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THE GLOBAL ATTRACTOR OF THE 2D G-NAVIER-STOKES EQUATIONS ON SOME UNBOUNDED DOMAINS

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Abstract. In this paper, we study the two dimensional g-Navier-Stokes equations on some unbounded domain $\Omega \subset R^2$. We prove the existence of the global attractor for the two dimensional g-Navier-Stokes equations under suitable conditions. Also, we estimate the dimension of the global attractor. For this purpose, we exploit the concept of asymptotic compactness used by Rosa for the usual Navier-Stokes equations.

1. Introduction

We study the existence of a global attractor of the g-Navier-Stokes equations on some unbounded domain $\Omega \subset R^2$ which has the following form,

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.1}$$

$$\frac{1}{g}\nabla \cdot (gu) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{1.2}$$

where $g = g(x_1, x_2)$ is a suitable real-valued smooth function. Here, ν and f are given and the velocity u and the pressure p are the unknowns. When $g = 1$, the equations (1.1)–(1.2) become the usual two dimensional Navier-Stokes equations. The motivation of g-Navier-Stokes equations arose from the study of usual Navier-Stokes equations on 3-dimensional thin domains. Among many results about these equations (for example, see [11], [14], [15], [16] and [17]) on the thin domain, we focus on Raugel and Sell's work. Raugel and Sell considered the Navier-Stokes

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equations on a three dimensional thin domain $\Omega_\epsilon = \Omega \times (0, \epsilon)$, for the purely Periodic boundary conditions and the Periodic-Dirichlet boundary conditions, that is, periodic conditions in the thin vertical direction and homogeneous Dirichlet conditions on the lateral boundary condition $\Gamma_l = \partial\Omega \times (0, \epsilon)$, where $\Omega \subset R^2$.

As in [10], an essential tool in their proof is the vertical mean operator M which allows the decomposition of every function $\mathbf{U}$ on $\Omega_\epsilon = \Omega \times (0, \epsilon)$ into the sum of a function $M\mathbf{U} = \mathbf{v}(x_1, x_2)$ which does not depend on the vertical variable and a function $(I - M)\mathbf{U} = \mathbf{w}(x_1, x_2, x_3)$ with vanishing vertical mean and thus to use more precise Sobolev and Poincaré inequalities. Then, they showed that the reduced 3D Navier-Stokes evolutionary equations by $\mathbf{v}$ incorporates the 2D navier-Stokes equations on Ω.

Later, by using the same tool as Raugel and Sell with improved Agmon inequalities, Temam and Ziane([24], [25]) generalized the results of [16] and [17] to other boundary conditions and, in the case of the free boundary conditions, to thin spherical domains. In [19], Roh applied same method for the domain $\Omega_g = \Omega_2 \times (0, g)$, where Ω_2 is a bounded region in the plane and $g = g(y_1, y_2)$ is a smooth function defined on Ω_2 with $g(y_1, y_2) > 0$ for $(y_1, y_2) \in \Omega_2$. Now, we consider the $3D$ Navier-Stokes equations,

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} - \nu\Delta\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla\Phi = \mathbf{F} \ \text{ in } \ \Omega_g, \\ \nabla\cdot\mathbf{U} = 0 \ \text{ in } \ \Omega_g, \end{aligned}$$

with the boundary condition

$$\mathbf{U}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial_t\Omega_g \cup \partial_b\Omega_g, \tag{1.3}$$

where

$$\begin{aligned} \partial_t\Omega_g &= \{(y_1, y_2, y_3) \in \Omega_g \ : \ y_3 = g(y_1, y_2)\}, \\ \partial_b\Omega_g &= \{(y_1, y_2, y_3) \in \Omega_g \ : \ y_3 = 0\}. \end{aligned}$$

One note that the lateral boundary conditions corresponding to $\partial\Omega_2$ were not used for the derivation of the g-Navier-Stokes equations.

Next, we define a vector field $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ given by

$$\begin{aligned} \mathbf{v}_i(x_1, x_2) &= M\mathbf{U}_i \\ &= \frac{1}{g(x_1, x_2)} \int_0^{g(x_1,x_2)} \mathbf{U}_i(x_1, x_2, y_3)\, dy_3, \quad \text{for } \ i = 1, 2, 3. \end{aligned}$$

Then, Roh[18] showed that $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) = (\mathbf{v}_1, \mathbf{v}_2)$ satisfies

$$\nabla \cdot (g\mathbf{u}) = \frac{\partial(g\mathbf{u}_1)}{\partial x_1} + \frac{\partial(g\mathbf{u}_2)}{\partial x_2} = \nabla g \cdot \mathbf{u} + g\ (\nabla \cdot \mathbf{u}) = 0 \ \text{ in } \Omega_2,$$

where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$.

This theorem means that good properties of the 2D g-Navier-Stokes equations can lead to initiate the study of the Navier-Stokes equations on thin three dimensional domain Ω_g, because the reduced 3D Navier-Stokes evolutionary equations by

$$\mathbf{v}(x_1, x_2, x_3) = \frac{1}{g(x_1, x_2)} \int_0^{g(x_1,x_2)} \mathbf{U}(x_1, x_2, x_3)\ dx_3,$$

can incorporate with the 2D g-Navier-Stokes equations. This is the basis for our study of the 2D g-Navier-Stokes equations.

In [18], Roh derived the g-Navier-Stokes equations to study the 3D Navier-Stokes equations on thin domain $\Omega_g = \Omega_2 \times (0, g)$, where Ω_2 is bounded domain and many good properties of the 2D g-Navier-Stokes equations were proved in [18], [19].

But we note that the derivation of the g-Navier-Stokes equations can be generalized to the unbounded domain(see [18]). So, in [5], Bae and Roh proved the existence of the solutions on the whole space R^2.

In this paper, for the restricted unbounded domain, we prove the existence of the global attractor of the 2D g-Navier-Stokes equations and estimate the dimension of the global attractor.

For the Navier-Stokes equations, the global attractor was first obtained for bounded domains in the works of Ladyzhenskaya[12] and Foias and Temam[9]. Then, the latter work showed the finite dimensionality of the attractor in the sense of the Hausdorff dimension(see [6], [7] and [22]).

The existence of the global attractor for dissipative evolution equations has relied on some kind of compactness of the semigroup generated by such equation together with the compact imbedding of the relevant Sobolev spaces. This approach is suitable only for bounded domain since the Sobolev imbedding are no longer compact otherwise. Nevertheless, the existence of the global attractor for unbounded domain was obtained by Abergel[1], Babin[3], Babin and Vishik[4] considering weighted spaces. But in their work, the forcing term and in some cases even the initial condition had to be restricted to the weighted spaces.

However, Rosa[20] proved that for suitable unbounded domains, the semigroup generated by solutions of the Navier-Stokes equations has a global attractor $\mathcal{A}$ when the external forcing term f even lies in V'.

For his result, Rosa[20] exploited the energy equation to prove that the semigroup becomes the asymptotic compactness which was already used by Abergel([1], [2]) and by Ladyzhenskaya[13].

In this paper, we prove the existence of a global attractor of 2D g-Navier-Stokes equation using the same argument. Also, we prove the finite dimensionality of the global attractor.

2. Preliminaries

We consider the flow of fluid enclosed in a region $\Omega \subset R^2$ with rigid boundary $\partial\Omega$ and governed by the g-Navier-Stokes equations. We denote by $u(x,t) \in R^2$ and $p(x,t) \in R$, respectively, the velocity and the pressure of initial-boundary value problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f \quad \text{in } \Omega, \\ \dfrac{1}{g}\nabla\cdot(gu) = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ u(\cdot,0) = u_0 \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\nu > 0$ is the kinematic viscosity of the fluid, $f = f(x,t) \in R^2$ is the external body force (assumed to be time independent), and the function $g = g(x_1,x_2)$ is positive real-valued smooth function. We assume that the function g satisfies

$$0 < m_0 \le g(x_1,x_2) \le M_0 \quad \text{for all} \quad (x_1,x_2) \in \Omega$$

for some constants m_0 and M_0.

The domain Ω can be an arbitrary unbounded open set in R^2 without any regularity assumption on its boundary $\partial\Omega$. But, we assume that the Poincaré inequality holds on Ω. More precisely, we assume only the following : There exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 g dx \le \frac{1}{\lambda_1}\int_\Omega |\nabla\phi|^2 g dx, \quad \forall\ \phi \in H_0^1(\Omega). \tag{2.2}$$

The mathematical frameworks of (2.1) is now following : first we let $\mathbb{L}^2(g) = (L^2(\Omega))^2$ and $\mathbb{H}_0^1(g) = (H_0^1(\Omega))^2$ endowed, respectively, with the inner products

$$(u,v) = \int_\Omega u\cdot v\ g dx, \quad u,v \in \mathbb{L}^2(g),$$

and

$$((u,v)) = \int_\Omega \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j \; gdx, \quad u=(u_1,u_2),\ v=(v_1,v_2) \in \mathbb{H}_0^1(g),$$

and norms $|\cdot| = (\cdot,\cdot)^{1/2}$, $||\cdot|| = ((\cdot,\cdot))^{1/2}$. Note that thanks to (2.2) the norm $||\cdot||$ is equivalent to the usual one in $\mathbb{H}_0^1(\Omega)$. Let $\mathcal{D}(\Omega)$ be the space of $\mathcal{C}^\infty$ functions with compact support contained in Ω and let

$$\begin{aligned} \mathcal{V} &= \{v \in (\mathcal{D}(\Omega))^2 : \nabla \cdot gv = 0 \ \text{ in } \Omega\}, \\ V &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{H}_0^1(g), \\ H &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{L}^2(g), \end{aligned}$$

with H and V endowed with the inner product and norm of , respectively, $\mathbb{L}^2(g)$ and $\mathbb{H}_0^1(g)$. It follows from (2.2) that

$$|u|^2 \le \frac{1}{\lambda_1}||u||^2, \quad \forall\, u \in V. \tag{2.3}$$

Now, we define a g-Laplacian operator as follows:

$$-\Delta_g u \equiv -\frac{1}{g}\Big(\nabla \cdot (g\nabla u)\Big) = -\Delta u - \Big(\frac{\nabla g}{g}\cdot \nabla\Big)u.$$

Using the g-Laplacian operator, we rewrite the first equation of (2.1) as follows :

$$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\Big(\frac{\nabla g}{g}\cdot\nabla\Big)u + (u\cdot\nabla)u + \nabla p = f. \tag{2.4}$$

Also, we define a g-orthogonal projection

$$P : \mathbb{L}^2(g) \to H$$

and g-Stokes operator

$$Au = -P\Big(\frac{1}{g}(\nabla \cdot (g\nabla u))\Big).$$

By applying the projection P into the equation (2.4), we then obtain the following weak formulation of (2.1) : for $f \in V'$ and $u_0 \in H$ given, find u satisfying

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V), \quad \forall\, T > 0, \tag{2.5}$$

such that

$$\begin{aligned} \frac{d}{dt}(u,v) + \nu((u,v)) + b_g(u,u,v) &+ \nu(Ru,v) \\ &= < f, v > \quad \forall v \in V,\ \forall\, t > 0, \end{aligned} \tag{2.6}$$

and

$$u(0) = u_0, \tag{2.7}$$

where $b_g : V \times V \times V \to R$ is given by

$$b_g(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, g dx, \tag{2.8}$$

and $Ru = P(\frac{\nabla g}{g} \cdot \nabla)u$ for all $u \in V$, and $< \cdot, \cdot >$ is the duality product between V' and V when we identify H with its dual. Then, the weak formulation (2.6) is equivalent to the functional equation

$$u' + \nu Au + Bu + \nu Ru = f \quad \text{in } V', \quad \text{for } t > 0, \tag{2.9}$$

where $u' = du/dt$, $A : V \to V'$ is the g-Stokes operator defined by

$$< Au, v >= ((u, v)), \quad \forall\, u, v \in V, \tag{2.10}$$

and $B(u) = B(u, u) = P(u \cdot \nabla)u$ is a bilinear operator $B : V \times V \to V'$ defined by

$$< B(u, v), w >= b_g(u, v, w), \quad \forall\, u, v, w \in V.$$

The g-Stokes operator A is an isomorphism from V into V', while B and R satisfy the following inequalities (see Sell and You[21] and Roh[19]):

$$||B(u)||_{V'} \le C\, |u|||u||, \quad ||Ru||_{V'} \le \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} ||u|| \quad \forall\, u \in V. \tag{2.11}$$

We have the following result (see Bae and Roh[5], and Temam[23]).

PROPOSITION 2.1. *Given $f \in V'$ and $u_0 \in H$, there exists a unique $u \in L^\infty(R^+; H) \cap L^2(0, T; V)$, $\forall\, T > 0$, such that (2.6) and (2.7) hold. Moreover, $u' \in L^2(0, T; V')$, $\forall\, T > 0$ and $u \in C(R^+; H)$.*

Now, let $u = u(t), t \ge 0$, be a solution given by Proposition 2.1. Since $u \in L^2(0, T; V)$ and $u' \in L^2(0, T; V')$, we obtain

$$\frac{1}{2} \frac{d}{dt} |u|^2 =< u', u > \tag{2.12}$$

and we have from (2.9) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u|^2 &= < f - \nu Au - B(u) - \nu Ru, u > \\ &= < f, u > - \nu ||u||^2 - b_g(u, u, u) - \nu \Big(\Big(\frac{\nabla g}{g} \cdot \nabla \Big) u, u \Big). \end{aligned}$$

Since $b_g(u,v,v) = 0, \forall\ u, v \in V$, we have

$$\frac{d}{dt}|u|^2 + 2\nu||u||^2 = 2 < f, u > -2\nu\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u, u\right) \tag{2.13}$$

in the distribution sense on R^+.

Moreover, using (2.3), we have

$$\frac{d}{dt}|u|^2 + 2\nu||u||^2 \le \frac{||f||^2_{V'}}{\nu} + \nu||u||^2 + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}||u||^2$$

and hence

$$\frac{d}{dt}|u|^2 + \nu\lambda_1\gamma_0|u|^2 \le \frac{d}{dt}|u|^2 + \nu\gamma_0||u||^2 \le \frac{||f||^2_{V'}}{\nu}, \tag{2.14}$$

where $\gamma_0 = 1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$ for sufficiently small $|\nabla g|_\infty$....

Therefore, from (2.14), we obtain

$$|u(t)|^2 \ \le\ |u_0|^2 e^{-\nu\lambda_1\gamma_0 t} \ +\ \frac{||f||^2_{V'}}{\nu^2\lambda_1\gamma_0}, \quad \forall\, t > 0, \tag{2.15}$$

and

$$\frac{1}{t}\int_0^t ||u(s)||^2 ds \ \le\ \frac{|u_0|^2}{t\nu\gamma_0} \ +\ \frac{||f||^2_{V'}}{\nu^2\gamma_0}, \quad \forall\, t > 0. \tag{2.16}$$

Due to Proposition 2.1, we can define a continuous semigroup $\{S(t)\}_{t\ge 0}$ in H by

$$S(t)u_0 = u(t), \quad \forall\, t > 0,$$

where $u(t)$ is a solution of (2.6) with $u(0) = u_0 \in H$.... We see also that the map $S(t) : H \to H$, for $t \ge 0$, is Lipschitz continuous on bounded subsets of H. Moreover, it follows form (2.15) that the set

$$\mathcal{B} = \left\{u \in H : |u| \le \rho_0 \equiv \frac{1}{\nu}\sqrt{\frac{2}{\lambda_1\gamma_0}}||f||_{V'}\right\} \tag{2.17}$$

is absorbing in H for the semigroup.

Then, we can also prove the following weak continuity of the semigroup $\{S(t)\}_{t\ge 0}$.

LEMMA 2.2. *Let $\{u_{0,n}\}_n$ be a sequence in H converging weakly in H to an element $u_0 \in H$. Then*

$$S(t)u_{0,n} \ \to\ S(t)u_0 \quad \text{weakly in } H, \quad \forall\, t \ge 0, \tag{2.18}$$

and

$$S(\cdot)u_{0,n} \ \to\ S(\cdot)u_0 \quad \text{weakly in } L^2(0,T;V), \quad \forall\, T > 0. \tag{2.19}$$

PROOF. Since the proof of this lemma is almost the same as in Rosa[20], we omit the proof. □

3. Existence of the global attractor

For the existence of the global attractor, we will prove the asymptotic compactness of the semigroup $\{S(t)\}_{t\geq 0}$. A semigroup is said to be asymptotically compact in a given metric space if

$$\{S(t_n)u_n\} \text{ is precompact,} \tag{3.1}$$

whenever

$$\{u_n\} \text{ is bounded and } t_n \rightarrow \infty. \tag{3.2}$$

To prove that $\{S(t)\}_{t\geq 0}$ is asymptotically compact in H, we use the energy equation (2.13).

First, we define a function $[\cdot,\cdot] : V \times V \rightarrow R$ by

$$[u,v] = \nu((u,v)) + \frac{\nu}{2}\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u,v\right) + \frac{\nu}{2}\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)v,u\right) - \frac{\nu\lambda_1}{4}(u,v) \tag{3.3}$$

for all $u, v \in V$. Then $[\cdot,\cdot]$ is bilinear and symmetric. Moreover, we have from (2.3) that

$$\begin{aligned}
[u]^2 = [u,u] &= \nu||u||^2 + \nu\left(\left(\frac{\nabla g}{g}\cdot\nabla\right)u,u\right) - \frac{\nu\lambda_1}{4}|u|^2 \\
&\geq \nu||u||^2 - \nu\left(\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} + \frac{1}{4}\right)||u||^2 \\
&\geq \frac{\nu}{2}||u||^2,
\end{aligned}$$

whenever $|\nabla g|_\infty$ is sufficiently small so that $\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} < 1/4$. Hence

$$\frac{\nu}{2}||u||^2 \leq [u]^2 \leq \frac{3}{2}\nu||u||^2, \quad \forall\, u \in V. \tag{3.4}$$

Thus $[\cdot,\cdot]$ is an inner product on V with the norm $[\cdot] = [\cdot,\cdot]^{1/2}$ equivalent to $||\cdot||$.

Now, by adding and subtracting $\nu\lambda_1|u|^2/2$ to (2.13), we obtain

$$\frac{d}{dt}|u|^2 + \frac{\nu\lambda_1}{2}|u|^2 + 2[u]^2 = 2 < f, u >, \tag{3.5}$$

for any solution $u = u(t) = S(t)u_0, u_0 \in H$. Then, by the variation of constant formula,

$$|u(t)|^2 = |u_0|^2 e^{-\nu\lambda_1 t/2} + 2\int_0^t e^{-\nu\lambda_1(t-s)/2}(< f, u(s) > -[u(s)]^2)ds,$$

which can be written

(3.6)

$$|S(t)u_0|^2 = |u_0|^2 e^{-\nu\lambda_1 t/2} + 2\int_0^t e^{-\nu\lambda_1(t-s)/2}(< f, S(s)u_0 > -[S(s)u_0]^2)ds,$$

for all $u_0 \in H$, and $t \geq 0$.

We are now to prove the following :

THEOREM 3.1. *Assume that the function g satisfies $|\nabla g|_\infty < m_0\lambda_1^{1/2}/4$ Then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically compact in H.*

PROOF. Let $B \subset H$ be a bounded set and consider sequences $\{u_n\} \subset B$ and $\{t_n\}$ such that $t_n \geq 0,\ t_n \to \infty$. Then it suffices to show that the set $\{S(t_n)u_n\}$ is precompact in H. Since the set $\mathcal{B}$ defined in (2.17) is absorbing, there exists a time $T(B) > 0$ such that

$$S(t)B \subset \mathcal{B}, \quad \forall\, t \geq T(B),$$

so that for t_n large enough ($t_n \geq T(B)$),

$$S(t_n)u_n \in \mathcal{B}. \tag{3.7}$$

Thus $\{S(t_n)u_n\}$ is weakly compact in H and hence there are subsequences, which we relabel as u_n and t_n, and an element $w \in \mathcal{B}$ such that

$$S(t_n)u_n \rightarrow w, \quad \text{weakly in } H. \tag{3.8}$$

Similarly for each $T > 0$, we also have

$$S(t_n - T)u_n \in \mathcal{B} \tag{3.9}$$

for $t_n \geq T + T(B)$. Therefore, $\{S(t_n - T)u_n\}$ is also weakly precompact in H and hence there are further subsequences, which we relabel as u_n and t_n, and an element $w_T \in \mathcal{B}$ such that

$$S(t_n - T)u_n \to w_T, \quad \text{weakly in } H, \quad \forall\, T \in \mathbb{N}. \tag{3.10}$$

Note that by the weak continuity of $S(t)$, proved in Lemma 2.2,

$$\begin{aligned} w = \lim_{n\to\infty}{}_{\mathrm{H_w}}\, S(t_n)u_n &= \lim_{n\to\infty}{}_{\mathrm{H_w}}\, S(T)S(t_n-T)u_n \\ &= S(T) \lim_{n\to\infty}{}_{\mathrm{H_w}}\, S(t_N - T)u_n \\ &= S(T)w_T, \end{aligned}$$

where $\lim_{\mathrm{H_w}}$ denotes the limit taken in the weak topology of H. Thus

$$w = S(T)w_T, \quad \forall\, T \in \mathbb{N}. \tag{3.11}$$

From (3.8), we have

$$|w| \le \liminf_{n\to\infty} |S(t_n)u_n|, \tag{3.12}$$

and we shall now prove that

$$\limsup_{n\to\infty} |S(t_n)u_n| \le |w|.$$

For $T \in \mathbb{N}$ and $t_n > T$ we have by (3.6)

$$\begin{aligned} |S(t_n)u_n|^2 &= |S(T)S(t_n-T)u_n|^2 \\ &= |S(t_n-T)u_n|^2 e^{-\nu\lambda_1 T/2} \\ &\quad + 2\int_0^T e^{-\nu\lambda_1(T-s)/2} < f, S(s)S(t_n-T)u_n > \\ &\quad - 2\int_0^T e^{-\nu\lambda_1(T-s)/2} [S(s)S(t_n-T)u_n]^2 ds. \end{aligned} \tag{3.13}$$

From (3.9), we obtain

$$\limsup_{n\to\infty} \left(e^{-\nu\lambda_1 T/2} |S(t_n-T)u_n|^2 \right) \le \rho_0 e^{-\nu\lambda_1 T/2}. \tag{3.14}$$

Also, by the weak continuity of (2.19) we deduce from (3.10) that

$$S(\cdot)S(t_n-T)u_n \rightarrow S(\cdot)w_T \quad \text{weakly in } L^2(0,T;V). \tag{3.15}$$

Then since

$$s \rightarrow e^{-\nu\lambda_1(T-s)/2} f \in L^2(0.T; V'),$$

we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T e^{-\frac{\nu\lambda_1(T-s)}{2}} < f, S(s)S(t_n-T)u_n > ds \\ &= \int_0^T e^{-\frac{\nu\lambda_1(T-s)}{2}} < f, S(s)w_T > ds. \end{aligned} \tag{3.16}$$

Moreover, since $[\cdot]$ is a norm on V equivalent $\|\cdot\|$, we see that

$$\left(\int_0^T e^{-\nu\lambda_1(T-s)/2}[\cdot]^2 ds\right)^{1/2}$$

is a norm in $L^2(0,T;V)$ equivalent to the usual norm. Therefore, from (3.15) we have

$$\begin{aligned}&\int_0^T e^{-\frac{\nu\lambda_1(T-s)}{2}}[S(s)w_T]^2 ds\\ &\leq \liminf_{n\to\infty}\int_0^T e^{-\frac{\nu\lambda_1(T-s)}{2}}[S(s)S(t_n-T)u_n]^2 ds.\end{aligned} \tag{3.17}$$

Hence

$$\begin{aligned}&\limsup_{n\to\infty}\left(-2\int_0^T e^{-\nu\lambda_1(T-s)/2}[S(s)S(t_n-T)u_n]^2 ds\right)\\ &=-2\liminf_{n\to\infty}\int_0^T e^{-\nu\lambda_1(T-s)/2}[S(s)S(t_n-T)u_n]^2 ds\\ &\leq -2\int_0^T e^{-\nu\lambda_1(T-s)/2}[S(s)w_T]^2 ds.\end{aligned} \tag{3.18}$$

We now pass to the limsup as n goes to infinity in (3.13), taking (3.14), (3.16), and (3.18) into account to obtain

(3.19)

$$\begin{aligned}&\limsup_{n\to\infty}|S(t_n)u_n|^2 \leq |\rho_0|^2 e^{-\nu\lambda_1 T/2}\\ &\qquad + 2\int_0^T e^{-\nu\lambda_1(T-s)/2}\Big\{< f, S(s)w_T > -[S(s)w_T]^2\Big\}ds.\end{aligned}$$

On the other hand, we obtain from (3.6) applied to $w = S(T)w_T$ that

$$\begin{aligned}|w|^2 &= |S(T)w_T|^2\\ &= e^{-\nu\lambda_1 T/2}|w_T|^2\\ &\quad + 2\int_0^T e^{-\nu\lambda_1(T-s)/2}\Big\{< f, S(s)w_T > -[S(s)w_T]^2\Big\}ds.\end{aligned} \tag{3.20}$$

Hence, from (3.19)–(3.20), we obtain

$$\begin{aligned}\limsup_{n\to\infty}|S(t_n)u_n|^2 &\leq |w|^2 + (\rho_0^2 - |w_T|^2)e^{-\nu\lambda_1 T/2}\\ &\leq |w|^2 + \rho_0^2 e^{-\nu\lambda_1 T/2}, \quad \forall\, T \in \mathbb{N}.\end{aligned} \tag{3.21}$$

Letting $T \to \infty$ in (3.21), we obtain

$$\limsup_{n\to\infty} |S(t_n)u_n|^2 \leq |w|^2. \tag{3.22}$$

Since H is a Hilbert space, (3.12) and (3.22) imply that

$$S(t_n)u_n \to w \quad \text{strongly in } H. \tag{3.23}$$

□

Using the result of Theorem 3.1, we prove the existence of the global attractor.

THEOREM 3.2. *Let Ω be an unbounded open set satisfying* (2.2). *Assume $\nu > 0$, $f \in V'$, and $|\nabla g|_\infty < m_0\lambda_1^{1/2}/4$. Then the dynamical system $\{S(t)\}_{t\geq 0}$ associated to the evolution equation* (2.6) *possesses a global attractor in H, i.e., a compact invariant set $\mathcal{A} \subset H$ which attracts all bounded sets in H. Moreover, $\mathcal{A}$ is connected in H and is maximal for the inclusion among all functional invariant sets bounded in H.*

PROOF. For the proof of this result, we refer Sell and You[21], and Temam[22]. □

4. The dimension of the attractor

We have shown the existence of the global attractor $\mathcal{A}$ for that g-Navier-Stokes equations on $\Omega \subset R^2$. In this section, we want to estimate the dimension of the global attractor $\mathcal{A}$ of 2D g-Navier-Stokes equations. We use the general theory developed by Constantin et al.[8], and follow the presentation given by Temam[22].

We rewrite the equation (2.9) in the abstract form

$$u' = F(u), \tag{4.1}$$

where $F(u) = -\nu Au - Bu - \nu Ru + f$. Then we see that the first variation equation

$$U' = F'(u)U \tag{4.2}$$

is equivalent to

$$\frac{dU}{dt} + \nu AU + B(u,U) + B(U,u) + \nu RU = 0. \tag{4.3}$$

Equation (4.3) is supplemented as usual by the initial condition

$$U(0) = \xi, \quad \xi \in H. \tag{4.4}$$

We can prove the following properties rigorously (see Temam[22]):

If u is the solution of (4.1) then the initial- and boundary-value problem (4.3)–(4.4) possesses a unique solution

$$U \in L^2(0,T;V) \cap C([0,T];H), \quad \forall\, T > 0.$$

For every $t > 0$, the function $u_0 \to S(t)u_0$ is Fréchet differentiable in H at u_0 with differential $L(t,u_0) : \xi \in H \to U(t) \in H$, where U is the solution of (4.3)–(4.4).

Before proceeding with the estimate of the Lyapunov exponents, we compute here a bound for the energy dissipation flux ϵ defined by

$$\epsilon = \nu\lambda_1 \limsup_{t\to\infty} \frac{1}{t}\int_0^t ||u(s)||^2 ds, \tag{4.5}$$

where u is the velocity. Also we estimate this quantity in terms of the data, and more specifically in terms of the generalized Grashof number G

$$G = \frac{||f||_{V'}}{\nu^2\lambda_1^{1/2}},$$

where λ_1 is given in (2.3).

By taking an inner product (4.1) with u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|^2 + \nu||u||^2 &\le ||f||_{V'}||u|| + \Big(-\frac{\nu}{g}(\nabla g\cdot\nabla u), u\Big)\\ &\le \frac{||f||_{V'}^2}{2\nu} + \frac{\nu}{2}||u||^2 + \frac{\nu}{m_0}|\nabla g|_\infty||u|||u|\end{aligned}$$

which implies, with (2.3)

$$\frac{d}{dt}|u|^2 + \nu||u||^2 \le \frac{||f||_{V'}^2}{\nu} + \frac{2\nu}{m_0\lambda_1^{1/2}}|\nabla g|_\infty||u||^2$$

and then

$$\nu\Big(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)\frac{1}{t}\int_0^t ||u(s)||^2 ds \le \frac{||f||_{V'}^2}{\nu} + \frac{|u_0|^2}{t}.$$

Therefore, we have

$$\epsilon \le \frac{\lambda_1}{\nu}\Big(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^{-1}||f||_{V'}^2 = \nu^3\lambda_1^2 G^2\Big(1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big)^{-1}, \tag{4.6}$$

whenever $\gamma_0 = 1 - \frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$.

More generally, we can consider, instead of a specific trajectory $S(t)u_0$ all the trajectories for u_0 in a bounded functional-invariant set $X \subset H$. In this case ϵ is defined as

$$\epsilon = \nu\lambda_1 \limsup_{t\to\infty} \operatorname{Sup}_{u_0\in X} \frac{1}{t}\int_0^t ||u(s)||^2 ds$$

and (4.6) remains valid.

We consider, for $m \in N$,

$$\begin{aligned}&|U_1(t)\wedge\cdots\wedge U_m(t)|_{\wedge^m H}\\ &= |\xi_1\wedge\cdots\wedge\xi_m|_{\wedge^m H}\exp\int_0^t \operatorname{Tr} F'(S(\tau)u_0)\circ Q_m(\tau)d\tau,\end{aligned}$$

where $S(\tau)u_0 = u(\tau)$ is the solution of (4.1); $U_1,\ldots,U_m$ are m solutions of (4.3)–(4.4) corresponding to the initial data $\xi_1\ldots,\xi_m; Q_m(\tau) = Q_m(\tau,u_0;\xi_1,\ldots,\xi_m)$ is the orthogonal projector in H onto the space the spanned by $U_1(\tau),\ldots,U_m(\tau)$.

At a given time τ, let $\varphi_j(\tau)$, $j = 1,\ldots,m$, be an orthonormal basis of $Q_m(\tau)H = \operatorname{Span}[U_1(\tau),\ldots,U_m(\tau)] : \varphi_j(\tau)\in V$ for $j = 1,\ldots,m$ since $U_1(\tau),\ldots,U_m(\tau)\in V$ (a.e. $\tau\in R_+$), and we have

$$\begin{aligned}\operatorname{Tr} F'(S(\tau)u_0)\circ Q_m(\tau) &= \sum_{j=1}^m (\operatorname{Tr} F'(u(\tau))\circ Q_m(\tau)\varphi_j(\tau),\varphi_j(\tau))\\ &= \sum_{j=1}^m (F'(u(\tau))\varphi_j(\tau),\varphi_j(\tau)).\end{aligned} \tag{4.7}$$

Omitting temporarily the dependence on τ, we write

$$\begin{aligned}&(F'(u)\varphi_j,\varphi_j)\\ &= -\nu(A\varphi_j,\varphi_j) - (B(\varphi_j,u),\varphi_j) - \left(\frac{\nu}{g}(\nabla g\cdot\nabla\varphi_j),\varphi_j\right)\\ &= -\nu||\varphi_j||^2 - b_g(\varphi_j,u,\varphi_j) - \left(\frac{\nu}{g}(\nabla g\cdot\nabla)\varphi_j,\varphi_j\right),\end{aligned} \tag{4.8}$$

$$\begin{aligned}\sum_{j=1}^m (F'(u)\varphi_j,\varphi_j) = &-\nu\sum_{j=1}^m ||\varphi_j||^2\\ &-\sum_{j=1}^m b_g(\varphi_j,u,\varphi_j) - \sum_{j=1}^m\left(\frac{\nu}{g}\nabla g\cdot\nabla\varphi_j,\varphi_j\right).\end{aligned}$$

Using the explicit expression of b_g, we obtain

$$\Big|\sum_{j=1}^{m} b_g(\varphi_j, u, \varphi_j)\Big| = \Big|\int_\Omega \sum_{j=1}^{m}\sum_{i,k=1}^{2} \varphi_{ji}(x) D_i u_k(x) \varphi_{jk}(x) g(x) dx\Big|$$
$$\leq \int_\Omega |\text{grad } u(x)| \rho(x) g(x) dx,$$

where

$$|\text{grad } u(x)| = \Big\{\sum_{i,k=1}^{2} |D_i u_k(x)|^2\Big\}^{1/2},$$
$$\rho(x) = \sum_{j=1}^{m}\sum_{i=1}^{2} (\varphi_{ji}(x))^2.$$

Therefore,

$$\begin{aligned} \Big|\sum_{j=1}^{m} b_g(\varphi_j, u, \varphi_j)\Big| &\leq \int_\Omega |\text{grad } u(x)| \rho(x) g(x) dx \\ &\leq \text{(with the Schwarz inequality)} \\ &\leq ||u|||\rho|. \end{aligned} \tag{4.9}$$

Also, we obtain

$$\begin{aligned} \left|\sum_{j=1}^{m}\left(\frac{\nu}{g}\nabla g \cdot \nabla \varphi_j, \varphi_j\right)\right| &\leq \sum_{j=1}^{m} \frac{\nu|\nabla g|_\infty}{m_0} ||\varphi_j|||\varphi_j| \\ &\leq \text{(with the property 2.2)} \\ &\leq \sum_{j=1}^{m} \frac{\nu|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} ||\varphi_j||^2. \end{aligned} \tag{4.10}$$

We recall that the dependence on τ has been omitted and in fact $u = u(x,\tau), \rho = \rho(x,\tau)$, etc... . From (4.8)–(4.10), we have established the following inequality

$$\text{Tr } F'(u(\tau)) \circ Q_m(\tau) \leq -\nu\left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) \sum_{j=1}^{m} ||\varphi_j||^2 + ||u|||\rho|. \tag{4.11}$$

Since the φ_js are orthonormal in H, hence in $\mathbb{L}^2(g)$, and belong to $V \subset \mathbb{H}_0^1(g)$, we have the Lieb-Thirring inequality (see Theorem A.3.1 in Temam[22]) there exists a dimensionless constant c depending only on the shape of Ω such that

$$|\rho(\tau)|^2 = \int_\Omega \rho^2(x,\tau)g(x)dx \le c\sum_{j=1}^m ||\varphi_j||^2. \tag{4.12}$$

Hence

$$\begin{aligned}
&\mathrm{Tr}\ F'(u(\tau))\circ Q_m(\tau)\\
&\le -\nu\left(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\sum_{j=1}^m||\varphi_j||^2 + ||u||\left(c\sum_{j=1}^m||\varphi_j||^2\right)^{1/2}\\
&\le \text{(with the Schwarz inequality)}\\
&\le -\frac{\nu}{2}\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)\sum_{j=1}^m||\varphi_j||^2 + \frac{c}{2\nu}||u||^2.
\end{aligned} \tag{4.13}$$

Now (2.2) allows us to majorize $\mathrm{Tr}\ F'(u)\circ Q_m$ in (4.13) as follows

$$\mathrm{Tr}\ F'(u(\tau))\circ Q_m(\tau) \le -\frac{\nu\lambda_1}{2}\left(1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right)m + \frac{c}{2\nu}||u||^2$$

and

$$\frac{1}{t}\int_0^t \mathrm{Tr}\ F'(u(\tau))\circ Q_m(\tau)d\tau \le -\frac{\nu}{2}\gamma_0\lambda_1 m + \frac{c}{2\nu}\frac{1}{t}\int_0^t ||u(\tau)||^2d\tau,$$

where $\gamma_0 = 1-\frac{2|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$.

Now we define

$$q_m(t) = \operatorname*{Sup}_{u_0\in\mathcal{A}}\ \operatorname*{Sup}_{\substack{\xi_j\in H\\ |\xi_j|\le 1\\ j=1,\dots,m}}\left(\frac{1}{t}\int_0^t \mathrm{Tr}F'(u(\tau))\circ Q_m(\tau)d\tau\right),$$

$$q_m(t) \le -\frac{\nu}{2}\gamma_0\lambda_1 m + \frac{c}{2\nu}\operatorname*{Sup}_{u_0\in\mathcal{A}}\frac{1}{t}\int_0^t ||u(\tau)||^2d\tau,$$

$$q_m = \limsup_{t\to\infty} q_m(t) \le -\kappa_1 m + \kappa_2,$$

where

$$\kappa_1 = \frac{\nu}{2}\gamma_0\lambda_1, \quad \kappa_2 = \frac{c}{2}\frac{\epsilon}{\nu^2\lambda_1},$$

$$\epsilon = \nu\lambda_1 \limsup_{t\to\infty}\ \operatorname*{Sup}_{u_0\in\mathcal{A}}\frac{1}{t}\int_0^t ||u(\tau)||^2d\tau.$$

Then we obtain the following bound on the Lyapunov exponents μ_j, $j \in \mathbb{N}$,

$$\mu_1 + \cdots + \mu_j \leq q_j \leq -\kappa_1 m + \kappa_2, \quad \forall j \in \mathbb{N}.$$

Using Lemma VI.2.2 in Temam[22], we see that if m is defined by

$$m - 1 < \frac{2\kappa_2}{\kappa_1} = \frac{2c\epsilon}{\nu^3 \lambda_1^2 \gamma_0} \leq m, \tag{4.14}$$

then $\mu_1 + \cdots + \mu_m \leq 0$ and

$$\frac{(\mu_1 + \cdots + \mu_j)_+}{|\mu_1 + \cdots + \mu_m|} \leq 1, \quad \forall\, j = 1, ..., m-1.$$

We have now proved the following

THEOREM 4.1. *We consider the dynamical system associated with the two dimensional g-Navier-Stokes equations when* $|\nabla g|_\infty < m_0 \lambda_1^{1/2}/4$. *We define* m

$$m - 1 < \frac{2\kappa_2}{\kappa_1} = \frac{2c\epsilon}{\nu^3 \lambda_1^2 \gamma_0} \leq m, \tag{4.15}$$

where c *is a dimensionless constant depending only on the shape of* Ω. *Then the corresponding global attractor* $\mathcal{A}$ *has a Hausdorff dimension less than or equal to* m *and a fractal dimension less than or equal to* $2m$.

REMARK 4.2. Thanks to (4.6) and (4.15),

$$\frac{2\kappa_2}{\kappa_1} = \frac{2c\epsilon}{\nu^3 \lambda_1^2 \gamma_0} \leq \frac{C}{\gamma_0^2} G^2$$

for some constant $C > 0$ and hence we can replace, in the statement of Theorem 4.1, m by another larger number m_1 given by

$$m_1 - 1 < \frac{C}{\gamma_0^2} G^2 \leq m_1.$$

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