$, there exist a quadratic form $C'$ and a form $Q'$ defined by

$$C'(\Psi) = Re < D'_{tr}\Psi, \Psi >:= (D'_{tr}\Psi, \Psi) \tag{5.7}$$

$$Q'(\Psi) = |\Psi|^2 |D'_{tr}\Psi|^2 - C'(\Psi)^2 - \sum_a (D'_{tr}\Psi, E_a \cdot \Psi)^2 \tag{5.8}$$

for $\Psi \in KerP'_{tr}$ (see [5] for the point foliation). It is obvious that $C'(\Psi) = (D_{tr}\Psi, \Psi)$. By using (5.3), (5.4), and (5.5), a direct calculation implies

$$\bar{C}'(e^{u/2}\bar{\Psi}) = C'(\Psi), \quad \bar{Q}'(e^{u/2}\bar{\Psi}) = Q'(\Psi). \tag{5.9}$$

Hence we have the following theorem.

THEOREM 5.3. *Let $(M, g_M, \mathcal{F})$ be a compact connected Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$ such that $\delta\kappa = 0$. Then for any transversal W-twistor spinor $\Psi$, $C'(\Psi)$ and $Q'(\Psi)$ are transversally conformal invariants with respect to $\Psi \to e^{u/2}\bar{\Psi}$. Moreover they are constant.*

*Proof.* The first statement follows from (5.9). Next, if we differentiate $C'(\Psi)$ with respect to $X \in \Gamma TM$, then

$$\nabla_X C'(\Psi) = (\nabla_X D'_{tr}\Psi, \Psi) + (D'_{tr}\Psi, \nabla_X \Psi).$$

From (3.14) and Proposition 3.4, we have

$$\begin{aligned}\nabla_X C'(\Psi) =& \frac{q\sigma^\nabla}{4(q-1)(q-2)}(\pi(X)\cdot\Psi, \Psi) \\ &- \frac{q}{2(q-2)}(\rho^\nabla(\pi(X))\cdot\Psi, \Psi) - \frac{1}{q}(D'_{tr}\Psi, \pi(X)\cdot D'_{tr}\Psi),\end{aligned}$$

which deduces that $C'(\Psi)$ is constant. Moreover,

$$\begin{aligned}\nabla_X Q'(\Psi) &= 2(\nabla_X\Psi, \Psi)|D'_{tr}\Psi|^2 + 2|\Psi|^2(\nabla_X D'_{tr}\Psi, D'_{tr}\Psi) \\ &\quad - 2\sum_a (D'_{tr}\Psi, E_a\cdot\Psi)(\nabla_X D'_{tr}\Psi, E_a\cdot\Psi) \\ &\quad - \frac{2}{q}\sum_a (D'_{tr}\Psi, E_a\cdot\Psi)(E_a\cdot D'_{tr}\Psi, \pi(X)\cdot D'_{tr}\Psi).\end{aligned}$$

By a long calculation together with Propositon 3.4, we obtain $\nabla_X Q'(\Psi) = 0$. □

Given a spinor $\Psi$, we define the associated vector field $T^\Psi$ by

$$T^\Psi = 2\sum_a (\Psi, E_a\cdot D'_{tr}\Psi)E_a. \tag{5.10}$$

Assume that $\Psi$ is a non-vanishing transversal W-twistor spinor. Then

$$T^{\Psi} = -q \,\mathrm{grad}_{\nabla} w, \tag{5.11}$$

where $w = |\Psi|^2$. On the other hand, we have from (5.10)

$$|C'(\Psi)\Psi - wD'_{tr}\Psi - \frac{1}{2}T^{\Psi} \cdot \Psi|^2 = wQ'(\Psi). \tag{5.12}$$

Hence if $\Psi$ satisfies $C'(\Psi) = 0 = Q'(\Psi)$, then (5.11) and (5.12) imply

$$wD'_{tr}\Psi = \frac{q}{2}\,\mathrm{grad}_{\nabla}(w) \cdot \Psi. \tag{5.13}$$

Hence we have the following proposition.

PROPOSITION 5.4. *Under the same condition as in Theorem 5.3, if $M$ admits a non-vanishing transversal W-twistor spinor $\Psi$ such that $C'(\Psi) = 0 = Q'(\Psi)$, then $\mathcal{F}$ is transversally conformally equivalent to a transversally Ricci-flat foliation on $(M, \bar{g}_M)$ with parallel basic spinor.*

*Proof.* Consider the metric $\bar{g}_M = g_L + \bar{g}_Q$, where $\bar{g}_Q = e^{2u}g_Q$ with $u = -\ln w$ and $\omega = |\Psi|^2$. Then Proposition 5.1(4) gives rise to

$$\begin{aligned}\bar{\nabla}_X(w^{-\frac{1}{2}}\bar{\Psi}) &= \overline{\nabla_X(w^{-\frac{1}{2}}\Psi)} - \frac{1}{2}w^{-\frac{1}{2}}\overline{\pi(X)\cdot \mathrm{grad}_{\nabla}(u)\cdot\Psi} \\ &\quad - \frac{1}{2}w^{-\frac{1}{2}}X(u)\bar{\Psi} \\ &= w^{-\frac{1}{2}}\overline{\{\nabla_X\Psi + \frac{1}{2w}\pi(X)\cdot\mathrm{grad}_{\nabla}(w)\cdot\Psi\}}\end{aligned}$$

for $X \in \Gamma TM$. It follows from (3.14) and (5.13) that

$$\bar{\nabla}_X(w^{-\frac{1}{2}}\bar{\Psi}) = 0.$$

That is, $\bar{\Phi} = w^{-\frac{1}{2}}\bar{\Psi}$ is a parallel basic spinor with respect to the metric $\bar{g}_M$. From (3,3), $\rho^{\bar{\nabla}}(\pi(X))\bar{\cdot}\bar{\Phi} = \sum_a \bar{E}_a\bar{\cdot}R^{\bar{S}}(X,\bar{E}_a)\bar{\Phi} = 0$. Therefore $\mathcal{F}$ on $(M, \bar{g}_M)$ is transversally Ricci-flat. □

PROPOSITION 5.5. *Under the same condition as in Theorem 5.3, every non-vanishing transversal W-twistor spinor $\Psi$ satisfies*

$$\begin{aligned}\frac{q}{2}\{w\Delta_B(\ln w) - \kappa^{\sharp}(w)\} &+ \frac{q(q-2)}{4w}|grad_{\nabla} w|^2 \\ &= \frac{q}{4(q-1)}\sigma^{\nabla}w - \frac{1}{w}(Q'(\Psi) + C'(\Psi)^2),\end{aligned}$$

*where $w = |\Psi|^2$.*

*Proof.* Let $(0 \neq)\Psi \in \mathrm{Ker}P'_{tr}$ be a transversal W-twistor spinor. From (5.8) and (5.10), we have

$$Q'(\psi) = w|D'_{tr}\Psi|^2 - C'(\Psi)^2 - \frac{1}{4}|T^\Psi|^2. \tag{5.14}$$

Since $\Delta_B(\ln w) = \frac{1}{w^2}|\mathrm{grad}_\nabla w|^2 + \frac{1}{w}\Delta_B w$, (5.11) implies

$$\Delta_B(\ln w) = \frac{1}{q^2w^2}|T^\Psi|^2 + \frac{1}{w}\Delta_B w. \tag{5.15}$$

On the other hand,

$$\begin{aligned} &< \kappa \cdot D'_{tr}\Psi, \Psi > + < \Psi, \kappa \cdot D'_{tr}\Psi > \\ =& - q\{\nabla_{\kappa^\sharp}\Psi, \Psi > + < \Psi, \nabla_{\kappa^\sharp}\Psi >\} \\ =& - q\kappa^\sharp(\omega), \end{aligned}$$

and thus

$$|D_b\Psi|^2 = |D'_{tr}\Psi|^2 + \frac{1}{4}|\kappa|^2 w - \frac{q}{2}\kappa^\sharp(w). \tag{5.16}$$

From (3.23), (5.15), and (5.16), we have

$$\Delta_B(\ln w) = \frac{1}{q^2w^2}|T^\Psi|^2 + \frac{1}{2(q-1)}\sigma^\nabla + \frac{1}{w}\kappa^\sharp(w) - \frac{2}{qw}|D'_{tr}\Psi|^2, \tag{5.17}$$

which completes the proof by using (5.13) and (5.14). □

A spinor field $\Psi$ is said to be *transversally conformally equivalent to a transversal Killing spinor* if there exists a transversally conformal change $\bar g_M = g_L + e^{2u}g_Q$ such that $e^{\frac{u}{2}}\bar\Psi$ is a transversal Killing spinor with respect to $\bar g_M$. This is equivalent that for any $X \in \Gamma TM$

$$\bar\nabla_X(e^{\frac{u}{2}}\bar\Psi) + a\pi(X)\bar\cdot(e^{\frac{u}{2}}\bar\Psi) = 0, \tag{5.18}$$

where $a(\neq 0)$ is a real number. Then we have the following theorem.

THEOREM 5.6. *Let $\Psi \in \mathrm{Ker}P'_{tr}$ be a non-vanishing transversal W-twistor spinor. Then $\Psi$ is transversally conformally equivalent to a transversal Killing spinor if and only if $C'(\Psi) \neq 0$ and $Q'(\psi) = 0$.*

*Proof.* Let $\Psi \in \mathrm{Ker}P'_{tr}$ be transversally conformally equivalent to a transversal Killing spinor with respect to $\bar g_Q = e^{2u}g_Q$. (5.18) is equivalent to

$$\nabla_X\Psi = \frac{1}{2}\pi(X)\cdot \mathrm{grad}_\nabla u \cdot \Psi - ae^u\pi(X)\cdot\Psi \tag{5.19}$$

for $X \in \Gamma TM$, where $a(\neq 0)$ is a real number. Now if we choose $u = -\ln w$, then (5.19) implies

$$\frac{1}{q}\omega D'_{tr}\Psi = \frac{1}{2}\mathrm{grad}_\nabla w \cdot \Psi + a\Psi, \tag{5.20}$$

which follows from (5.7) and (5.20) that

$$C'(\Psi) = qa \neq 0. \tag{5.21}$$

On the other hand, (5.20) gives rise to

$$\omega|D'_{tr}\Psi|^2 = \frac{q^2}{4}|\mathrm{grad}_\nabla w|^2 + q^2a^2. \tag{5.22}$$

From (5.8), (5.11), (5.21), and (5.22), we obtain $Q'(\Psi) = 0$.

Conversely, we consider a non-vanishing transversal W-twistor spinor $\Psi$ with $C'(\Psi) \neq 0$ and $Q'(\Psi) = 0$. Then (5.12) implies

$$C'(\Psi)\Psi - wD'_{tr}\Psi - \frac{1}{2}T^\Psi \cdot \Psi = 0, \tag{5.23}$$

and hence by (5.11)

$$C'(\Psi)\Psi - wD'_{tr}\Psi + \frac{q}{2}\mathrm{grad}_\nabla w \cdot \Psi = 0. \tag{5.24}$$

If we choose $u$ so that $w = \frac{C'(\Psi)}{qa}e^{-u}$, then (5.23) and (5.24) show that $\Psi$ satisfies (5.18). This means that $\Psi$ is transversally conformally equivalent to a transversal Killing spinor. □

## 6. Eigenvalue estimates

In this section, let $(M, g_M, \mathcal{F})$ be a compact connected Riemannian manifold with a transverse spin foliation $\mathcal{F}$ and a bundle-like metric $g_M$ such that the mean curvature form $\kappa$ satisfies $\Delta_B\kappa = 0$. The existence of a bundle-like metric $g_M$ for $(M, \mathcal{F})$ such that $\kappa$ is basic, i.e., $\kappa \in \Omega^1_B(\mathcal{F})$, is proved in [4]. Given a bundle-like metric $g_M$ with $\kappa \in \Omega^1_B(\mathcal{F})$, it is assured ([16], [17]) that there exists another bundle-like metric whose mean curvature form is basic harmonic.

By a straightforward calculation, we have that for $\Psi \in \Gamma S(\mathcal{F})$

$$|P'_{tr}\Psi|^2 = |\nabla_{tr}\Psi|^2 - \frac{1}{q}|D'_{tr}\Psi|^2. \tag{6.1}$$

It follows from (3.7) that

$$\int_M |P'_{tr}\Psi|^2 = \int_M \left\{|D_{tr}\Psi|^2 - \frac{1}{4}K^\sigma|\Psi|^2 - \frac{1}{q}|D'_{tr}\Psi|^2\right\}, \tag{6.2}$$

where $K^\sigma = \sigma^\nabla + |\kappa|^2$. Since

$$|D_{tr}\Psi|^2 = |D'_{tr}\Psi|^2 - \frac{1}{4}|\kappa|^2|\Psi|^2 - Re < D_{tr}\Psi, \kappa\cdot\Psi >, \tag{6.3}$$

we have from (6.2) and (6.3) that

(6.4)

$$\int_M |P'_{tr}\Psi|^2 = \frac{q-1}{q}\int_M \left\{|D_{tr}\Psi|^2 - \frac{q}{4(q-1)}\left(K^\sigma + \frac{1}{q}|\kappa|^2\right)|\Psi|^2\right\} + \int_M Re < D_{tr}\Psi, \kappa\cdot\Psi > .$$

Let $D_b\Psi = \lambda\Psi$. Then (6.4) becomes

$$\int_M |P'_{tr}\Psi|^2 = \frac{q-1}{q}\int_M \left\{\lambda^2 - \frac{q}{4(q-1)}\left(K^\sigma + \frac{1}{q}|\kappa|^2\right)\right\}|\Psi|^2. \tag{6.5}$$

Hence we have the following theorem (cf.[10]).

THEOREM 6.1. *Let $(M, g_M, \mathcal{F})$ be a compact connected Riemannian manifold with a transverse spin foliation $\mathcal{F}$ of codimension $q \geq 2$ and bundle-like metric $g_M$ such that $\Delta_B\kappa = 0$. Then any eigenvalue $\lambda$ of the basic Dirac operator $D_b$ satisfies*

$$\lambda^2 \geq \frac{q}{4(q-1)}\inf_M\left(K^\sigma + \frac{1}{q}|\kappa|^2\right). \tag{6.6}$$

*In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature $\sigma^\nabla$.*

*Proof.* It suffices to investigate the limiting case that $\mathcal{F}$ admits a non-vanishing spinor field $\Psi$ such that $D_b\Psi = \lambda\Psi$. From (6.5), we see $P'_{tr}\Psi = 0$. Since $D'_{tr}\Psi = \lambda\Psi + \frac{1}{2}\kappa\cdot\Psi$, we have from (3.14)

$$\nabla_X\Psi = -\frac{\lambda}{q}\pi(X)\cdot\Psi - \frac{1}{2q}\pi(X)\cdot\kappa\cdot\Psi \tag{6.7}$$

for $X \in \Gamma TM$. It follows from (3.10) and (3.16) that $\mathcal{F}$ is minimal. Therefore Theorem 4.3 says that $\mathcal{F}$ is transversally Einsteinian with constant transversal scalar curvature $\sigma^\nabla > 0$. □

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Seoung Dal Jung and Yeong Bong Moon
Department of Mathematics
Cheju National University
Cheju 690-756, Korea
*E-mail*: sdjung@cheju.cheju.ac.kr(S. D. Jung)
myb555@choll.com(Y. B. Moon)