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STABLE MINIMAL HYPERSURFACES IN A CRITICAL POINT EQUATION

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Abstract. On a compact $n-$dimensional manifold M^n, a critical point of the total scalar curvature functional, restricted to the space of metrics with constant scalar curvature of volume 1, satisfies the critical point equation (CPE), given by $z_g = s_g'^*(f)$. It has been conjectured that a solution (g, f) of CPE is Einstein. The purpose of the present paper is to prove that every compact stable minimal hypersurface is in a certain hypersurface of M^n under an assumption that $Ker(s_g'^*) \neq 0$.

1. Introduction

Let M^n be an $n-$dimensional compact manifold and $\mathcal{M}_1$ the set of smooth Riemannian structures on M^n of volume 1. Given a metric $g \in \mathcal{M}_1$, let $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ be the total scalar curvature functional defined by

$$\mathcal{S}(g) = \int_{M^n} s_g dv_g,$$

where s_g is the scalar curvature of g and dv_g the volume form determined by the metric and orientation. Due to the resolution of Yamabe problem, we may consider the set $\mathcal{C}$ of constant scalar curvature(csc, hereafter) metrics

$$\mathcal{C} = \{g \in \mathcal{M}_1 \,|\, s_g : \text{constant}\}.$$

It has been conjectured in Conjecture A, introduced in [1] and [4], that the critical points of $\mathcal{S}$ restricted to $\mathcal{C}$ are Einstein metrics.

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The Euler-Lagrange equations for a critical point g of this restricted variational problem may be written as the following critical point equation(CPE, hereafter):

$$z_g = s_g'^*(f), \tag{1}$$

where z_g is the traceless Ricci tensor, f is a function on M^n, and

$$s_g'^*(f) = D_g df - g\Delta_g f - f r_g,$$

where r_g is the Ricci tensor.

Conjecture A implies that $z_g = 0$, or $f \in \mathrm{Ker}(s_g'^*)$. Hence, it is natural to assume that $\mathrm{Ker}(s_g'^*) \neq 0$; otherwise, the validity of Conjecture A fails, which is not true in some cases. Therefore, we may assume that $\mathrm{Ker}(s_g'^*) \neq 0$ *throughout the present paper.*

In this paper, under the assumption that $\mathrm{Ker}(s_g'^*) \neq 0$, we study about compact oriented stable minimal hypersurfaces of M^n, and prove the following main theorem:

Main Theorem. *Let $\varphi \in Ker(s_g'^*)$ and $\Gamma = \varphi^{-1}(0)$. Then every compact oriented stable minimal hypersurfaces of M^n should be contained in Γ.*

It was shown in [2] that the set Γ is a totally geodesic submanifold of M^n. Therefore, it follows immediately from Main Theorem that

Corollary 1. *Every compact oriented stable minimal hypersurface of M^n is totally geodesic* [3].

Remark 1. In view of the following two remarks, we may conclude that our main theorem will be useful in understanding the topology of M^n and the structure of Γ:

(i) For $n \leq 7$, it is well known that each element in $H_{n-1}(M^n, \mathrm{Z})$ can be represented by sums of embedded compact oriented stable minimal hypersurfaces [6], p.51. Therefore, the informations about the topology of M^n for $n \leq 7$ may be obtained by studying such hypersurfaces.

(ii) For $n = 3$, it was proved in [5] that $H_2(M^3, \mathrm{Z}) = 0$ if and only if Γ is connected and that M^3 is diffeomorphic to S^3 in this case; in fact, this theorem gives a relationship between $H_2(M^3, \mathrm{Z})$ and the submanifold Γ.

2. The proof of main theorem

This section is devoted to the proof of the Main Theorem. Let f be a solution of CPE (1), $\varphi \in \mathrm{Ker}(s_g'^*)$, and $\Gamma = \varphi^{-1}(0) = \{x \in M^n | \varphi(x) =$

$0\}$. Also let Σ be a compact oriented stable minimal hypersurface of M^n. Then our Main Theorem may be restated as $\Sigma \subset \Gamma$.

Now, assume that Σ is not contained in Γ. Our Main Theorem will be proved by showing that this assumption leads to a contradiction. Under the assumption, it will be proved after the following two Lemmas.

LEMMA 2. *The oriented stable minimal hypersurface Σ is properly contained in M_0, where $M_0 = \{x \in M^n | f(x) < -1\}$. In other words, $f < -1$ on Σ.*

PROOF. Consider the following three cases, as in the proof of the Main Theorem of [4]:

CASE A. $\Sigma \subset M_0 \cup \partial M_0$.
CASE B. $\Sigma \subset (M^n \setminus M_0)$.
CASE C. $\Sigma \cap M_0 \neq \phi$ and $\Sigma \cap (M^n \setminus (M_0 \cup \partial M_0)) \neq \phi$.

Using the stability condition and co-area formula, it may be easily shown that the last two cases do not occur(Refer to [5] for the detailed proof). Therefore, the only possible remaining case is Case A. Hence, our Lemma is proved. □

The proof of the following Lemma is essentially same with the Contention 1 in the proof of Lemma 3 of [5], except that the dimension is not restricted to $n = 3$.

LEMMA 3. *We have $\int_\Sigma \varphi = 0$.*

PROOF. Under our assumption, the Laplacian Δ_g and the intrinsic Laplacian Δ_Σ on the minimal hypersurface Σ are related by

$$\Delta_g \varphi = \Delta_\Sigma \varphi + D_g d\varphi(\nu, \nu), \tag{2}$$

where ν is a normal vector field on Σ. On the other hand, the equation $s_g'^*(\varphi) = 0$ is equivalent to

$$0 = D_g d\varphi - (\Delta_g \varphi) g - \varphi r_g \tag{3}$$

with $\Delta_g \varphi = -\frac{s_g}{n-1}\varphi$, from which we have

$$D_g d\varphi(\nu, \nu) = \varphi r_g(\nu, \nu) + \Delta_g \varphi. \tag{4}$$

Hence, substitution of (4) into (2) gives

$$\varphi r_g(\nu, \nu) = -\Delta_\Sigma \varphi. \tag{5}$$

Replacing φ by f in (2) also gives

$$\Delta_g f = \Delta_\Sigma f + D_g df(\nu, \nu) \tag{6}$$

and

$$D_g df(\nu,\nu) = (1+f)r_g(\nu,\nu) - \frac{s_g}{n} + \Delta_g f, \tag{7}$$

since

$$r_g - \frac{s_g}{n} = D_g df - g\Delta_g f - f r_g \tag{8}$$

in virtue of (1). Thus, substitution of (7) into (6) gives

$$h r_g(\nu,\nu) = -\Delta_\Sigma f + \frac{s_g}{n}, \tag{9}$$

where $h = 1 + f$. In virtue of (5) and (9), we have

$$\int_\Sigma h\Delta_\Sigma \varphi = -\int_\Sigma \varphi h r_g(\nu,\nu) = \int_\Sigma \varphi\Delta_\Sigma f - \frac{s_g}{n}\varphi. \tag{10}$$

On the other hand, since Σ is a manifold without boundary, it follows from the Green's theorem and Stoke's theorem that

$$\int_\Sigma h\Delta_\Sigma\varphi - \varphi\Delta_\Sigma f = \int_\Sigma div_\Sigma(hd\varphi) - div_\Sigma(\varphi df) = 0.$$

Hence, (10) may be reduced to the following equation, proving our Lemma:

$$\frac{s_g}{n}\int_\Sigma \varphi = 0. \tag{11}$$

□

Now, we are ready to prove our Main Theorem.

PROOF OF MAIN THEOREM. Assume that Σ is not contained in Γ. Then in virtue of Lemma 3, φ has positive and negative values on Σ. Let Σ_+ be defined by

$$\Sigma_+ = \{x \in \Sigma \mid \varphi(x) > 0\}.$$

Then our assumption implies that Σ_+ is not empty. Let ν be a tangent vector of Σ which is an outward normal vector field along $\partial\Sigma_+$. It is clear from the definition of Σ_+ that we have $\nu(\varphi) < 0$ along $\partial\Sigma_+$.

On the other hand, it follows from (5) and (9) that

$$-h\Delta_\Sigma\varphi = \varphi h r_g(\nu,\nu) = -\varphi\Delta_\Sigma h + \frac{s_g}{n}\varphi. \tag{12}$$

Hence integration over Σ_+ gives

$$-\int_{\Sigma_+} h\Delta_\Sigma\varphi = -\int_{\Sigma_+}\varphi\Delta_\Sigma h + \frac{s_g}{n}\int_{\Sigma_+}\varphi$$

with

$$\int_{\Sigma_+} h\Delta_\Sigma\varphi = \int_{\Sigma_+} div_\Sigma(hd\varphi) - g_\Sigma(dh, d\varphi)$$
$$= \int_{\partial\Sigma_+} h\nu(\varphi) - \int_{\Sigma_+} g_\Sigma(dh, d\varphi)$$

and

$$\int_{\Sigma_+} \varphi\Delta_\Sigma h = \int_{\Sigma_+} div_\Sigma(\varphi dh) - g_\Sigma(d\varphi, dh) = -\int_{\Sigma_+} g_\Sigma(d\varphi, dh).$$

Consequently,

$$-\int_{\partial\Sigma_+} h\nu(\varphi) = \frac{s_g}{n}\int_{\Sigma_+} \varphi. \tag{13}$$

The right-hand side of (13) is positive in virtue of the definition of Σ_+, while the left-hand side of (13) is negative since $h < 0$ on Σ in virtue of Lemma 3 and $\nu(\varphi) < 0$ on $\partial\Sigma_+$ in virtue of the discussion of the previous paragraph. Hence, the assumption that Σ is not contained in Γ leads to a contradiction (13), completing the proof of our Main Theorem.

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