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EXISTENCE AND ALGORITHM OF SOLUTIONS FOR GENERALIZED MIXED QUASI-VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. In this paper, we introduce and study a new class of generalized mixed quasi-variational-like inequalities. Using the auxiliary principle technique, we construct a new iterative algorithm for finding the approximate solutions of the generalized mixed quasi-variational-like inequality. An existence result of solutions for the generalized mixed quasi-variational-like inequality and the convergence of the iterative algorithm are also established. Our results extend, unify and improve many recent known results.

1. Introduction

It is well known that the auxiliary principle technique plays an important role in variational inequality theory. In 1990, Bose[4] studied a class of general nonlinear variational inequalities by using auxiliary principle technique. Afterwards, Ding[10], Ding and Luo[13], Huang and Fang[16] and others extended the results in [4] to several classes of generalized mixed variational inequalities, generalized mixed quasi-variational inequalities and generalized set-valued nonlinear quasi-variational-like inequalities.

Inspired and motivated by the results [1–29] and [31], in this paper, we introduce and study a new class of generalized mixed quasi-variational-like inequalities. Using the auxiliary principle technique, we construct a new iterative algorithm for finding the approximate solutions of the generalized mixed quasi-variational-like inequality. An existence result

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of solutions for the generalized mixed quasi-variational-like inequality and the convergence of the iterative algorithm are also established. Our results extend, unify and improve many recent known results.

2. Preliminaries

Let H be a real Hilbert space with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Assume that I and 2^H denote the identity mapping on H and the family of nonempty subsets of H, respectively. Let $K : H \to 2^H$ be a set-valued mapping such that for each $x \in H$, $K(x)$ is a closed convex subset of H. Suppose that $N : H\times H\times H \to H$, $\eta : H\times H \to H$, A, B, C and $g : H \to H$ be mappings. Let $b : H \times H \to R$ be a real mapping satisfying the following conditions:

(C1) b is linear in the first argument;

(C2) b is convex in the second argument;

(C3) b is bounded, that is, there exists a constant $\gamma > 0$ satisfying

$$|b(u,v)| \leq \gamma\|u\|\|v\|, \quad \forall u,v \in H;$$

(C4) $b(u,v) - b(u,w) \leq b(u,v-w)$, $\forall u,v,w \in H$.

For a given $f \in H$, we consider the following generalized mixed quasi-variational-like inequality: Find $x \in H$ such that $x \in K(x)$ and

$$\begin{aligned} &\langle N(A(x),B(x),C(x)) - f, \eta(g(y),g(x))\rangle \\ &\quad + b(x,y) - b(x,x) \geq 0, \quad \forall y \in K(x). \end{aligned} \tag{2.1}$$

Now we consider some special cases of the problem (2.1):

(A) If $f = 0$, $K = I$, $b(x,y) = f(y)$, $N(A(x),B(x),C(x)) = A(x) - B(x)$ and $\eta(g(y),g(x)) = g(y) - g(x)$ for all $x \in H$, then the problem (2.1) is equivalent to finding $x \in H$ such that

$$\langle A(x) - B(x), g(y) - g(x)\rangle \geq f(x) - f(y), \quad \forall y \in H,$$

which was introduced and studied by Yao[31].

(B) If $f = 0$, $N(A(x),B(x),C(x)) = A(x)$, $\eta(g(y),g(x)) = \eta(y,x)$, $b = 0$ and $K(x) = K$ for all $x \in H$, where K is a closed convex subset of H, then the problem (2.1) is equivalent to finding $x \in H$ such that

$$\langle A(x), \eta(y,x)\rangle \geq 0, \quad \forall y \in K.$$

which was introduced and studied by Parida, Sahoo and Kumar[29].

In order to get our main results, we need the following definitions and Lemmas.

DEFINITION 2.1. Let A, $B : H \to H$ and $N : H \times H \times H \to H$ be mappings. The mapping N is said to be

(1) *strongly monotone* with respect to A in the first argument if there exists a constant $\alpha > 0$ satisfying

$$\langle N(A(x), u, v) - N(A(y), u, v), x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y, u, v \in H;$$

(2) *relaxed monotone* with respect to B in the second argument if there exists a constant $\beta > 0$ satisfying

$$\langle N(u, B(x), v) - N(u, B(y), v), x - y\rangle \geq -\beta\|x - y\|^2, \quad \forall x, y, u, v \in H;$$

(3) *Lipschitz continuous* with respect to the first argument if there exists a constant $a > 0$ satisfying

$$\|N(x, u, v) - N(y, u, v)\| \leq a\|x - y\|, \quad \forall x, y, u, v \in H.$$

Similarly, we can define the Lipschitz continuity of N with respect to the second and third arguments, respectively.

DEFINITION 2.2. A mapping $g : H \to H$ is said to be *Lipshitz continuous* if there exists a constant $b > 0$ satisfying

$$\|g(x) - g(y)\| \leq b\|x - y\|, \quad \forall x, y \in H.$$

DEFINITION 2.3. Let $\eta : H \times H \to H$ and $g : H \to H$ be mappings. The mapping η is said to be

(1) *strongly monotone* with respect to g if there exists a constant $\xi > 0$ such that

$$\langle \eta(g(x), g(y)), x - y\rangle \geq \xi\|x - y\|^2, \quad \forall x, y \in H;$$

(2) *Lipschitz continuous* if there exists a constant $c > 0$ satisfying

$$\|\eta(g(x), g(y))\| \leq c\|x - y\|, \quad \forall x, y \in H.$$

HYPOTHESIS 2.1. *Let $\eta : H \times H \to H$ and $g : H \to H$ be mappings and*

(1) $\eta(x, y) + \eta(y, z) = \eta(x, z)$, $\forall x, y, z \in H$;

(2) $x - y = u - v$ *implies that* $\eta(g(x), g(y)) = \eta(g(u), g(v))$, $\forall x, y, u, v \in H$;

(3) *For each* $x \in H$, *the mapping* $y \mapsto \langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(x))\rangle$ *is convex and lower semicontinuous on* H.

In order to solve the generalized mixed quasi-variational-like inequality (2.1), we consider the following auxiliary variational inequality problem: for any given $f \in H$ and $x \in H$, find a unique $w \in K(x)$ such that

$$\begin{aligned}\langle w, y - w\rangle \geq{} & \langle x, y - w\rangle \\ & - \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(w))\rangle \\ & - \rho b(x, y) + \rho b(x, w), \quad \forall y \in K(x),\end{aligned} \tag{2.2}$$

where ρ is positive constant.

LEMMA 2.1. *Let $K : H \to 2^H$ be such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H. Let A, B, C and $g : H \to H$, $N : H \times H \times H \to H$, $\eta : H \times H \to H$ be mappings and $b : H \times H \to R$ be such that for any given $x \in H$, the functional $y \mapsto b(x, y)$ is proper convex and lower semicontinuous on H. Suppose that Hypothesis 2.1 holds. Then for any given $x \in H$ and $f \in H$, the functional $J : K(x) \to R$ defined by*

$$\begin{cases} J(y) = \frac{1}{2}\langle y, y\rangle + j(y), \\ j(y) = \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(x))\rangle \\ \qquad\quad + \rho b(x, y) - \langle x, y\rangle. \end{cases} \tag{2.3}$$

has a unique minimum point $w \in K(x)$ and w is the unique minimum point of J on $K(x)$ if and only if w is a unique solution of the auxiliary variational inequality problem (2.2).

PROOF. Since the functionals $y \mapsto \eta(g(y), g(x))$ and $y \mapsto b(x, y)$ are proper convex and lower semicontinuous for each $x \in H$, it is easy to show that $j(y)$ is proper convex lower semicontinuous on $K(x)$ and $J(y)$ is strictly convex lower semicontinuous on $K(x)$. It follows from Theorem 2.5 of [30, p. 25] that j is bounded from below by a hyperplane

$r(y) = \langle h, y\rangle + r$ for any $y \in H$, where $h \in H$ and $r \in R$. Hence we have

$$\begin{aligned} J(y) &= \frac{1}{2}\langle y, y\rangle + j(y) \geq \frac{1}{2}\|y\|^2 + \langle h, y\rangle + r \\ &= \frac{1}{2}\|h + y\|^2 - \frac{1}{2}\|h\|^2 + r, \end{aligned}$$

which implies that

$$J(y) \to \infty \quad \text{as } \|y\| \to \infty. \tag{2.4}$$

Let $\{y_n\}_{n\geq 0} \subseteq K(x)$ is a minimizing sequence of J on $K(x)$, that is,

$$\lim_{n\to\infty} J(y_n) = d \quad \text{and} \quad d = \inf_{y\in K(x)} J(y).$$

We claim that $\{y_n\}_{n\geq 0}$ is bounded. Otherwise, there exists a subsequence $\{y_{n_k}\}_{k\geq 0}$ of $\{y_n\}_{n\geq 0}$ such that $\|y_{n_k}\| \geq k,\ k = 1, 2, \cdots$. In light of (2.4), we infer that

$$J(y_{n_k}) \to \infty \quad \text{as } k \to \infty,$$

which is a contradiction. Therefore there exists a constant $r_1 > 0$ such that

$$\{y_n\}_{n\geq 0} \subseteq K(x) \cap B(0, r_1) = \{y \in K(x) : \|y\| \leq r_1\}.$$

The Weierstrass Theorem (see [30, p. 24]) ensures that there exists $w \in K(x)$ such that $J(w) = \min_{y\in K(x)} J(y)$. It follows from that strict convexity of J that w is a unique minimizing point of J on $K(x)$.

Now assume that w is a solution of the auxiliary variational inequality problem (2.2). It follows from (2.3) that

$$\begin{aligned} &\frac{1}{2}(\langle y, y\rangle - \langle w, w\rangle) \\ &= \frac{1}{2}[\langle w + y - w, w + y - w\rangle - \langle w, w\rangle] \\ &= \langle w, y - w\rangle + \frac{1}{2}\langle y - w, y - w\rangle \\ &\geq \langle w, y - w\rangle \\ &\geq \langle x, y - w\rangle - \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(w))\rangle \\ &\quad + \rho b(x, w) - \rho b(x, y) \\ &= \langle x, y\rangle - \langle x, w\rangle - \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(x))\rangle \\ &\quad + \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(w), g(x))\rangle \\ &\quad + \rho b(x, w) - \rho b(x, y), \end{aligned}$$

which implies that $J(y) \geq J(w)$ for all $y \in K(x)$. That is, $J(w) = \min_{y \in K(x)} J(y)$.

Conversely, suppose that w is a unique minimizing point of J on $K(x)$. For any $y \in K(x)$ and $t \in [0,1]$, we have

$$\begin{aligned} J(w) &= \frac{1}{2}\langle w, w\rangle + j(w) \\ &\leq J(w + t(y-w)) \\ &= \frac{1}{2}\langle w + t(y-w), w + t(y-w)\rangle + j(w + t(y-w)) \\ &\leq \frac{1}{2}\langle w, w\rangle + \frac{t^2}{2}\langle y-w, y-w\rangle + t\langle w, y-w\rangle + j(w) \\ &\quad + t(j(y) - j(w)). \end{aligned}$$

It follows that

$$\frac{t}{2}\langle y-w, y-w\rangle + \langle w, y-w\rangle + j(y) - j(w) \geq 0.$$

Letting $t \to 0$ in the above inequality, we deduce that

$$\begin{aligned} &\langle w, y-w\rangle + \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(x))\rangle \\ &\quad + \rho b(x,y) - \langle x, y\rangle - \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(w), g(x))\rangle \\ &\quad - \rho b(x,w) + \langle x, w\rangle \geq 0. \end{aligned}$$

This is,

$$\begin{aligned} \langle w, y-w\rangle \geq{}& \langle x, y-w\rangle - \rho\langle N(A(x), B(x), C(x)) - f, \eta(g(y), g(w))\rangle \\ & - \rho b(x,y) + \rho b(x,w), \quad \forall y \in K(x). \end{aligned}$$

Hence w is a solution of the auxiliary variational inequality problem (2.2). This completes the proof. □

Based on Lemma 2.1 and the auxiliary variational inequality problem (2.2), we now suggest and analyze the following iterative algorithm for finding the approximate solutions of the generalized mixed quasi-variational-like inequality (2.1).

ALGORITHM 2.1. *Let $K : H \to 2^H$ be such that for each $x \in H$, $K(x)$ is a nonempty closed convex subset of H. Let A, B, C and $g : H \to H$, $N : H \times H \times H \to H$, $\eta : H \times H \to H$ and $b : H \times H \to R$ be* *mappings. For given $x_0 \in H$ and $f \in H$, compute sequence $\{x_n\}_{n\geq 0}$ by the following scheme*

$$\begin{aligned}\langle x_{n+1}, y - x_{n+1}\rangle \geq \langle x_n, y - x_{n+1}\rangle \\ &- \rho\langle N(A(x_n), B(x_n), C(x_n)) - f, \eta(g(y), g(x_{n+1}))\rangle \\ &- \rho b(x_n, y) + \rho b(x_n, x_{n+1}), \quad \forall y \in K(x_n),\ n \geq 0,\end{aligned}$$

where ρ is a positive constant.

3. Main result

In this section, we prove the existence of solutions of the generalized mixed quasi-variational-like inequality (2.1) and the convergence of the sequence generated by Algorithm 2.1.

THEOREM 3.1. *Let A, B, C, m, $g : H \to H$ be Lipschitz continuous with constants a, p, c, h, l, respectively. Assume that $N : H \times H \times H \to H$ is Lipschitz continuous with respect to the first, second and third arguments with constants t, s, k, respectively, and is strongly monotone with respect to A in the first argument with constant α, relaxed monotone with respect to B in the second argument with constant β, respectively. Suppose that $\eta : H \times H \to H$ is Lipschitz continuous and strongly monotone with constants d and ξ, respectively and Hypothesis 2.1 holds. Let $b : H \times H \to R$ satisfy* (C1)–(C4) *and $K : H \to 2^H$ be a set-valued mapping such that $K(x) = m(x) + K$ for each $x \in H$, where K is a closed convex subset of H. Let*

$$L = (ta + sp)\sqrt{1 - 2\xi + d^2h^2} + cdhk + \gamma.$$

Suppose that there exists a constant $\rho > 0$ satisfying

$$2l + \rho L < 1 \tag{3.1}$$

and one of following conditions:

$$\begin{aligned}&(sp + ta)^2 > L, \\ &(\alpha - \beta - (L - 2l))^2 > ((sp + ta)^2 - L)4l(1 - l), \\ &\left|\rho - \frac{\alpha - \beta - L(1 - 2l)}{(sp + ta)^2 - L}\right| \\ &< \frac{\sqrt{(\alpha - \beta - (L - 2l))^2 - ((sp + ta)^2 - L)4l(1 - l)}}{(sp + ta)^2 - L};\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&(sp+ta)^2 < L,\\
&\left|\rho - \frac{\alpha-\beta-L(1-2l)}{(sp+ta)^2-L}\right| \\
&> \frac{\sqrt{(\alpha-\beta-(L-2l))^2-((sp+ta)^2-L)4l(1-l)}}{L-(sp+ta)^2}.
\end{aligned}
\tag{3.3}
$$

Then for each $f \in H$, the sequence $\{x_n\}_{n\geq 0}$ generated by Algorithm 2.1 converges strongly to x^* and x^* is a solution of the generalized mixed quasi-variational-like inequality (2.1).

PROOF. It follows from (C1)–(C4) that $y \mapsto b(x,y)$ is convex and continuous for each $x \in H$. In view of Lemma 2.1, we infer that the auxiliary variational inequality problem (2.2) has a unique solution w in $K(x)$. It follows from Algorithm 2.1 that

$$
\begin{aligned}
&\langle x_{n+1}, y - x_{n+1}\rangle\\
&\geq \langle x_n, y - x_{n+1}\rangle\\
&\quad - \rho\langle N(A(x_n), B(x_n), C(x_n)) - f, \eta(g(y), g(x_{n+1}))\rangle\\
&\quad - \rho b(x_n, y) + \rho b(x_n, x_{n+1}), \quad \forall y \in K(x_n)
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
&\langle x_{n+2}, y - x_{n+2}\rangle\\
&\geq \langle x_{n+1}, y - x_{n+2}\rangle\\
&\quad - \rho\langle N(A(x_{n+1}), B(x_{n+1}), C(x_{n+1})) - f, \eta(g(y), g(x_{n+2}))\rangle\\
&\quad - \rho b(x_{n+1}, y) + \rho b(x_{n+1}, x_{n+2}), \quad \forall y \in K(x_{n+1})
\end{aligned}
\tag{3.5}
$$

for all $n \geq 0$. Adding $\langle -m(x_n), y - x_{n+1}\rangle$ to two sides of the inequality (3.4) and taking $y = x_{n+2} - m(x_{n+1}) + m(x_n) \in K(x_n)$, we get that

$$
\begin{aligned}
&\langle x_{n+1} - m(x_n), x_{n+2} - m(x_{n+1}) + m(x_n) - x_{n+1}\rangle\\
&\geq \langle x_n - m(x_n), x_{n+2} - m(x_{n+1}) + m(x_n) - x_{n+1}\rangle\\
&\quad - \rho\langle N(A(x_n), B(x_n), C(x_n)) - f, \eta(g(x_{n+2} - m(x_{n+1})\\
&\qquad + m(x_n)), g(x_{n+1}))\rangle - \rho b(x_n, x_{n+2} - m(x_{n+1}) + m(x_n))\\
&\quad + \rho b(x_n, x_{n+1}), \quad \forall n \geq 0.
\end{aligned}
\tag{3.6}
$$

Adding $\langle -m(x_{n+1}), y - x_{n+2}\rangle$ to two sides of the inequality (3.5) and substituting $y = x_{n+1} - m(x_n) + m(x_{n+1}) \in K(x_n)$ into it, we derive that

$$
\begin{aligned}
&\langle x_{n+2} - m(x_{n+1}), x_{n+1} - m(x_n) + m(x_{n+1}) - x_{n+2} \rangle \\
&\geq \langle x_{n+1} - m(x_{n+1}), x_{n+1} - m(x_n) + m(x_{n+1}) - x_{n+2} \rangle \\
&\quad - \rho \langle N(A(x_{n+1}), B(x_{n+1}), C(x_{n+1})) - f, \\
&\qquad \eta(g(x_{n+1} - m(x_n) + m(x_{n+1})), g(x_{n+2})) \rangle \\
&\quad - \rho b(x_{n+1}, y) + \rho b(x_{n+1}, x_{n+2}), \quad \forall n \geq 0.
\end{aligned} \tag{3.7}
$$

It follows from (C1)–(C4), Hypothesis 2.1, (3.6) and (3.7) that

$$
\begin{aligned}
&\langle x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1}), x_{n+1} - x_{n+2} \\
&\quad - m(x_n) + m(x_{n+1}) \rangle \\
&\leq \langle x_n - x_{n+1} - m(x_n) + m(x_{n+1}), x_{n+1} - x_{n+2} \\
&\quad - m(x_n) + m(x_{n+1}) \rangle - \rho \langle N(A(x_n), B(x_n), C(x_n)) - f \\
&\quad - (N(A(x_{n+1}), B(x_{n+1}), C(x_{n+1})) - f), \\
&\qquad \eta(g(x_{n+1} - x_{n+2}), g(m(x_n) - m(x_{n+1}))) \rangle \\
&\quad - \rho b(x_{n+1} - x_n, x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1})) \\
&\leq [\|m(x_n) - m(x_{n+1})\| \\
&\quad + \rho\gamma \|x_{n+1} - x_n\| \|x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1})\| \\
&\quad + \|x_n - x_{n+1} - \rho(N(A(x_n), B(x_n), C(x_n)) \\
&\qquad - N(A(x_{n+1}), B(x_{n+1}), C(x_n)))\| \\
&\quad + \rho \|N(A(x_n), B(x_n), C(x_n)) \\
&\qquad - N(A(x_{n+1}), B(x_{n+1}), C(x_n))\| \\
&\quad \times \|x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1}) \\
&\qquad - \eta(g(x_{n+1} - x_{n+2}), g(m(x_n) - m(x_{n+1})))\| \\
&\quad + \rho \|N(A(x_{n+1}), B(x_{n+1}), C(x_{n+1})) \\
&\qquad - N(A(x_{n+1}), B(x_{n+1}), C(x_n))\| \\
&\quad \times \|\eta(g(x_{n+1} - x_{n+2}), g(m(x_n) - m(x_{n+1})))\|, \quad \forall n \geq 0.
\end{aligned} \tag{3.8}
$$

Since $N : H \times H \times H \to H$ is Lipschitz continuous with respect to the first, second and third arguments, respectively, and is strongly monotone with respect to A in the first argument, relaxed monotone with respect to B in the second argument, and A, B, and C are Lipschitz continuous, we have

$$
\begin{aligned}
&\|x_n - x_{n+1} - \rho(N(A(x_n), B(x_n), C(x_n)) \\
&\quad - N(A(x_{n+1}), B(x_{n+1}), C(x_n)))\|^2 \\
&= \|x_n - x_{n+1}\|^2 - 2\rho\langle x_n - x_{n+1}, N(A(x_n), B(x_n), C(x_n)) \\
&\quad - N(A(x_{n+1}), B(x_n), C(x_n))\rangle \\
&\quad - 2\rho\langle x_n - x_{n+1}, N(A(x_{n+1}), B(x_n), C(x_n)) \\
&\quad - N(A(x_{n+1}), B(x_{n+1}), C(x_n))\rangle \\
&\quad + \rho^2(\|N(A(x_n), B(x_n), C(x_n)) \\
&\quad - N(A(x_{n+1}), B(x_n), C(x_n))\| \\
&\quad + \|N(A(x_{n+1}), B(x_n), C(x_n)) \\
&\quad - N(A(x_{n+1}), B(x_{n+1}), C(x_n))\|)^2 \\
&\leq (1 - 2(\alpha - \beta)\rho + \rho^2(ta + sp)^2)\|x - y\|^2
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
&\|N(A(x_{n+1}), B(x_{n+1}), C(x_n)) \\
&\quad - N(A(x_{n+1}), B(x_{n+1}), C(x_{n+1}))\| \\
&\leq ck\|x_n - x_{n+1}\|
\end{aligned} \tag{3.10}
$$

for all $n \geq 0$. It follows from the Lipschitz continuity of η, g and m that

$$
\begin{aligned}
&\|x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1}) \\
&\quad - \eta(g(x_{n+1} - x_{n+2}), g(m(x_n) - m(x_{n+1})))\|^2 \\
&\leq (1 - 2\xi + d^2h^2)\|x_{n+1} - x_{n+2} - m(x_n) + m(x_{n+1})\|^2
\end{aligned} \tag{3.11}
$$

and

$$\|m(x_n) - m(x_{n+1})\| \leq l\|x_n - x_{n+1}\| \tag{3.12}$$

for all $n \geq 0$. In view of (3.8)–(3.12), we conclude that

$$\|x_{n+1} - x_{n+2}\| \leq \theta\|x_n - x_{n+1}\|, \quad \forall n \geq 0,$$

where

$$\theta = 2l + \rho L + \sqrt{1 - 2(\alpha - \beta)\rho + \rho^2(ta + sp)^2}. \tag{3.13}$$

It is easy to verify that (3.1) and one of (3.2) and (3.3) yield that $\theta < 1$. This implies that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in H. Let $x_n \to x^* \in H$ as $n \to \infty$. Lemma 2.1 ensures that there exists a unique $w \in K(x^*)$ satisfying

$$\begin{aligned}&\langle w, y - w\rangle \\ \geq\ &\langle x^*, y - w\rangle - \rho\langle N(A(x^*), B(x^*), C(x^*)) - f, \eta(g(y), g(w))\rangle \\ &- \rho b(x^*, y) + \rho b(x^*, w), \quad \forall y \in K(x^*).\end{aligned} \tag{3.14}$$

In view of (3.4), (3.14) and the above proof, we infer that

$$\|x_{n+1} - w\| \leq \theta\|x_n - x^*\|, \quad \forall n \geq 0,$$

which implies that $x_n \to w$ as $n \to \infty$. That is, $x^* = w$. Therefore, x^* is a solution of the generalized mixed quasi-variational-like inequality (2.1). This completes the proof. □

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