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FINITE ORTHOGONAL POLYNOMIALS SATISFYING A SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT. The orthogonality of polynomials plays an important role in many areas and in many cases only finite orthogonalities are used. Concerning this fact we find characterizations of a finite orthogonal polynomial system satisfying a second order differential equation and then give several examples.

1. Introduction

Consider a second order Sturm-Liouville differential equation of the form

$$L[y](x) := \ell_2(x)y''(x) + \ell_1(x)y'(x) = \lambda y(x), \tag{1.1}$$

where ℓ_i's are analytic functions on the real line $\mathbb{R}$ and λ is the eigenvalue parameter. It is well known that if the differential equation (1.1) has a polynomial system as solutions, then $\ell_2(x) = \ell_{22}x^2 + \ell_{21}x + \ell_{20}$ and $\ell_1(x) = \ell_{11}x + \ell_{10}$ must be polynomials of degree ≤ 2 and ≤ 1, respectively, and the eigenvalue $\lambda = \lambda_n = \ell_{22}n(n-1) + \ell_{11}n$.

In 1929, Bochner[1] showed that there exist, up to a linear change of variable, only five polynomial systems satisfying the differential equation of the form (1.1). They are (see [7]) :

- Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$;
- Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$;
- Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$;
- $\{x^n\}_{n=0}^{\infty}$;
- Bessel polynomials $\{B_n^{(\alpha)}(x)\}_{n=0}^{\infty}$.

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The orthogonality of polynomials plays a very important role in many areas such as numerical analysis, approximation theory, dynamics, computer science and so on. Therefore there are many results on orthogonal polynomial systems, in particular, which satisfy the differential equation (1.1). We refer to [2, 4, 5, 6, 8, 9].

Concerning the fact that only finite orthogonalities of polynomials are used in many cases, we find characterizations of a finite orthogonal polynomial system satisfying a second order differential equation (1.1) and then give several examples.

2. Main Theorems

All polynomials are assumed to be real polynomials in the real variable x. We denote the degree of polynomial ϕ by $\deg(\phi)$ with convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ with $\deg(P_n) = n$, $n \geq 0$. Note that any PS forms a basis of the space of all polynomials. Any linear functional on the space of all polynomials is called a moment functional. For a moment functional σ, we denote its action on a polynomial ϕ by

$$\langle \sigma, \phi \rangle$$

and call $\{\sigma_n := \langle \sigma, x^n \rangle\}_{n=0}^{\infty}$ the moments of σ.

Definition 2.1. A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called a finite Tchebychev polynomial system (FTPS) (a finite orthogonal polynomial system (FOPS)) of order m_0, $0 \leq m_0 < \infty$, if there exists a non-zero moment functional σ such that

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn}, \quad 0 \leq n,\, m \leq m_0, \tag{2.1}$$

where K_n is a non-zero (positive) real constant and δ_{mn} is the Kronecker delta function. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is a FTPS (FOPS) of order m_0 relative to σ and σ a finitely orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

Any PS may be considered as a FTPS of order 0 and any Tchebychev PS (orthogonal PS) as a FTPS (FOPS) of order infinite. If K_n in (2.1) is allowed to be zero, then the PS $\{P_n(x)\}_{n=0}^{\infty}$ is called a finite weak TPS of order m_0 and we define a weak TPS (WTPS) if the order of orthogonality m_0 is infinite.

For a moment functional σ and a polynomial ψ, we let σ', $\psi\sigma$ be the moment functionals defined by

$$\langle \sigma', \phi \rangle = -\langle \sigma, \phi' \rangle \quad \text{and} \quad \langle \psi\sigma, \phi \rangle = \langle \sigma, \psi\phi \rangle,$$

for any polynomial ϕ.

LEMMA 2.1. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a FTPS of order* m_0 *relative to* σ *and* ψ *be a polynomial of degree* $\deg(\psi) \le m_0$. *Then we have*

(a) $\sigma' = 0$ *if and only if* $\sigma = 0$,
(b) $\psi\sigma = 0$ *if and only if* $\psi \equiv 0$.

PROOF. (a) If $\sigma' = 0$, then $\langle \sigma', x^n \rangle = -\langle \sigma, nx^{n-1} \rangle = 0$, $n \ge 0$. Hence, $\sigma = 0$ since $\{x^{n-1}\}_{n=1}^{\infty}$ is also a PS. The converse is trivial.

(b) Assume that $\psi\sigma = 0$ but $\psi \not\equiv 0$. Then, we may write ψ as $\psi(x) = \sum_{j=0}^{k} c_j P_j(x)$, where c_j are constants with $c_k \neq 0$ and $0 \le k = deg(\psi) \le m_0$. Acting $\psi\sigma$ on $P_k(x)$, we have

$$0 = \langle \psi\sigma, P_k \rangle = \sum_{j=0}^{k} c_j \langle \sigma, P_j P_k \rangle = c_k \langle \sigma, P_k^2 \rangle.$$

Since $\langle \sigma, P_k^2 \rangle \neq 0$, we have $c_k = 0$ which is a contradiction. Hence, we have that $\psi \equiv 0$. The converse is trivial. □

THEOREM 2.2. *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be a FOPS order* m_0. *Then the zeros of* P_n, $1 \le n \le m_0$, *are all real and simple.*

PROOF. Let π be a polynomial of degree $\le m_0$. Then we may write $\pi(x) = \sum_{j=0}^{\deg(\pi)} c_j P_j(x)$ and so

$$\langle \sigma, \pi^2 \rangle = \sum_{j=0}^{\deg(\pi)} c_j^2 \langle \sigma, P_j^2 \rangle > 0.$$

Note that every polynomial π which is nonnegative on the real line $\mathbb{R}$ can be written by $\pi = p^2 + q^2$, where p and q are polynomials with $\deg(p^2) \le \deg(\pi)$ and $\deg(q^2) \le \deg(\pi)$. Hence, $\langle \sigma, \pi \rangle > 0$ for every polynomial π which is nonnegative on $\mathbb{R}$ with $\deg(\pi) \le m_0$. Since $\langle \sigma, P_n \rangle = 0$, P_n has at least one real zero. Now assume that P_n has k real zeros, say $x_1, x_2, \cdots, x_k$, with odd multiplicity. Then $(x - x_1)(x - x_2)\cdots(x - x_k)P_n(x)$ is nonnegative on $\mathbb{R}$ so that

$$\langle \sigma, (x - x_1)(x - x_2)\cdots(x - x_k)P_n(x) \rangle > 0.$$

On the other hand, if $k < n$, then by the orthogonality, $\langle \sigma, (x - x_1)(x - x_2)\cdots(x - x_k)P_n(x) \rangle = 0$, which is a contradiction. Hence, $k = n$ and so P_n has n real zeros with odd multiplicity. Since $\deg(P_n) = n$, all the zeros of P_n should be real and simple. □

Any PS $\{P_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ, called a canonical moment functional of the PS $\{P_n(x)\}_{n=0}^{\infty}$, uniquely up to a non-zero constant multiple by the conditions

$$\langle \sigma, P_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, P_n \rangle = 0, \quad n \geq 1.$$

Note that if $\{P_n(x)\}_{n=0}^{\infty}$ is a Tchebychev PS relative to σ, then σ is a canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

LEMMA 2.3. *If the differential equation* (1.1) *has a PS* $\{P_n(x)\}_{n=0}^{\infty}$ *of solutions, then any canonical moment functional* σ *of* $\{P_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$(\ell_2 \sigma)' = \ell_1 \sigma. \tag{2.2}$$

PROOF. See Theorem 2.6 and Remark 2.4 in [7]. □

LEMMA 2.4. *Let* $L[\cdot]$ *be the differential operator in* (1.1). *If* $L[p] = \lambda p$ *and* $L[q] = \mu q$ *for some polynomials* p *and* q *with* $\lambda \neq \mu$, *then* $\langle \sigma, pq \rangle = 0$ *for any solution* σ *of the moment equation* (2.2).

PROOF. Since σ satisfies the moment equation (2.2), we have by Lemma 2.3,

$$\begin{aligned}
(\lambda - \mu)\langle \sigma, pq \rangle &= \langle \sigma, L[p]q - pL[q] \rangle \\
&= \langle (\ell_2 q\sigma)'' - (\ell_1 q\sigma)', p \rangle - \langle \sigma, pL[q] \rangle \\
&= \langle (q'(\ell_2 \sigma))', p \rangle - \langle \sigma, pL[q] \rangle \\
&= \langle \sigma, pL[q] \rangle - \langle \sigma, pL[q] \rangle \\
&= 0.
\end{aligned}$$

Hence, we have that $\langle \sigma, pq \rangle = 0$ since $\lambda \neq \mu$. □

THEOREM 2.5. *If the differential equation* (1.1) *has a FTPS* $\{P_n(x)\}_{n=0}^{\infty}$ *of order* m_0 *as solutions, then* $\lambda_n \neq 0$, $1 \leq n \leq m_0 - 1$.

PROOF. Assume that the differential equation (1.1) has a FTPS $\{P_n(x)\}_{n=0}^{\infty}$ of order m_0 as solutions and let σ be a finitely orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. By Lemma 2.3, σ satisfies the equation (2.2). Suppose that $\lambda_n = 0$ for some $1 \leq n \leq m_0 - 1$. Then we have

$$\begin{aligned}(\ell_2 P_n'\sigma)' &= P_n''\ell_2\sigma + P_n'(\ell_2\sigma)' \\ &= \ell_2 P_n''\sigma + \ell_1 P_n'\sigma \\ &= \lambda_n P_n \sigma \\ &= 0.\end{aligned}$$

Hence, $\ell_2 P_n'\sigma = 0$ by Lemma 2.1 (a) and so we have $\ell_2 P_n' \equiv 0$ by $\deg(\ell_2 P_n') \le n+1 \le m_0$ and Lemma 2.1 (b). Since $\ell_2 \not\equiv 0$, $P_n' \equiv 0$ and so $n = 0$ which is impossible. □

Note that if $\deg(\ell_2) \le 1$ in (1.1), then the eigenvalue $\lambda_n \neq 0$ for $n \ge 1$. Note also that if $\{P_n(x)\}_{n=0}^\infty$ satisfies the differential equation (1.1), then $\{P_n^{(r)}(x)\}_{n=r}^\infty$ also satisfies the following differential equation

$$\ell_2(x)y'' + [r\ell_2'(x) + \ell_1(x)]y' = [\lambda_n - r\ell_{11} - r(r-1)\ell_{22}]y.$$

The orthogonality of a PS satisfying a differential equation of the form (1.1) is also preserved by the differentiation. More precisely, we have

THEOREM 2.6. *If $\{P_n(x)\}_{n=0}^\infty$ is a FTPS of order m_0 satisfying the differential equation* (1.1), *then $\{P_n'(x)\}_{n=1}^\infty$ is a FTPS of order $m_0 - 1$.*

PROOF. Let σ be the canonical moment functional for $\{P_n(x)\}_{n=0}^\infty$ so that σ satisfies the functional equation (2.2). Thus,

$$\begin{aligned}\langle \ell_2\sigma, P_m'P_n'\rangle &= -\langle (\ell_2 P_n'\sigma)', P_m\rangle \\ &= -\langle (\ell_2\sigma)'P_n' + \ell_2\sigma P_n'', P_m\rangle \\ &= -\langle (\ell_1 P_n' + \ell_2 P_n'')\sigma, P_m\rangle \qquad (2.3) \\ &= -\langle L[P_n]\sigma, P_m\rangle \\ &= -\lambda_n\langle \sigma, P_n P_m\rangle.\end{aligned}$$

Since $\lambda_n \neq 0$ for $1 \le n \le m_0 - 1$ and $\langle \sigma, P_n^2\rangle \neq 0$ for $0 \le n \le m_0$, $\{P_n'(x)\}_{n=1}^\infty$ is a FTPS of order $m_0 - 1$ relative to $\ell_2\sigma$. □

By Theorem 2.6, we can inductively see that if $\{P_n(x)\}_{n=0}^\infty$ is a FTPS of order m_0 satisfying the differential equation (1.1), then $\{P_n^{(r)}(x)\}_{n=r}^\infty$, $0 \le r \le m_0 - 1$, is a FTPS of order $m_0 - r$.

From here on, $\{P_n(x)\}_{n=0}^\infty$ is always assumed to be a monic PS, that is,

$$P_n(x) = x^n + \text{lower degree terms}, \qquad n \ge 0.$$

Now, assume that a PS $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation

$$P_{n+1}(x) = (x - b_n)P_n(x) - c_nP_{n-1}(x), \quad n \geq 1, \tag{2.4}$$

where b_n and c_n are constants.

It is well known that $\{P_n(x)\}_{n=0}^{\infty}$ is a Tchebychev PS (orthogonal PS) if and only if $c_n \neq 0$ ($c_n > 0$), $n \geq 1$. See [2, 3, 12] for examples. Here we prove a more generalized result.

THEOREM 2.7. *If $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation of the form* (2.4), *then $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS.*

PROOF. Let σ be the canonical moment functional of $\{P_n(x)\}_{n=0}^{\infty}$. Then, by definition, $\langle \sigma, P_n \rangle = 0$, $n \geq 1$. Acting σ on (2.4), we have that $\langle \sigma, xP_n \rangle = 0$, $n \geq 2$. Multiplying x on (2.4) and then acting σ again, we have that $\langle \sigma, x^2P_n \rangle = 0$, $n \geq 3$. Continuing the same process, we have

$$\langle \sigma, x^kP_n \rangle = 0, \quad 0 \leq k < n$$

so that $\{P_n(x)\}_{n=0}^{\infty}$ is a WTPS relative to σ. □

Suppose that $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation (2.4). Then we can easily see by the orthogonality that

$$\langle \sigma, P_n^2 \rangle = c_n \langle \sigma, P_{n-1}^2 \rangle, \quad n = 1, 2, \cdots .$$

If N is the smallest integer such that $c_N = 0$, we obtain inductively that

$$\langle \sigma, P_n^2 \rangle = 0, \quad n \geq N$$

and

$$\langle \sigma, P_n^2 \rangle = c_nc_{n-1} \cdots c_1 \langle \sigma, 1 \rangle, \quad n = 1, 2, \cdots, N-1. \tag{2.5}$$

More precisely, we can prove

THEOREM 2.8. *Let $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the three term recurrence relation* (2.4). *Then $\{P_n(x)\}_{n=0}^{\infty}$ is a FTPS(FOPS) of order m_0 if and only if $c_n \neq 0$ ($c_n > 0$), $1 \leq n \leq m_0$.*

PROOF. Since $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the three term recurrence relation (2.4), it is a WTPS. Since $\{P_n(x)\}_{n=0}^{\infty}$ is a FTPS of order m_0, $\langle \sigma, P_{m_0}^2 \rangle \neq 0$. By the equation (2.5), $c_n \neq 0$ for $1 \leq n \leq m_0$. By the equation (2.5) again, $c_n \neq 0$ for $1 \leq n \leq m_0$ implies $\langle \sigma, P_n^2 \rangle \neq 0$ for $1 \leq n \leq m_0$. Since $\langle \sigma, P_0^2 \rangle \neq 0$ trivially, the converse was proved. □

By Theorem 2.8, we can conclude the following: Let $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the three term recurrence relation (2.4). Let $m_0 \geq 1$ be the smallest integer such that $\langle \sigma, P_{m_0}^2 \rangle = 0$ and $N \geq 1$ be the smallest integer such that $c_N = 0$. Then $m_0 = N$.

THEOREM 2.9. *Let $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the three term recurrence relation* (2.4) *and let $N \geq 1$ be the smallest integer such that $c_N = 0$. If $\{P_n(x)\}_{n=0}^{\infty}$ is a solution of a second order differential equation* (1.1), *then $P_N(x)$ must be of the form*

$$P_N(x) = (x - x_1)^m (x - x_2)^{N-m}, \quad x_1, x_2 \in \mathbb{C}, \tag{2.6}$$

where x_1 and x_2 are the (complex) *zeros of ℓ_2, and m is a non-negative integer.*

PROOF. Since $c_N = 0$, we have $P_{N+1}(x) = (x - b_N)P_N(x)$. Since P_{N+1} satisfies the differential equation (1.1), we obtain

$$2\ell_2(x)P_N'(x) = [(\lambda_{N+1} - \lambda_N)(x - b_N) - \ell_1(x)]P_N(x) \tag{2.7}$$

so that $\ell_2 P_N'$ must P_N as a factor. Note that if z_0 is a (complex) zero of P_N with multiplicity m, then it is a zero of P_N' with multiplicity $m-1$. By the equation (2.7), ℓ_2 must have all zeros of P_N which can have at most two zeros. Hence, P_N have at most two distinct zeros, which can be written by the form (2.6). □

Note that if $c_N = 0$ in the three term recurrence relation, we can inductively see that P_{N+k}, $k \geq 0$, has P_N as a factor. Let's write $P_{N+k}(x) = P_N(x)q_k(x)$, where q_k is a monic polynomial with $\deg(q_k) = k$. If we assume moreover that $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation (1.1), then q_k also satisfies a differential equation

$$\ell_2(x)q_k''(x) + \tilde{\ell}_1(x)q_k'(x) = \mu_k q_k(x),$$

where $\mu_k = \lambda_{N+k} - \lambda_N$ and $\tilde{\ell}_1$ is a polynomial such that $2\ell_2 P_N' = \tilde{\ell}_1 P_N$.

3. Examples

EXAMPLE 3.1. Consider a Laguerre type differential equation of form

$$xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x). \tag{3.1}$$

It is well known that the equation (3.1) has a unique monic polynomial solution

$$L_n^{(\alpha)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad n \geq 0,$$

where $\binom{\alpha}{\beta} = 0$ if α or $\beta < 0$, or $\alpha < \beta$. It is easy to see that $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation

$$L_{n+1}^{(\alpha)}(x) = (x - b_n)L_n^{(\alpha)}(x) - c_n L_n^{(\alpha)}(x),$$

where $b_n = (-1)^n n!(n + \alpha + (n+1)(n+1+\alpha))$ and $c_n = n(n+\alpha)$. The case $\alpha = 0$ is the one originally studied by Laguerre[9]. The case $\alpha > -1$ is due to Sonine[11] and the generalized Laguerre PS for $\alpha < -1$ and $-\alpha \notin \mathbb{N}$ is due to Morton and Krall[10]. By the result in section 2, we have shown that for $-\alpha \in \mathbb{N}$, $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is a FTPS but never a FOPS of order $m_0 = -\alpha - 1$.

EXAMPLE 3.2. Consider a Jacobi type differential equation of the form

$$(3.2)\quad (1-x^2)y''(x)+[(\beta-\alpha)-(\alpha+\beta+2)x]y'(x) = -n(n+\alpha+\beta+1)y(x).$$

It is well known that if $\alpha + \beta \notin \{-1, -2, \cdots\}$, then the equation (3.2) has a unique monic polynomial solution

$$P_n^{(\alpha,\beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k,\ n \geq 0.$$

It is easy to see that $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation of the form (2.4). As a special case, if $\alpha = -2$ and $\beta \notin \{-1, -2, \cdots\}$,

$$P_{n+1}^{(-2,\beta)}(x) = (x - b_n)P_n^{(-2,\beta)}(x) - c_n P_{n-1}^{(-2,\beta)}(x),$$

where

$$b_n = \frac{\beta^2 - 4}{(2n-2+\beta)(2n+\beta)}, \quad n \geq 1,$$

and

$$c_n = \frac{4n(n-2)(n+\beta)(n-2+\beta)}{(2n-2+\beta)^2(2n-1+\beta)(2n-3+\beta)}, \quad n \geq 1.$$

Hence, we have $c_1 = -\frac{4}{\beta^2} < 0$ and $c_2 = 0$. By the result in section 2, we have shown that $\{P_n^{(-2,\beta)}(x)\}_{n=0}^{\infty}$ is a FTPS but never a FOPS of order $m_0 = 1$.

EXAMPLE 3.3. Consider a Jacobi type differential equation of the form

$$(3.3)\qquad (1-x^2)y''(x) + (x+1)y'(x) = -n(n-2)y(x),$$

which is a special case of Jacobi polynomials with $\alpha = -2$, $\beta = -1$. It can be shown that the equation (3.3) has monic polynomial solutions

$$P_n^{(-2,-1)}(x) = \binom{2n-3}{n}^{-1} \sum_{k=0}^{n} \binom{n-2}{k} \binom{n-1}{n-k} (x-1)^{n-k}(x+1)^k \; n \neq 2,$$

and $P_2^{(-2,-1)}(x) = x^2 - 2x + \gamma$, where γ is a constant. It is easy to see that $\{P_n^{(-2,-1)}(x)\}_{n=0}^{\infty}$ satisfies a three term recurrence relation

$$P_{n+1}^{(-2,-1)}(x) = (x - b_n)P_n^{(-2,-1)}(x) - c_n P_{n-1}^{(-2,-1)}(x),$$

where $b_1 = 3$, $c_1 = -(\gamma + 3)$, $b_2 = -1$, $c_2 = \gamma - 1$,

$$b_n = \frac{-3}{(2n-3)(2n-1)}, \quad n \geq 3$$

and

$$c_n = \frac{n(n-3)}{(2n-3)^2}, \quad n \geq 3.$$

The smallest integer $N \geq 1$ such that $c_N = 0$ is 1 when $\gamma = -3$, 2 when $\gamma = 1$, and 3 when $\gamma \neq 1$. Hence, $\{P_n^{(-2,-1)}(x)\}_{n=0}^{\infty}$ is a FTPS but never a FOPS of order m_0 depending on γ. More precisely, $m_0 = 0$ if $\gamma = -3$, $m_0 = 1$ is $\gamma = 1$, and $m_0 = 2$ if $\gamma \neq 1, -3$.

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