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COMPOSITION OPERATORS FROM HARDY SPACES INTO α-BLOCH SPACES ON THE POLYDISK

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Abstract. Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$, where $\mathbb{D}^n$ is the unit polydisk of $\mathbb{C}^n$. The sufficient and necessary conditions for a composition operator to be bounded and compact from the Hardy space $H^2(\mathbb{D}^n)$ into α-Bloch space $\mathcal{B}^\alpha(\mathbb{D}^n)$ on the polydisk are given.

1. Introduction

Let $\mathbb{D}^n$ be the unit polydisk of $\mathbb{C}^n$ and the class of all holomorphic functions with domain $\mathbb{D}^n$ will be denoted by $H(\mathbb{D}^n)$. Let φ be a holomorphic self-map of $\mathbb{D}^n$ and the composition operator C_φ induced by φ is defined by $(C_\varphi f)(z) = f(\varphi(z))$ for z in $\mathbb{D}^n$ and $f \in H(\mathbb{D}^n)$. It is interesting to characterize the composition operator on various analytic function spaces, the book [1] contains plenty of information.

Let $0 < \alpha$, a function f holomorphic in $\mathbb{D}^n$ is said to belong to the α-Bloch space $\mathcal{B}^\alpha(\mathbb{D}^n)$ if

$$\|f\|_\alpha = |f(0)| + \sup_{z \in \mathbb{D}^n} \sum_{k=1}^{n} |\frac{\partial f}{\partial z_k}(z)|(1 - |z_k|^2)^\alpha < +\infty.$$

It is easy to show that $\mathcal{B}^\alpha(\mathbb{D}^n)$ is a Banach space with the above norm $\|\cdot\|_\alpha$.

For $z, w \in \mathbb{C}^n$, write $z \cdot w = (z_1 w_1, \cdots, z_n w_n)$, $e^{i\theta} = (e^{i\theta_1}, \cdots, e^{i\theta_n})$. When we write $0 \le r < 1$, where $r = (r_1, \cdots, r_n)$, means $0 \le r_i < 1 (i = 1, \cdots, n)$. The Hardy space $H^p(\mathbb{D}^n)$ is defined on $\mathbb{D}^n$ by

$$H^p = H^p(\mathbb{D}^n) = \{f \in H(\mathbb{D}^n), \quad \|f\|_p < \infty\},$$

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where

$$\|f\|_p = \frac{1}{(2\pi)^n} \sup_{0\leq r<1} \int_{[0,2\pi]^n} |f(r\cdot e^{i\theta})|^p dt_1 \cdots dt_n.$$

When $p=2$, the Hardy space $H^2(\mathbb{D}^n)$ is a Hilbert space whose reproducing kernel and the normalized reproducing kernel are respectively

$$K_a(z) = \frac{1}{\prod_{k=1}^n (1-\overline{a_k}z_k)}, \quad k_a(z) = \frac{\prod_{k=1}^n (1-|a_k|^2)^{1/2}}{\prod_{k=1}^n (1-\overline{a_k}z_k)}.$$

About the details of the Hardy space on polydisk, you can refer Rudin's book([3]).

Recently, in the setting of unit disk, Pérez-González and Xiao studied composition operators from the Hardy space into the Bloch space in [2]. In the several complex variable case, Shi and Luo first studied Composition operator on the Bloch space in [4], then Zhou and Shi studied composition operator on the Bloch space on the polydisk in [5] and [6]. In this paper, we study the boundedness and compactness of composition operators from the Hardy space into the α-Bloch space in the setting of polydisk. In this paper, C always denote positive constant and may be different at different occurrences.

2. Main results and proof

In this section, we give our main results and their proofs:

THEOREM 2.1 *Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$, then $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is bounded if and only if there exists a constant $M>0$, such that for any $z \in \mathbb{D}^n$,*

$$\sum_{k,l=1}^n \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}} \leq M. \tag{2.1}$$

PROOF. Suppose that (2.1) holds. For a function $f \in H^2$ and $z \in \mathbb{D}^n$, we have

$$\begin{aligned}
&\sum_{k=1}^{n} |\frac{\partial(f\circ\varphi)}{\partial z_k}|(1-|z_k|^2)^\alpha \\
\leq &\sum_{k=1}^{n}\sum_{l=1}^{n} |\frac{\partial f}{\partial w_l}(\varphi(z))||\frac{\partial \varphi_l}{\partial z_k}(z)|(1-|z_k|^2)^\alpha \\
\leq &\sum_{k=1}^{n}\sum_{l=1}^{n} |\frac{\partial f}{\partial w_l}(\varphi(z))||\frac{\partial \varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}}(1-|\varphi_l(z)|^2)^{3/2} \\
\leq &\, C\|f\|_2 \sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}} \\
\leq &\, CM\|f\|_2.
\end{aligned}$$

Therefore $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is bounded.

Conversely, suppose that $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is bounded, then for any $f \in H^2(\mathbb{D}^n)$, we have $\|C_\varphi f\|_\alpha \leq C\|f\|_2$. For any fixed $l (1 \leq l \leq n)$ and $w \in \mathbb{D}$, let

$$f_w(z) = \frac{\sqrt{1-|w|^2}}{1-\overline{w}z_l}.$$

It is easy to check that $f_w \in H^2(\mathbb{D}^n)$ and therefore $\|C_\varphi f_w\|_\alpha \leq C\|f_w\|_2 = C$, i.e.,

$$\sum_{k=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)|(1-|z_k|^2)^\alpha|\frac{\partial f_w}{\partial w_l}(\varphi(z))| \leq C.$$

For any $z \in \mathbb{D}^n$, replacing w in the above inequality with $\varphi_l(z)$, we get

$$\sum_{k=1}^{n} |\varphi_l(z)||\frac{\partial \varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}} \leq C.$$

Consequently, by the above arguments, we have

$$\sum_{k,l=1}^{n} |\frac{\partial \varphi_l}{\partial z_k}(z)|\frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}} \leq C$$

for any $z \in \mathbb{D}^n$. □

Next, we characterize compact composition operator $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$. Before we give the second main result, we give some lemmas as following:

LEMMA 2.2 *Suppose $f \in H^2(\mathbb{D}^n)$, then for $z = (z_1, z_2, ..., z_n) \in \mathbb{D}^n$,*

$$|f(z)| \leq \|f\|_2 \sum \frac{1}{(1-|z_i|^2)^{1/2}}.$$

LEMMA 2.3 *Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$, then $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is compact if and only if whenever $\{f_j\}$ is bounded in $H^2(\mathbb{D}^n)$ and $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}^n$, then $C_\varphi f_j \to 0$ in $\mathcal{B}^\alpha(\mathbb{D}^n)$.*

PROOF. Using Lemma 2.2 and Montel theorem, modify the proof of the Proposition 3.11 in [1], we can give the proof. Since the proof is routine, we omit it. □

THEOREM 2.4 *Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$, then $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is compact if and only if for any $\epsilon > 0$, there exists a constant $\delta > 0$ such that for any $z \in \mathbb{D}^n$ and $dist(\varphi(z), \partial\mathbb{D}^n) < \delta$, the following inequality is satisfied*

$$\sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}} \leq \epsilon. \tag{2.2}$$

PROOF. First, suppose that (2.2) holds. Let $\{f_j\}$ are satisfied

(a) $\|f_j\|_2 \leq M (j = 1, 2, ...)$,

(b) $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}^n$.

By Lemma 2.3, we only need to prove $\|f_j \circ \varphi\|_\alpha \to 0$. In fact

$$\begin{aligned} &\sum_{k=1}^{n} \left|\frac{\partial (f_j \circ \varphi)}{\partial z_k}\right| (1-|z_k|^2)^\alpha \\ \leq &\sum_{k=1}^{n}\sum_{l=1}^{n} \left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right| \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| (1-|z_k|^2)^\alpha \\ \leq &\sum_{k=1}^{n}\sum_{l=1}^{n} \left|\frac{\partial f_j}{\partial w_l}(\varphi(z))\right| \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}} (1-|\varphi_l(z)|^2)^{3/2} \\ \leq &\|f_j\|_2 \sum_{k,l=1}^{n} \left|\frac{\partial \varphi_l}{\partial z_k}(z)\right| \frac{(1-|z_k|^2)^\alpha}{(1-|\varphi_l(z)|^2)^{3/2}}. \end{aligned} \tag{2.3}$$

By (a), (b) and (2.2), we know that when $dist(\varphi(z), \partial\mathbb{D}^n) < \delta$,

$$\sum_{k=1}^{n} \left|\frac{\partial (f_j \circ \varphi)}{\partial z_k}\right| (1-|z_k|^2)^\alpha \leq C\epsilon. \tag{2.4}$$

Let $K = \{w \in \mathbb{D}^n : \text{dist}(w, \partial\mathbb{D}^n) \geq \delta\}$, then it is obvious that K is compact subset of $\mathbb{D}^n$. By (b), for every $l = 1, 2..., n$, $\frac{\partial f_j}{\partial w_l} \to 0$ uniformly on K. Therefore,

$$\sum_{k=1}^{n} \left| \frac{\partial (f_j \circ \varphi)}{\partial z_k} \right| (1 - |z_k|^2)^\alpha \to 0 \tag{2.5}$$

is uniformly on $\text{dist}(\varphi(z), \partial\mathbb{D}^n) \geq \delta$. Consequently, by (2.4) and (2.5), we get

$$\|C_\varphi f_j\|_\alpha = |f_j(\varphi(0))| + \sup_{z \in \mathbb{D}^n} \sum_{k=1}^{n} \left| \frac{\partial (f_j \circ \varphi)}{\partial z_k} \right| (1 - |z_k|^2)^\alpha \to 0.$$

Therefore by Lemma 2.3, we see that $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is compact.

Conversely, suppose that $C_\varphi : H^2(\mathbb{D}^n) \to \mathcal{B}^\alpha(\mathbb{D}^n)$ is compact. Let λ^m be a sequence in $\mathbb{D}^n$ such that $\varphi(\lambda^m) \to \partial\mathbb{D}^n$ as $m \to \infty$. Let $w^m = \varphi(\lambda^m) = (w_1^m, w_2^m, ..., w_n^m)$ and without loss of generality, we assume that l is a positive integer with $|w_l^m| \to 1$ as $m \to \infty$. We take functions

$$f_{m,l}(z) = \frac{\sqrt{1 - |w_l^m|^2}}{1 - \overline{w_l^m} z_l}.$$

Then it is obvious that $f_{m,l}$ is bounded and convergence to 0 uniformly on compact subsets of $\mathbb{D}^n$. By Lemma 2.3, $\|C_\varphi f_{m,l}\|_\alpha \to 0$. By some computations, we get

$$\left| \frac{\partial \varphi_l}{\partial {\lambda^m}_k}(\lambda^m) \right| \frac{(1 - |{\lambda^m}_k|^2)^\alpha}{(1 - |\varphi_l(\lambda^m)|^2)^{3/2}} \to 0 \quad \text{as} \quad m \to \infty.$$

Repeating the above arguments, we can get

$$\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial \lambda_k^m}(\lambda^m) \right| \frac{(1 - |{\lambda^m}_k|^2)^\alpha}{(1 - |\varphi_l(\lambda^m)|^2)^{3/2}} \to 0.$$

Therefore, for any $z \in \mathbb{D}^n$ such that $\text{dist}(\varphi(z), \partial\mathbb{D}^n) < \delta$, we have

$$\sum_{k,l=1}^{n} \left| \frac{\partial \varphi_l}{\partial z_k}(z) \right| \frac{(1 - |z_k|^2)^\alpha}{(1 - |\varphi_l(z)|^2)^{3/2}} \leq \epsilon.$$

The conclusion follows. □

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