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R-HOMOMORPHISMS AND R-HOMOGENEOUS MAPS

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ABSTRACT. In this paper, all rings and all near-rings R are associative, all modules are right R-modules. For a near-ring R, we consider representations of R as R-groups. We start with a study of AGR rings and their properties.

Next, for any right R-module M, we define a new concept GM module and investigate the commutative property of faithful GM modules and some characterizations of GM modules. Similarly, for any near-ring R, we introduce an R-group with MR-property and some properties of MR groups.

1. Introduction

Throughout this paper, for an associative ring R(a near-ring R), we will consider M is a right R-module(G an R-group). We begin with a study of AGR rings and their properties.

Next, for any group G and a nonempty subset S of End(G), we know the centralizer near-ring of S and G as $C(S;G) = \{f \in M(G) | \ \alpha f = f\alpha, \ \forall \alpha \in S\}$, and for a nonempty subset S of the distributive elements on G, we can define the centralizer near-ring of S and a unitary R-group G. Furthermore, we will show that if R is a unitary semisimple ring and M a unitary right R-module, then $M_R(M)$ is a semisimple near-ring.

Finally, for any right R module M, we define a new concept GM module and investigate the commutative property of faithful GM modules and some characterizations of GM modules. Also, for any near-ring R, we introduce an R-group with MR-property and some properties of MR groups as analogous properties of GM modules.

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A near-ring R with $(R,\ +)$ abelian is called *abelian.* An element d in R is called *distributive* if $(a+b)d = ad + bd$ for all a and b in R.

We consider the following notations: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ is called the *zero symmetric part* of R, $R_c = \{a \in R \mid 0a = a\}$ is called the *constant part* of R, and $R_d = \{a \in R \mid a \text{ is distributive}\}$ is called the *distributive part* of R. We note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R.

Let $(G, +)$ be a group(not necessarily abelian). We will use right operations(that is, operations on the right side of the variables) in the near-ring case to distinguish from left operations in the ring case in this paper. In the set

$$M(G) := \{f \mid f : G \longrightarrow G\}$$

of all self maps of G, if we define the sum $f+g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf+xg$ for all $x \in G$ (called the *pointwise addition of maps*) and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* of the group G. Also, if we define the set

$$M_0(G) := \{f \in M(G) \mid of = o\}$$

for the additive group G with identity o, then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

Let R be a near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write xr (right scalar multiplication in R) for $x(\theta_r)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary.* Note that R itself is an R-group called the *regular group.*

Naturally, every group G has an $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

An R-group G with the property that for each $x,\ y \in G$ and $a \in R$, $(x+y)a = xa + ya$ is called a *distributive R-group*, and also an R-group G with $(G,\ +)$ is abelian is called an *abelian R-group.* For example, if $(G,\ +)$ is abelian, then $M(G)$ is an abelian near-ring and moreover, G is an abelian $M(G)$-group. On the other hand, every distributive near-ring R is a distributive R-group.

A near-ring R is called *distributively generated* (briefly, *D.G.*) by S if $(R,+) = gp\langle S\rangle = gp\langle R_d\rangle$ where S is a semigroup of distributive elements in R, in particular, $S = R_d$(this is multiplicatively closed and contain the unity of R if R is unitary), and $gp\langle S\rangle$ is a group generated by S. This D.G. near-ring R which is generated by S is denoted by (R, S).

On the other hand, the set of all distributive elements of $M(G)$ are obviously the set $End(G)$ of all endomorphisms of the group G, that is,

$$(M(G))_d = \mathrm{End}(G)$$

which is a semigroup under composition, but not yet a near-ring. Here we denote that $E(G)$ is the D.G. near-ring generated by $\mathrm{End}(G)$, that is,

$$E(G) = gp\langle \mathrm{End}(G)\rangle.$$

Obviously, $E(G)$ is a subnear-ring of $(M_0(G),+,\cdot)$. Thus we say that $E(G)$ is the *endomorphism near-ring* of the group G.

For the remainder basic concepts and results on ring case and near-rings case, we refer to [1], [16], and [18].

2. Properties of generalized AR rings

We begin to study a class of rings in which all the additive endomorphisms or only the left multiplication endomorphisms are generated by ring endomorphisms. This research was motivated by the work on Sullivan's Research Problem (that is, characterize those rings in which every additive endomorphism is a ring endomorphism, these rings are called *AR rings* [3], [4], [6], [7], [8], [10] and [19], and the investigation of LSD-generated rings and SD-generated rings [3] and [9]. Now, we introduce some generalizations of AR rings.

At first, a ring R is said to be an *AGR ring* if every additive endomorphism is generated by ring endomorphisms, that is,

$$\mathrm{End}(R,+) = gp\langle \mathrm{End}(R,+,\cdot)\rangle$$

Clearly, we see that every AR ring is AGR, but not conversely from Example 2.4. Note if the left regular representation of R into $\mathrm{End}(R,+)$ is surjective, then R is an AGR ring.

Putting $\mathcal{L}(R)$ is the $\{x \in R|\ xab = xaxb,\ \forall\ a,\ b \in R\}$ of all left self distributive elements in R and $\mathcal{R}(R)$ is the $\{x \in R|\ abx = axbx,\ \forall\ a,\ b \in$

$R\}$ of all right self distributive elements in R. $(\mathcal{L}(R), \cdot)$ and $\mathcal{R}(R)$ are subsemigroups of $(R, \cdot)$. Note that $\mathcal{L}(R)$ contains all one-sided unities and all central idempotents of R.

A ring R is called *LSD* (resp. *LSD-generated*) if $R = \mathcal{L}(R)$ (resp. $R = gp(\mathcal{L}(R))$), similarly for *RSD* and *RSD-generated*. R is called *SD* (resp. *SD-generated*) if $R = \mathcal{L}(R) \cap \mathcal{R}(R)$ (resp. $R = gp(\mathcal{L}(R) \cap \mathcal{R}(R))$. The classes of LSD, LSD-generated, SD and SD-generated rings are closed with respect to homomorphisms and direct sums, and the class of AGR rings is not contained in the class of SD-generated rings [3], [9]. $\mathcal{I}(R)$ and $\mathcal{N}(R)$ denote the set of idempotent elements of R and the set of nilpotent elements of R, respectively.

Notice that R is an AGR ring if and only if there exists a subsemigroup S of $\text{End}(R, +, \cdot)$ such that $\text{End}(R, +) = gp\langle S\rangle$. Sometimes, we use the other notations: $\text{End}_{\mathbb{Z}}(R)$ instead of $\text{End}(R, +)$, $\text{End}(R)$ instead of $\text{End}(R, +, \cdot)$ and $GE(R)$ instead of $gp\langle \text{End}(R, +, \cdot)\rangle$.

We will use the following several notations: For each $x \in R$, ${}_x\tau$ denotes the left multiplication mapping(that is, $a \longmapsto xa, \forall a \in R$) and $\mathcal{T}(R)$ is the set $\{{}_x\tau \mid x \in R\}$. Clearly ${}_x\tau \in \text{End}(R, +)$ and $x \in \mathcal{L}(R)$ if and only if ${}_x\tau \in \text{End}(R, +, \cdot)$. $\mathcal{LGE}(R)$ is the $\{x \in R \mid {}_x\tau \in gp\langle \text{End}(R, +, \cdot)\rangle\}$. Also, $\mathcal{L}(R) \subset \mathcal{LGE}(R)$.

A ring R is called *almost AR ring* if every left multiplication endomorphism is a ring endomorphism, that is, $\mathcal{T}(R) \subset \text{End}(R, +, \cdot)$.

The following two statements are very easily proved, but these are the basis of the notions of almost AR rings and AGR rings.

LEMMA 2.1. *Let R be any ring. Then we have the following:*

(1) *$\mathcal{T}(R)$ is a subring of $\text{End}_{\mathbb{Z}}(R)$;*
(2) *$GE(R)$ is a subring of $\text{End}_{\mathbb{Z}}(R)$;*
(3) *$\mathcal{LGE}(R)$ is a subring of R.*

PROPOSITION 2.2.

(1) *Every LSD ring is an almost AR ring.*
(2) *Every AR ring is an almost AR ring.*

The following statement is a special case of a characterization of AGR rings,

PROPOSITION 2.3. *For every AGR ring R, and for any positive integer n, we get that $\oplus_{i=1}^{n} R_i$ is an AGR ring, where $R_i \cong R$, for all $i = 1, 2, \ldots, n$.*

Proof. We prove the case for $n = 2$, that is, $R \oplus R$. Similarly, we can prove for the case $n > 2$. We must show that

$$\mathrm{End}_{\mathbb{Z}}(R \oplus R) = GE(R \oplus R).$$

Since $\mathrm{End}_{\mathbb{Z}}(R \oplus R) \cong \mathrm{Mat}_2(\mathrm{End}_{\mathbb{Z}}(R))$, we obtain that

$$\mathrm{End}_{\mathbb{Z}}(R \oplus R) \cong \begin{bmatrix} \mathrm{End}_{\mathbb{Z}}(R) & \mathrm{End}_{\mathbb{Z}}(R) \\ \mathrm{End}_{\mathbb{Z}}(R) & \mathrm{End}_{\mathbb{Z}}(R) \end{bmatrix} = \begin{bmatrix} GE(R) & GE(R) \\ GE(R) & GE(R) \end{bmatrix}.$$

Let $f \in \mathrm{End}_{\mathbb{Z}}(R \oplus R)$ such that

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \; f_{ij} \in GE(R).$$

Then

$$f_{11} = \sum_i \lambda_i h_i, \; f_{12} = \sum_j \lambda_j h_j, \; f_{21} = \sum_k \lambda_k h_k, \; f_{22} = \sum_t \lambda_t h_t,$$

where, $\lambda's \in \mathbb{Z}$ and $h's \in \mathrm{End}(R)$. Thus f is expressed of the form

$$f = \sum_i \lambda_i \begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix} + \sum_j \lambda_j \begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix} + \sum_k \lambda_k \begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix} + \sum_t \lambda_t \begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}.$$

Since all $\begin{bmatrix} h_i & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & h_j \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ h_k & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & h_t \end{bmatrix}$ are ring endomorphisms of $R \oplus R$, $R \oplus R$ is an AGR ring. □

EXAMPLE 2.4. [3] Rings additively generated by central idempotents and one sided unities are LSD-generated and RSD-generated, so that SD-generated. In particular, since the rings $\mathbb{Z}$ and $\mathbb{Z}_n$ are additively generated by 1, and $\mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) \cong \mathbb{Z}, \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_n) \cong \mathbb{Z}_n$, we see that $\mathbb{Z}$ and $\mathbb{Z}_n$ are both AGR, LSD-generated and SD-generated rings. However, $\mathbb{Z}$ and $\mathbb{Z}_n$ are not AR rings except for the cases $\mathbb{Z}_1$ and $\mathbb{Z}_2$, because any nontrivial endomorphism on $\mathbb{Z}$ or $\mathbb{Z}_n$ is additive but which is not a ring endomorphism. On the other hand, if $x \in \mathcal{L}(R)$ implies $x^3 = x^n$ for $n > 3$, then $\mathcal{L}(S) = \{0\}$ for any nonzero proper subring S of $\mathbb{Z}$. Hence any nonzero proper subring of $\mathbb{Z}$ is an AGR ring which is not LSD-generated and SD-generated.

Proposition 2.3 and Example 2.4 show that there are many examples of AGR rings and LSD-generated rings. Obviously, we get the following useful lemma:

LEMMA 2.5. *For any surjective ring endomorphism h, $\mathcal{L}(R)$ and $\mathcal{R}(R)$ are all fully invariant under h.*

From this lemma, we get the following statement.

PROPOSITION 2.6. *Let R be a ring with unity. If R is an AGR ring with $S \subset End(R)$ such that $End_{\mathbb{Z}}(R) = gp\langle S\rangle$, and each element of S is onto, then R is LSD-generated, moreover SD-generated.*

Proof. Let $x \in R$. Consider the left translation mapping $\phi_x : R \longrightarrow R$ by $\phi_x(a) = xa$ for all $a \in R$, which is a group endomorphism. Since R is an AGR ring, $\phi_x = \sum_i^n \lambda_i h_i$, where $\lambda_i \in \mathbb{Z}$ and $h_i \in S$ such that h_i is onto, for $i = 1, 2, \cdots, n$. Since $1 \in R$, $\phi_x(1) = \sum_i^n \lambda_i h_i(1)$, that is, $x = \sum_i^n \lambda_i h_i(1)$, and since $1 \in \mathcal{L}(R) \cap \mathcal{R}(R)$ by Lemma 2.5, we have $h_i(1) \in \mathcal{L}(R) \cap \mathcal{R}(R)$. Hence R is LSD-generated and RSD-generated, so is SD-generated. □

3. R-homomorphisms and R-homogeneous maps

Hereafter, we can introduce similar notions of AR rings or almost AR rings in right R-modules and R-groups. First, we introduce a new concept GM-property of a right R-module and investigate it's properties.

For any ring R, right R-modules M and N, the set of all R-module homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$ and the set of all group homomorphisms from M to N is by

$$\mathrm{Hom}(M, N) := \mathrm{Hom}_{\mathbb{Z}}(M, N),$$

in particular, we denote that $\mathrm{End}_R(M) := \mathrm{Hom}_R(M, M)$ and $\mathrm{End}(M) : = \mathrm{End}_{\mathbb{Z}}(M) = \mathrm{Hom}_{\mathbb{Z}}(M, M)$. In this case, M is called a *GM module* over R if every group homomorphism of M is an R-module homomorphism, that is,

$$\mathrm{End}(M) = \mathrm{End}_R(M).$$

In particular, R is called a *GM ring* if R is a GM module as a right R-module, that is, for all $f \in \mathrm{End}_{\mathbb{Z}}(R)$, x, $r \in R$, we have $f(xr) = f(x)r$.

For example, $\mathbb{Q}$ is a GM module, because of $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q})=\mathbb{Q}=\mathrm{End}_{\mathbb{Q}}(\mathbb{Q})$.

PROPOSITION 3.1. *Let $\{M_i \mid i \in \Lambda\}$ be any family of right R-modules. Then each M_i is a GM module for all $i \in \Lambda$ if and only if $M := \oplus M_i$ is a GM module.*

Proof. Suppose each M_i is a GM module for all $i \in \Lambda$. Let $f \in \mathrm{End}(M)$. Consider canonical epimorphism $\pi_i : \oplus M_i \longrightarrow M_i$ and canonical monomorphism $\kappa_i : M_i \longrightarrow \oplus M_i$ as usual meaning in module theory. Define $f_i \in \mathrm{End}(M_i)$ as $f_i = \pi_i \circ f \circ \kappa_i$. Then by assumption, f_i is an R-module homomorphism.

To show that f is an R-module homomorphism, we must show that for all $x \in M, r \in R, i \in \Lambda$,

$$\pi_i(f(xr)) = \pi_i(f(x)r).$$

Indeed, since $f_i = \pi_i \circ f \circ \kappa_i$ and $\sum_{i\in\Lambda}(\kappa_i \circ \pi_i)x = x$ for any $x \in M$, we have

$$\begin{aligned}\pi_i \circ f(xr) &= \pi_i \circ f(\sum_{i\in\Lambda}(\kappa_i \circ \pi_i)(xr)) = \sum_{i\in\Lambda}\pi_i \circ f \circ \kappa_i \circ \pi_i(xr)\\ &= \sum_{i\in\Lambda} f_i \circ \pi_i(xr) = \sum_{i\in\Lambda} f_i(\pi_i(x)r) = \sum_{i\in\Lambda}(f_i \circ \pi_i)(x)r\\ &= \sum_{i\in\Lambda}\pi_i \circ f \circ \kappa_i \circ \pi_i(x)r = \pi_i \circ f(\sum_{i\in\Lambda}(\kappa_i \circ \pi_i)(x))r\\ &= \pi_i \circ f(x)r.\end{aligned}$$

Hence $(f(xr)) = (f(x)r)$, for all $x \in M, r \in R$. Consequently, $M = \oplus M_i$ is a GM module.

Conversely, let $g_i \in \mathrm{End}(M_i)$ and let $x_i \in M_i$, $r \in R$, Consider, $x = (x_i)_{i\in\Lambda} =: (x_i) \in \oplus M_i$, where $x_i \in M_i$, $x_i = 0$ except finitely many $i \in \Lambda$. We can define a function $f : \oplus M_i \longrightarrow \oplus M_i$ by $f(x) = (g_i(x_i))_{i\in\Lambda} =: (g_i(x_i))$.

Let x, $y \in M$, by the above notation. Then

$$f(x+y) = f((x_i+y_i)) = (g_i(x_i+y_i)) = (g_i(x_i))+(g_i(y_i)) = f(x)+f(y).$$

Since $M := \oplus M_i$ is a GM module, above equalities implies that $f \in \mathrm{End}(M) = \mathrm{End}_R(M)$.

From this fact, since

$$f(xr) = f((x_i)r) = f((x_ir)) = (g_i(x_ir))$$

and

$$f(x)r = f((x_i))r = (g_i(x_i))r = (g_i(x_i)r),$$

we derive that

$$g_i(x_ir) \;=\; (g_i(x_i)r)$$

for all $i \in \Lambda$; that is, $g_i \in \mathrm{End}_R(M_i)$. Therefore each M_i is a GM module. □

PROPOSITION 3.2. *Let R be a GM ring. Then for any $x \in R$, xR is a GM ring. Furthermore, this xR is also a GM module as an R-module.*

Proof. Let $f \in \text{End}(xR)$, and $g : R \longrightarrow R$ be defined by $g(a) = f(xa)$ for all $a \in R$. Then $g \in \text{End}_{\mathbb{Z}}(R)$. This implies that $g(axb) = g(a)xb$, because $\text{End}_{\mathbb{Z}}(R) = \text{End}_R(R)$. So we have

$$f(xaxb) = g(axb) = g(a)xb = f(xa)xb.$$

Hence, for any $x \in R$, xR is a GM ring. Obviously, we can check that xR is a GM module as an R-module. □

Applying Propositions 3.1 and 3.2, we obtain the following:

COROLLARY 3.3. *Let R be a GM unitary ring. Then all finitely generated right ideals and all direct sums of principle right ideals are GM rings.*

From the faithful GM-property, we get a commutativity of rings.

PROPOSITION 3.4. *Let M be a right R-module. If M is a faithful GM module, then R is a commutative ring.*

Proof. Let $f \in \text{End}(M)$ and let $a, b \in R$, where $f(x) = xa$, for all $x \in M$. Then

$$f(xb) = (xb)a.$$

On the other hand, since $f \in \text{End}(M) = \text{End}_R(M)$, we have that

$$f(xb) = f(x)b = (xa)b.$$

Hence $(xb)a = (xa)b$ for all $x \in M$. Since M is faithful, so we see that $ab = ba$. □

Next, we shall treat a D.G. near-ring R generated by S and a faithful R-group G. Furthermore, there is a module like concept as follows: Let (R, S) be a D.G. near-ring. Then an additive group G is called a *D.G. (R, S)-group* if there exists a D.G. near-ring homomorphism

$$\theta \; : \; (R, S) \longrightarrow (M(G), \text{End}(G)) = E(G)$$

such that $S\theta \subset \text{End}(G)$. If we write that xr instead of $x(\theta_r)$ for all $x \in G$ and $r \in R$, then an D.G. (R, S)-group is an additive group G satisfying the following conditions:

$$x(rs) = (xr)s, \;\; x(r + s) = xr + xs, \;\; (x + y)s = xs + ys,$$

for all $x,\ y \in G$ and all $r,\ s \in S$.

Such a homomorphism θ is called a *D.G. representation* of (R, S) on G. This D.G. representation is said to be *faithful* if $\mathrm{Ker}\theta = \{0\}$. In this case, we say that G is called a *faithful D.G. (R, S)-group* [11], [12], [17], [18].

Let $G,\ T$ be two additive groups (not necessarily abelian). Then the set

$$M(G,\ T) := \{f \mid f : G \longrightarrow T\}$$

of all maps from G to T becomes an additive group under pointwise addition of maps. Since $M(T)$ is a near-ring of self maps on T, we note that $M(G,\ T)$ is an $M(T)$-group with a scalar multiplication:

$$M(G,\ T) \times M(T) \longrightarrow M(G,\ T)$$

defined by $(f,\ g) \longmapsto f \cdot g$, where $x(f \cdot g) = (xf)g$ for all $x \in G$.

Let G and T be two R-groups. Then the mapping $f : G \longrightarrow T$ is called a *R-group homomorphism* if for all $x,\ y \in G$ and $a \in R$, (i) $(x + y)f = xf + yf$ and (ii) $(xa)f = (xf)a$. In this paper, we call that the mapping $f : G \longrightarrow T$ with the condition $(xa)f = (xf)a$ is an *R-homogeneous map* (or simply, *R-map*) [15]. We define the set

$$M_R(G,\ T) := \{f \in M(G,\ T) |\ (xr)f = (xf)r,\ \forall\ x \in G,\ r \in R\}$$

of all R-homogeneous maps from G to T.

For any near-ring R and R-group G, we write the set

$$M_R(G) := \{f \in M(G) |\ (xr)f = (xf)r,\ \forall\ x \in G,\ r \in R\}$$

of all R-homogeneous maps on G as defined previously.

On the other hand, an element $a \in R$ is said to *distributive on G* if $(x + y)a = xa + ya$ for all $x,\ y \in G$. Putting $D_R(G)$ the set of all distributive elements on G, $D_R(G)$ becomes a ring whenever G is abelian. For example, every unitary abelian near-ring contains a unitary ring.

The following two statements are motivation of MR-property of R-groups.

LEMMA 3.5. *Let G be an abelian D.G. (R, S)-group. Then the set $M_R(G) := \{f \in M(G) |\ (xr)f = (xf)r,\ \forall\ x \in G,\ r \in R\}$ is a subnear-ring of $M(G)$.*

Proof. Let $f,\ g \in M_R(G)$. For any $x \in G$ and $r \in R$, since R is a D.G. near-ring generated by S, consider that

$$r = \sum_{i=1}^{n} \delta_i s_i,$$

where $\delta_i = 1$, or -1 and $s_i \in S$ for $i = 1, \cdots, n$. We have that

$$\begin{aligned}
&(xr)(f+g)\\
&= (xr)f + (xr)g\\
&= (xf)r + (xg)r\\
&= xf\left(\sum_{i=1}^{n} \delta_i s_i\right) + xg\left(\sum_{i=1}^{n} \delta_i s_i\right)\\
&= xf\delta_1 s_1 + xg\delta_1 s_1 + xf\delta_2 s_2 + xg\delta_2 s_2 + \cdots + xf\delta_n s_n + xg\delta_n s_n\\
&= \delta_1 xfs_1 + \delta_1 xgs_1 + \delta_2 xfs_2 + \delta_2 xgs_2 + \cdots + \delta_n xfs_n + \delta_n xgs_n\\
&= \delta_1(xfs_1 + xgs_1) + \delta_2(xfs_2 + xgs_2) + \cdots + \delta_n(xfs_n + xgs_n)\\
&= \delta_1(xf + xg)s_1 + \delta_2(xf + xg)s_2 + \cdots + \delta_n(xf + xg)s_n\\
&= (xf + xg)\delta_1 s_1 + (xf + xg)\delta_2 s_2 + \cdots + (xf + xg)\delta_n s_n\\
&= (xf + xg)\left(\sum_{i=1}^{n} \delta_i s_i\right)\\
&= (xf + xg)r\\
&= x(f+g)r.
\end{aligned}$$

Similarly, we have the following equalities:

$$(xr)(-f) = -(xr)f = -(xf)r = x(-f)r$$

and

$$(xr)f \cdot g = ((xr)f)g = ((xf)r)g = (xf)gr = x(f \cdot g)r.$$

Thus $M_R(G)$ is a subnear-ring of $M(G)$. □

On the other hand, for a group G and a nonempty subset S of $\mathrm{End}(G)$, we define the *centralizer* of S in G as following:

$$C(S;G) = \{f \in M(G) \mid \alpha f = f\alpha \ \forall\ \alpha \in S\},$$

which is a subnear-ring of $M(G)$, we say that $C(S;G)$ is the *centralizer near-ring* of S in G. This is an extended concept of centralizer near-ring which is introduced in [13, 14], at there, S is a subsemigroup of $\mathrm{End}(G)$. Also, for any endomorphism α of G, the centralizer of α in G is $C(\{\alpha\};G)$ we denote it simply by $C(\alpha;G)$. Note that obviously, $C(\alpha;G)$ is a subnear-ring of $M(G)$ and

$$C(S;G) = \cap_{\alpha\in S}C(\alpha;G).$$

Also, we see that $C(1_G;G) = M(G)$ and $C(0;G) = M_0(G)$.

In ring and module theory, we obtain the following important structure for near-ring and R-group theory:

Considering each element $a \in R$ is an endomorphism of V and

$$M_R(V) := \{f \in M(V)|\ af = fa,\ \forall\ a \in R\},$$

we see that

$$M_R(V) = C(R;\ V)$$

is the centralizer near-ring of R and V. Also

$$M_R(V) = \cap_{a\in R}M_a(V).$$

PROPOSITION 3.6. *Let R be a semisimple ring with unity 1 and let M be a right R-module. Then $M_R(M)$ is a semisimple near-ring.*

Proof. Consider $R = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where each S_i is a simple ring. Let e_i denote the unity of S_i. If $M_i = \{x \in M|\ xe_i = x\}$, then

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

and $M_i f \subset M_i$ for each $f \in M_R(M)$. Moreover, if f_i denotes the restriction of f to M_i, then the mapping

$$\theta\ :\ M_R(M) \longrightarrow M_{S_1}(M_1) \oplus M_{S_2}(M_2) \oplus \cdots \oplus M_{S_1}(M_1)$$

defined by $\theta(f) = (f_1, f_2, \cdots, f_n)$ is a near-ring homomorphism. This mapping is surjective, for if $(f_1, f_2, \cdots, f_n) \in M_{S_1}(M_1) \oplus M_{S_2}(M_2) \oplus \cdots \oplus M_{S_n}(M_n)$, then we can extend each f_i to all of M defined by $(x_1+x_2+\cdots+x_n)\bar{f_i} = x_i f_i$. Then $f = \sum \bar{f_i}$ is an element of $M_R(M)$ such that $\theta(f) = (f_1, f_2, \cdots, f_n)$. To show that θ is injective, we note that $(x_1 + x_2 + \cdots + x_n)fe_i = (x_ie_i)f = x_if,\ i = 1, 2, \cdots, n$. This implies $(x_1 + x_2 + \cdots + x_n)f = x_1f_1 + x_2f_2 + \cdots + x_nf_n$. Thus $\theta(f)$ implies that $f = 0$. Hence θ is an isomorphism and from the Theorem 1 of [13], each $M_{S_i}(M_i)$ is a simple near-ring. □

Now we get a more general concept then centralization which is known till now.

PROPOSITION 3.7. *Let R be a near-ring with unity 1 and G a unitary R-group. Then for any nonempty subset S of $D_R(G)$,*

$$M_S(G) := \{f \in M(G) |\ af = fa,\ \forall\ a \in S\}$$

is a centralizer subnear-ring of $M(G)$ and

$$M_S(G) = \cap_{a \in S} M_{\{a\}}(G).$$

Moreover obviously, we see that $M_{\{1\}}(G) = M(G)$ and $M_{\{0\}}(G) = M_0(G)$.

In Proposition 3.7, $M_S(G)$ is called the *centralizer near-ring* of S and G which is a generalization of centralizer near-rings in [13, 14, 15]. We denote $M_{\{a\}}(G)$ by $M_a(G)$ for convenance. Then

$$M_S(G) = \cap_{a \in S} M_a(G).$$

COROLLARY 3.8. [13, 14, 15] *Let R be a ring with unity 1 and V a unitary right R-module. Then $M_R(V) := \{f \in M(V) |\ (xa)f = (xf)a,$ for all $x \in V,\ a \in R\}$ is a subnear-ring of $M(V)$.*

LEMMA 3.9. [18] *Let G be a faithful R-group. Then we have the following conditions:*

(1) *If $(G, +)$ is abelian, then $(R, +)$ is abelian;*
(2) *If G is distributive, then R is distributive.*

Applying this Lemma, we get the following Proposition:

PROPOSITION 3.10. *If G is a distributive abelian faithful R-group, then R is a ring.*

The following statement which is obtained from Lemma 3.9 and property of faithful D.G. (R, S)-group is a generalization of the Proposition 3.10.

PROPOSITION 3.11. *Let (R, S) be a D.G. near-ring. If G is an abelian faithful D.G. (R, S)-group, then R is a ring.*

Finally, we also introduce the MR-property of R-group, which is motivated by the Lemma 3.5. An R-group G is called an *MR group* over near-ring R, provided that every mapping on G is an R-homogeneous map of G, that is,

$$M(G) = M_R(G).$$

EXAMPLE 3.12. (1) If $R = \mathbb{Z}$ is the near-ring of integers, then every regular R-group is an MR group.

(2) If $R = M_S(G)$ is a centralizer near-ring, then R-group G is an MR group.

We also apply Proposition 3.1 for GM-property for module to MR-property for R-group. Thus we only introduce a characterization of MR groups for direct sum without proof as following.

PROPOSITION 3.13. *Let $\{G_i |\, i \in \Lambda\}$ be any family of R-groups. Then each G_i is an MR group if and only if $G := \oplus G_i$ is an MR group.*

A similar property of Proposition 3.4 for MR group is obtained, using the variables on the right side of maps on R-group as defined previously, together with Proposition 3.10. Thus we have the following:

PROPOSITION 3.14. *Let G be an R-group.*

(1) *If G is a faithful MR group, then R is a commutative near-ring.*
(2) *If G is a faithful distributive abelian MR group, then R is a commutative ring.*

Proof. Let $a,\ b \in R$. Define a mapping $f : G \longrightarrow G$ given by $xf = xa$, for all $x \in G$. Then clearly, $f \in M(G)$. Since G is an MR group, $f \in M_R(G)$. Thus we have the equalities: $(xb)f = (xb)a = x(ba)$ and since $f \in M(G) = M_R(G)$,

$$(xb)f = (xf)b = (xa)b = x(ab).$$

Since G is a faithful R-group, these two equalities implies that $ab = ba$. Hence R is a commutative near-ring. □

From the Propositions 3.9 and 3.14, we get the following statement.

COROLLARY 3.15. *If G is an abelian faithful MR group over near-ring R, then R becomes a commutative ring.*

The following are another characterization of MR groups and GM modules.

PROPOSITION 3.16. *Let G be an R-group with the representation*

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Then $R\theta \subset$ Center of $M(G)$ if and only if G is an MR group.

Proof. We will prove the only if part. Suppose that $R\theta \subset$ Center of $M(G)$. To show that $M(G) = M_R(G)$, let $f \in M(G)$, and let $x \in G$, $r \in R$. Then from the definitions of θ and the Center of $M(G)$, we have

$$(xr)f = (xr\theta)f = x(r\theta \circ f) = x(f \circ r\theta) = (xf)r\theta = (xf)r.$$

This implies that $f \in M_R(G)$, that is, $M(G) \subset M_R(G)$. Hence G is an MR group. □

COROLLARY 3.17. *Let M be a right R-module with the right regular representation*

$$\theta : (R, +, \cdot) \longrightarrow (End(M), +, \cdot).$$

Then $\theta(R) \subset$ Center of $End(M)$ if and only if M is a GM module.

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