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ON A GENERALIZED APERIODIC PERFECT MAP

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Abstract. An aperiodic perfect map(APM) is an array with the property that every array of certain size, called a window, arises exactly once as a contiguous subarray in the array. In this article, we deal with the generalization of APM in higher dimensional arrays. First, we reframe all known definitions onto the generalized n-dimensional arrays. Next, some elementary known results on arrays are generalized to propositions on n-dimensional arrays. Finally, with some devised integer representations, two constructions of infinite family of n-dimensional APMs are generalized from known 2-dimensional constructions in [7].

1. Introduction

The perfect window property is defined as follows; in a given c-ary $m \times n$ array, every c-ary $u \times v$ array occurs exactly once as a contiguous subarray called a window for fixed positive integers u and v with $u \leq m$ and $v \leq n$. An array satisfying the window property is called an aperiodic perfect map(APM) . If the window property holds in a single period of the array, it is called a periodic perfect map(PM).

The perfect window property has been studied firstly in sequences ([1])-a de Bruijn sequence is a c-ary sequence of period n with the window property that for a given positive integer v with $v \leq n$ every possible c-ary v-tuple occurs in a period of the sequence. Not only many construction methods for de Bruijn sequences have been devised ([4]), but also their existence has been settled ([1, 3, 11]).

For the case of array, many results on PM has been known using the theory of finite field. K. Paterson[9, 10] shows that a PM exists for

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each parameter set satisfying the necessary condition for the existence for any alphabet size c of prime power. G. Hurlbert, C. Mitchell, and K. Paterson[5] show that for each possible parameter set there exists a PM with 2×2 windows.

In contrast to the rich theory of PM, not many results of APMs are known in spite of the practical importance of APM for use on encoding schemes for a two dimensional position sending ([2]). For a given PM, an APM of slightly larger size can be obtained by a simple extension which is called the *closure* of the PM (S.Kanetkar and M.Wagh[6]). However, there are still sets of parameters left satisfying necessary conditions for the existence of APM if the array is "long and thin" enough or its alphabet size is not a prime power ([7]). The first result on the existence of APM is given by C. Mitchell only for the binary case with 2×2 windows. In 2002, the complete answer for the existence of APM for 2×2 windows is given by S. Kim[7] for any alphabet size c.

In this article, we generalize the theory of APM to the higher dimensional array. The generalized higher dimensional window property has not been studied while there could be an analogous application to communication such as determining positions in a cuboid with a device which can examine a certain size of subcuboid ([2]). We firstly reframe all known definitions onto the generalized n-dimensional arrays. Next, we generalize some elementary results on arrays to n-dimensional arrays. Finally, two constructions for infinite family of generalized APMs are given, which is the generalizations of constructions given by S. Kim[7]. We devise some representations of integers in terms of linear combination of consecutive multiples of given integers to prove the constructions being APMs.

2. Generalized definitions and preliminary results

For given positive integers m_j with $j = 1, \ldots, n$, let I_j be an m_j-set and let C be a non-empty finite set with $|C| = c$. A function $a : I_1 \times \ldots \times I_n \to C$ is called a c-ary $m_1 \times \ldots \times m_n$ array. Usually, we assume that $I_j = \{0, 1, \ldots, m_j - 1\}$ and the function a corresponds to an array $A = (a_{i_1 \ldots i_n})$, where

$$a_{i_1 \ldots i_n} = a(i_1, \ldots, i_n)$$

for all $(i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n$. The domain of the function a is called the *index set* and the codomain C is called the alphabet set.

A is called an n-dimensional c-ary $m_1 \times \ldots \times m_n$ array if A is c-ary $m_1 \times \ldots \times m_n$ array such that $m_j \geq 2$ for all $j = 1, \ldots, n$.

Let $A = (a_{i_1 \ldots i_n})$ be an n-dim c-ary $m_1 \times \ldots \times m_n$ array and let u_i be a positive integer with $u_i < m_i$ for $i = 1, \ldots, n$. For $(s_1, s_2, \ldots, s_n) \in I_1 \times \ldots \times I_n$, the $(s_1, \ldots, s_n)$-th $u_1 \times \ldots \times u_n$ window of A is a $u_1 \times \ldots \times u_n$ array $(\alpha_{i_1 \ldots i_n})$ written as $[a_{s_1 \ldots s_n}]_{u_1 \times \ldots \times u_n}$ to be defined as

$$\alpha_{i_1 \ldots i_n} = a_{j_1 \ldots j_n},$$

where $j_k \equiv i_k + s_k (mod\ m_k)$ for $0 \leq i_k \leq u_k - 1$, $k = 1, 2, \ldots, n$. For given integers γ_k with $1 \leq \gamma_k \leq m_k$ $(k = 1, 2, \ldots, n)$, we define

$$[A_{\gamma_1 \times \ldots \times \gamma_n}]_{u_1 \times \ldots \times u_n} = \{[a_{s_1 \ldots s_n}]_{u_1 \times \ldots \times u_n} \mid 0 \leq s_k \leq \gamma_k - 1,\ 1 \leq k \leq n\}$$

which is called the set of $u_1 \times \ldots \times u_n$ *windows* for $\gamma_1 \times \ldots \times \gamma_n$. If $\gamma_k = m_k$ for $k = 1, \ldots, n$, we denote $[A_{\gamma_1 \times \ldots \times \gamma_n}]_{u_1 \times \ldots \times u_n}$ by $[A]_{u_1 \times \ldots \times u_n}$. When the size of window is given, it will be simply written as $[A_{\gamma_1 \times \ldots \times \gamma_n}]$.

DEFINITION 1. Let c be a positive integer with $c \geq 2$. For $i = 1, 2, \ldots n$, let m_i and u_i be positive integers satisfying $m_i \geq 2$. An n-dimensional c-ary $m_1 \times \ldots \times m_n$ array $A = (a_{i_1 \ldots i_n})$ is called a $c - ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n)$ n-dimensional Periodic Perfect Map (or simply PM_n) if every c-ary $u_1 \times \ldots \times u_n$ array occurs exactly once as a window $[a_{s_1 \ldots s_n}]_{u_1 \times \ldots \times u_n}$, where s_i satisfies $0 \leq s_i \leq m_i - 1$ for $i = 1, 2, \ldots, n$.

The term "periodic" comes from the way that the windows "wrap-round" the array due to the modulo arithmetic on the indices in Definition 1. We immediately have the following lemma as necessary conditions for the parameters of an PM_n.

LEMMA 2. *If A is an $c - ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n) PM_n$, then the parameters satisfy the following:*

(i) $m_i > u_i$ *for each* $i = 1, 2, \ldots, n$.
(ii) $m_1 m_2 \ldots m_n = c^{u_1 u_2 \ldots u_n}$.

If the positive integers c, m_i and u_i satisfy the necessary conditions for existence of a PM_n in Lemma 2, the set of ordered integers $(c; m_1, \ldots, m_n; u_1, \ldots, u_n)$ are called an *admissible parameter set for* PM_n.

Note that the $c^{u_1 u_2 \ldots u_n}$ windows $[a_{s_1 \ldots s_n}]_{u_1 \times \ldots \times u_n}$ $(0 \leq s_i \leq m_i - 1, i = 1, \ldots, n)$ of a c-ary $(m_1, \ldots, m_n; u_1, \ldots, u_n)\mathrm{PM}_n$ are distinct. We call a 2-dimensional PM_2 just a Periodic Perfect Map (written as PM).

Next, we define an n-dimensional Aperiodic Perfect Map. In this case, the window does not wrap-round in any index.

DEFINITION 3. Let c be a positive integer with $c \geq 2$. For $i = 1, \ldots, n$, let m_i and u_i be positive integers satisfying $m_i \geq 2$. A n-dimensional c-ary $m_1 \times \ldots \times m_n$ array $A = (a_{i_1 \ldots i_n})$ is called a $c-ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n)$ n-dimensional Aperiodic Perfect Map (or simply APM_n) if every c-ary $u_1 \times \ldots \times u_n$ array occurs exactly once as a window $[a_{s_1 \ldots s_n}]_{u_1 \times \ldots \times u_n}$ of A with $0 \leq s_i \leq m_i - u_i$ for $i = 1, \ldots, n$.

The following necessary conditions for the existence of an APM_n are immediate from those of APM stated in [8].

LEMMA 4. *If A is a $c-ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n) APM_n$, then the parameters satisfy the following:*

(i) $m_i \geq u_i$ *for each* $i = 1, \ldots, n$,

(ii) $\prod_{i=1}^{n} (m_i - u_i + 1) = c^{u_1 u_2 \ldots u_n}$.

We call $(c; m_1, m_2, \ldots, m_n; u_1, u_2, \ldots, u_n)$ an *admissible parameter set* for APM_n if it satisfies the conditions given in the above lemma. Note that the simplified notation a $c-ary$ $(m, n; u, v)$APM means the $c-ary$ $(m_1, m_2; u_1, u_2)\mathrm{APM}_2$ where $m_1 = m$, $m_2 = n$, $u_1 = u$ and $u_2 = v$. We remark that, without loss of generality, we restrict our definition of admissible parameters sets for $c-ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n)\mathrm{PM}_n$ (or APM_n) under the condition that $m_1 \leq m_2 \leq \ldots \leq m_n$ (see [7]).

Next, we describe a construction of an APM_n from the simple extension of a PM_n which extends the definition of the closure of PM given in [7].

DEFINITION 5. Let $A = (a_{i_1 \ldots i_n})$ be a $c-ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n)$ PM_n. We define a n-dim c-ary $(m_1 + u_1 - 1) \times \ldots \times (m_n + u_n - 1)$ array $\bar{A} = (\bar{a}_{i_1 \ldots i_n})$ by

$$\bar{a}_{i_1 \ldots i_n} = a_{s_1 \ldots s_n}$$

for $0 \leq i_k \leq m_k + u_k - 1$, $k = 1, 2, \ldots, n$, where $i_k \equiv s_k \pmod{m_k}$. $\bar{A}$ is called the closure of A.

Now we state the following lemma which is generalized from the result due to S. Kanetkar and M. Wage[6].

LEMMA 6. *If A is a $c-ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n)PM_n$ then $\bar{A}$ is a $c-ary$ $(m_1 + u_1 - 1, \ldots, m_n + u_n - 1; u_1, \ldots, u_n)APM_n$.*

PROOF. Let A be a $c-ary$ $(m_1, \ldots, m_n; u_1, \ldots, u_n)\mathrm{PM}_n$. It is clear that

$$[A_{m_1 \times \ldots \times m_n}]_{u_1 \times \ldots \times u_n} = \left[\bar{A}_{m_1 \times \ldots \times m_n}\right]_{u_1 \times \ldots \times u_n}.$$

From the definition of PM_n and APM_n, we have $\tilde{A}$ is a $c-ary$ $(m_1 + u_1 - 1, ..., m_n + u_n - 1; u_1, ..., u_n)\mathrm{APM}_n$. □

REMARK 7. Note that a $c-ary$ $(m_1, ..., m_n; u_1, ..., u_n)\mathrm{APM}_n$ arises by this construction from a $c-ary$ $(m_1-u_1+1, ..., m_n-u_n+1; u_1, u_2, ..., u_n)$ PM_n. The latter must satisfies the conditions that $u_k = m_k - u_k + 1 = 1$ or $m_k - u_k + 1 > u_k$ for $k = 1, 2, ..., n$, i.e., $m_k = u_k = 1$ or $m_k \geq 2u_k$ for $k = 1, 2, ..., n$. Since a PM_n is an n-dim array, so $m_k > 1$ so that the parameters of the APM_n necessary satisfy $m_k \geq 2u_k$ for all $k = 1, 2, ..., n$.

3. Some generalized APM_ns from APM_2s in [7]

In this section two constructions of APMn is given from APM_2 which are devised to give a complete answer for the existence of APM for 2×2 windows (see S. Kim[7]). The one is a c-ary $(u_1, ..., u_{n-1}, m_n; u_1, ..., u_{n-1}, u_n)$APM, and the other is a c-ary $(2u_1 - 1, ..., 2u_{n-1} - 1, m_n ; u_1, ..., u_n)$ APM.

We first observe some properties on integers which immediately follow from representation of integers.

REMARK 8. For $1 \leq i \leq n$, let u_i be a positive integer with $2 \leq u_i$ and let c be any positive integer.

(1) Let $z = \prod_{s=1}^{n} u_s$ and let b be an integer with $0 \leq b \leq c^z - 1$. If we consider b as an integer in base c, then b may be uniquely written as

$$b = \sum_{s=0}^{z-1} b_s c^s,$$

where $0 \leq b_s \leq c - 1$ for each $s = 0, 1, ..., z - 1$.

(2) If t is a non-negative integer such that $0 \leq t \leq (\prod_{s=1}^{n} u_s) - 1$, then t may be written uniquely as

$$t = \pi_1(t) + \pi_2(t)u_1 + \pi_3(t)u_1u_2 + ... + \pi_n(t)u_1u_2...u_{n-1},$$

where $\pi_i(t)$ is an integer with $0 \leq \pi_i(t) \leq u_i - 1$ for $i = 1, 2, ..., n$.

Now we generalize the construction of c-ary $(u, n; u, v)$APM in [7] to the n-dimensional case.

CONSTRUCTION 9. *c-ary $(u_1, ..., u_{n-1}, m_n ; u_1, ..., u_{n-1}, u_n)APM_n$. Let $(c; u_1, ..., u_{n-1}, m_n; u_1, ..., u_{n-1}, u_n)$ be an admissible parameter set for APM_n. Note that from Lemma 4, we have*

$$m_n = c^{u_1u_2...u_n} + u_n - 1.$$

(1) *Let* $B = (b_s)$ *be a* $c^{u_1 u_2 \dots u_{n-1}}$*-ary span* u_n *de Bruijn sequence and* $\bar{B} = (\bar{b}_t)$ *be the closure of* B. *i.e.,* $\bar{B} = (\bar{b}_t)$ *is defined by* $\bar{b}_t = b_s$ *if and only if* $t = s \pmod{c^{u_1 u_2 \dots u_n}}$ *and it is a 1-dim* $c^{u_1 u_2 \dots u_{n-1}}$*-ary* m_i *array which is a* $c^{u_1 u_2 \dots u_{n-1}}$*-ary* $(c^{u_1 u_2 \dots u_n} + u_n - 1; u_n)APM_1$.

(2) *Let* $t = s \pmod{c^{u_1 u_2 \dots u_n}}$. *For any* $\bar{b}_t = b_s$, *from Remark 8,*

(i) b_s *can be written as*

$$b_s = \sum_{i=0}^{u_1 u_2 \dots u_{n-1} - 1} (b_s)_i c^i,$$

where $0 \le (b_s)_i \le c - 1$ *for all* $i \in \mathbb{N}_{u_1 u_2 \dots u_{n-1}}$.

(ii) *Each* $i \in \mathbb{N}_{u_1 u_2 \dots u_{n-1}}$ *may be uniquely written as*

$$i = i_1 + i_2 u_1 + i_3 u_1 u_2 + \dots + i_{n-1} u_1 u_2 \dots u_{n-2},$$

where $0 \le i_t \le u_t - 1$ *for* $t = 1, \dots, n-1$.

Hence, we write $(b_s)_i = (b_s)_{i_1 + i_2 u_1 + i_3 u_1 u_2 + \dots + i_{n-1} u_1 u_2 \dots u_{n-2}}$.

(3) Define a n-dim c-ary $u_1 \times \dots \times u_{n-1} \times m_n$ array $A = (a_{i_1 i_2 \dots i_{n-1} i_n})$ *by*

$$a_{i_1 i_2 \dots i_{n-1} i_n} = (b_{i_n})_{i_1 + i_2 u_1 + i_3 u_1 u_2 + \dots + i_{n-1} u_1 u_2 \dots u_{n-2}},$$

where $0 \le i_t \le u_t - 1$ *for* $t = 1, \dots, n-1$ *and* $0 \le i_n \le m_n = c^{u_1 u_2 \dots u_n} + u_n - 2$.

THEOREM 10. *Let* $(c; u_1, \dots, u_{n-1}, m_n\ ;\ u_1, \dots, u_{n-1}, u_n)$ *be an admissible parameter set for* APM_n. *Let* B *be a* $c^{u_1 u_2 \dots u_{n-1}}$*-ary span* u_n *de Bruijn sequence. Then the* n*-dim* $u_1 \times \dots \times u_{n-1} \times (c^{uv} + v - 1)$ *array constructed obtained from Construction 9 is a c-ary* $(u_1, \dots, u_{n-1}, m_n; u_1, \dots, u_{n-1}, u_n)APM_n$.

PROOF. We prove that $A = (a_{i_1 i_2 \dots i_{n-1} i_n})$ in Construction 9 is a c-ary $(u_1, \dots, u_{n-1}, m_n; u_1, \dots, u_n)$ APM$_n$. Note that since $m_i = u_i$ for $i = 1, \dots, n-1$ and $m_n = c^{u_1 u_2 \dots u_n} + u_n - 1$, any $u_1 \times \dots \times u_n$ window $[a_{k_1 k_2 \dots k_n}]$ in $[A_{1 \times 1 \times \dots \times 1 \times (m_n - u_n + 1)}]$ has $k_j = 0$ for $j = 1, \dots, n-1$ and $0 \le k_n \le c^{u_1 u_2 \dots u_n} - 1$. Let $[a_{0\,0 \dots 0\,l}] = (\alpha_{s_1 \dots s_n})$ and $[a_{0\,0 \dots 0\,k}] = (\beta_{s_1 \dots s_n})$ be any two $u_1 \times \dots \times u_n$ windows in $[A_{1 \times 1 \times \dots \times 1 \times (m_n - u_n + 1)}]$, where $0 \le l, k \le c^{u_1 u_2 \dots u_n} - 1$. Suppose that $[a_{0\,0 \dots 0\,l}] = [a_{0\,0 \dots 0\,k}]$. Then, for all $s_t = 0, 1, \dots, u_t - 1$, $t = 0, 1, \dots, n$, we have

$$\alpha_{s_1 \dots s_n} = a_{s_1 \dots s_{n-1}\, s_n + l} = a_{s_1 \dots s_{n-1}\, s_n + k} = \beta_{s_1 \dots s_n}.$$

From Construction 9, it implies

$$(b_{s_n + l})_{s_1 + s_2 u_1 + s_3 u_1 u_2 + \dots + s_{n-1} u_1 u_2 \dots u_{n-2}}$$
$$= (b_{s_n + k})_{s_1 + s_2 u_1 + s_3 u_1 u_2 + \dots + s_{n-1} u_1 u_2 \dots u_{n-2}}.$$

It immediately follows that for all $b_{l+s_n} = b_{k+s_n}$ for s_n with $0 \leq s_n \leq u_n - 1$, which implies $[b_l]_{u_n} = [b_k]_{u_n}$ in $[B_{c^{u_1 u_2 \dots u_{n-1}}}]_{u_n}$. From the definition of de Bruijn sequence, it follows that $l = k$.

Therefore, $\left|[A_{1\times\dots\times 1\times(m_n-u_n+1)}]\right| = c^{u_1 u_2 \dots u_n}$ which implies that each c-ary $u_1 \times \dots \times u_n$ array occurs exactly once in $[A_{1\times 1\times\dots\times 1\times(m_n-u_n+1)}]$ and A is a c-ary $(u_1, \dots, u_{n-1}, m_n; u_1, \dots, u_n)\mathrm{APM}_n$. $\square$

Next, we generalize the construction of c-ary $(2u-1, n; u, v)\mathrm{APM}_2$ given in [7] to the n-dimensional case.

CONSTRUCTION 11. *c-ary $(2u_1 - 1, \dots, 2u_{n-1} - 1, m_n; u_1, \dots, u_{n-1}, u_n)APM_n$. Let $(c; 2u_1-1, \dots, 2u_{n-1}-1, m_n; u_1, \dots, u_n)$ be an admissible parameter set for APM_n. Note that, from Lemma 4, we have*

$$m_n = \frac{c^{u_1 u_2 \dots u_n}}{u_1 u_2 \dots u_{n-1}} + u_n - 1$$

and so $c^{u_1 u_2 \dots u_n}$ is divisible by $u_1 u_2 \dots u_{n-1}$. Let $B = (b_i)$ be a c-ary span $u_1 u_2 \dots u_n$ de Bruijn sequence and let $\bar{B} = (\bar{b}_i)$ be the closure of B which is a c-ary $(1, c^{u_1 u_2 \dots u_n} + u_1 u_2 \dots u_n - 1; 1, u_1 u_2 \dots u_n)APM$. Define a n-dim $(2u_1 - 1) \times \dots \times (2u_{n-1} - 1) \times m_n$ array $A = (a_{i_1 \dots i_n})$ by

$$a_{i_1 i_2 \dots i_n} = \bar{b}_{i_1 + i_2 u_1 + i_3 u_1 u_2 + \dots + i_n u_1 u_2 \dots u_{n-1}}.$$

for $0 \leq i_j \leq 2u_j - 2$, $j = 1, \dots, n-1$ and $0 \leq u_j \leq m_n - 1 = \frac{c^{u_1 u_2 \dots u_n}}{u_1 u_2 \dots u_{n-1}} + u_n - 2$. Note that $\bar{b}_k = b_l$ for $l \equiv k \pmod{c^{u_1 u_2 \dots u_n}}$.

THEOREM 12. *Let B be a c-ary span $u_1 u_2 \dots u_n$ de Bruijn sequence. Let $(c; 2u_1-1, \dots, 2u_{n-1}-1, m_n; u_1, \dots, u_n)$ be an admissible parameter set for APM_n. Then the n-dim $(2u_1 - 1) \times \dots \times (2u_{n-1} - 1) \times m_n$ array constructed from B by Construction 11 is a c-ary $(2u_1-1, \dots, 2u_{n-1}-1, m_n; u_1, \dots, u_n)APM_n$.*

PROOF. Let $(c; 2u_1 - 1, \dots, 2u_{n-1} - 1, m_n; u_1, \dots, u_n)$ be an admissible parameter set for APM_n. Let $A = (a_{i_1 \dots i_n})$ be an array constructed as in Construction 11 from a $c-ary$ span $u_1 u_2 \dots u_n$ de Bruijn sequence $B = (b_i)$. Then, from Lemma 4, we have

$$m_n = \frac{c^{u_1 u_2 \dots u_n}}{u_1 u_2 \dots u_{n-1}} + u_n - 1$$

and $c^{u_1 u_2 \dots u_n}$ is divisible by $u_1 u_2 \dots u_{n-1}$. Let $\bar{B} = (\bar{b}_i)$ be the closure of B. For each i_k with $0 \leq i_k \leq u_k - 1$ $(k = 1, \dots, n)$, let $[a_{s_1 s_2 \dots s_n}] = (\alpha_{i_1 i_2 \dots i_n})$ and $[a_{t_1 t_2 \dots t_n}] = (\beta_{i_1 i_2 \dots i_n})$ be $u_1 \times \dots \times u_n$ windows in

$$[A_{u_1 \times \dots \times u_{n-1} \times (m_n - u_n + 1)}]_{u_1 \times \dots \times u_n}.$$

Note that $0 \le s_i, t_i \le u_i - 1$ for $i = 1, \dots, n-1$ and $0 \le s_n, t_n \le m_n - u_n = \frac{c^{u_1 u_2 \dots u_n}}{u_1 u_2 \dots u_{n-1}} - 1$.

Suppose that $[a_{s_1\, s_2\, \dots\, s_n}] = [a_{t_1\, t_2\, \dots\, t_n}]$. Then, for $0 \le i_k \le u_k - 1$ and $k = 1, \dots, n$, we have $\alpha_{i_1\, i_2\, \dots\, i_n} = \beta_{i_1\, i_2\, \dots\, i_n}$ so that

$$\begin{aligned}\alpha_{i_1\, i_2\, \dots\, i_n} &= a_{s_1+i_1\, s_2+i_2\, \dots\, s_n+i_n} \\ &= \bar{b}_{(s_1+i_1)+(s_2+i_2)u_1+(s_3+i_3)u_1u_2+\dots+(s_n+i_n)u_1u_2\dots u_{n-1}} \\ &= \bar{b}_{(t_1+i_1)+(t_2+i_2)u_1+(t_3+i_3)u_1u_2+\dots+(t_n+i_n)u_1u_2\dots u_{n-1}} \\ &= a_{t_1+i_1\, t_2+i_2\, \dots\, t_n+i_n} \\ &= \beta_{i_1\, i_2\, \dots\, i_n}.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\bar{b}_{(i_1+i_2u_1+i_3u_1u_2+\dots+i_nu_1u_2\dots u_{n-1})+(s_1+s_2u_1+s_3u_1u_2+\dots+s_nu_1u_2\dots u_{n-1})} \\ &= \bar{b}_{(i_1+i_2u_1+i_3u_1u_2+\dots+i_nu_1u_2\dots u_{n-1})+(t_1+t_2u_1+t_3u_1u_2+\dots+t_nu_1u_2\dots u_{n-1})}.\end{aligned}$$

Let $l = i_1 + i_2u_1 + i_3u_1u_2 + \dots + i_nu_1u_2\dots u_{n-1}$. Then, for all l with $0 \le l \le u_1u_2\dots u_n - 1$, we have

$$\bar{b}_{l+(s_1+s_2u_1+s_3u_1u_2+\dots+s_nu_1u_2\dots u_{n-1})} = \bar{b}_{l+(t_1+t_2u_1+t_3u_1u_2+\dots+t_nu_1u_2\dots u_{n-1})}$$

which implies that

$$\begin{aligned}&[b_{s_1+s_2u_1+s_3u_1u_2+\dots+s_nu_1u_2\dots u_{n-1}}]_{u_1u_2\dots u_n} \\ &= [b_{t_1+t_2u_1+t_3u_1u_2+\dots+t_nu_1u_2\dots u_{n-1}}]_{u_1u_2\dots u_n}.\end{aligned}$$

From the definition of a de Bruijn sequence B, since $0 \le s_k, t_k \le u_k - 1$ when $1 \le k \le n-1$ and $0 \le s_n, t_n \le c^{u_1u_2\dots u_n} - 1$, we have

$$\begin{aligned}&s_1 + s_2u_1 + s_3u_1u_2 + \dots + s_nu_1u_2\dots u_{n-1} \\ &= t_1 + t_2u_1 + t_3u_1u_2 + \dots + t_nu_1u_2\dots u_{n-1}.\end{aligned}$$

By Remark 8, we have $s_i = t_i$ for all $i = 1, \dots, 0$.

Therefore, we conclude that $[a_{s_1\, s_2\, \dots\, s_n}] = [a_{t_1\, t_2\, \dots\, t_n}]$ if and only if $s_i = t_i$ for all $i = 1, \dots, 0$. Thus, we have

$$\left|[A_{(m_1-u_1+1)\times\dots\times(m_n-u_n+1)}]\right| = \left|[A_{u_1\times\,\dots\,\times u_{n-1}\times(\frac{c^{u_1u_2\dots u_n}}{u_1u_2\dots u_{n-1}})}]\right| = c^{u_1u_2\dots u_n}$$

so that every c-ary $u_1 \times \dots \times u_n$ array occurs exactly once as a window i.e., A is a c-ary $(2u_1-1, \dots, 2u_{n-1}-1, m_n; u_1, \dots, u_n)\mathrm{APM}_n$. □

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