

RANK-PRESERVING OPERATORS OF NONNEGATIVE INTEGER MATRICES

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ABSTRACT. The set of all $m \times n$ matrices with entries in \mathbb{Z}_+ is denoted by $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. We say that a linear operator T on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ is a (U, V) -operator if there exist invertible matrices $U \in \mathbb{M}_{m \times m}(\mathbb{Z}_+)$ and $V \in \mathbb{M}_{n \times n}(\mathbb{Z}_+)$ such that either $T(X) = UXV$ for all X in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, or $m = n$ and $T(X) = UX^tV$ for all X in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. In this paper we show that a linear operator T preserves the rank of matrices over the nonnegative integers if and only if T is a (U, V) -operator. We also obtain other characterizations of the linear operator that preserves rank of matrices over the nonnegative integers.

1. Introduction

One of the most active and fertile subjects in matrix theory is the study of those linear operators on matrices that leave certain properties or relations of matrices invariant. Although the linear preservers concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings ([1]–[4], [8]).

Marcus and Moyls [6, 7], and Westwick [10] presented characterizations of the linear operators that preserve the ranks of matrices over the algebraically closed field. In [5], Lautemann extended this to an arbitrary field.

Also, Beasley and Pullman [1, 2] extended the results of fields to several semiring; the Boolean algebra and the chain semirings, etc.

In this paper, we characterize the linear operators that preserve the ranks of all matrices over the nonnegative integers.

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2. Preliminaries and basic results

Let \mathbb{Z}_+ be the nonnegative part of the ring \mathbb{Z} of integers. Let S be the subset of \mathbb{Z}_+^n , where n is a positive integer. Then $\text{span}(S)$ is the set of all finite linear combinations of members of S over \mathbb{Z}_+ . A *semidomain* \mathbb{V} generated by S is the $\text{span}(S)$. If there exists a finite subset S of a semidomain \mathbb{V} such that $\mathbb{V} = \text{span}(S)$, then \mathbb{V} is called a *finitely generated semidomain*. The elements of a semidomain are called *vectors*. A nonzero vector $\mathbf{x} = [x_i]$ in \mathbb{Z}_+^n is *irreducible* if the greatest common divisor of x_i 's is 1 (That is, $\gcd(x_1, \dots, x_n) = 1$).

The set of vectors $\{\mathbf{v}_i \mid i \in I\}$ is called a *basis* of a semidomain \mathbb{V} if $\text{span}(\{\mathbf{v}_i \mid i \in I\}) = \mathbb{V}$ and no proper subset of $\{\mathbf{v}_i \mid i \in I\}$ spans \mathbb{V} . A set S of vectors in a semidomain \mathbb{V} is called *linearly dependent* if there exists a vector \mathbf{x} in S such that $\mathbf{x} \in \text{span}(S \setminus \{\mathbf{x}\})$; S is called *linearly independent* if it is not linearly dependent. Thus an independent set cannot contain a zero vector. Also a basis of a semidomain is linearly independent.

LEMMA 2.1. *If \mathbb{V} is a finitely generated semidomain over \mathbb{Z}_+ , a basis of \mathbb{V} is uniquely determined.*

PROOF. Let $S = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_m\}$ and $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ be bases of \mathbb{V} , where m and n are some positive integers. Then we have $\text{span}(X) = \text{span}(S)$. We will claim that $S = X$. Let \mathbf{s} be any element in S . Then \mathbf{s} is a linear combination of members of X , each of which is a linear combination of members of S . Thus, there exist scalars $y_i, \alpha_{ji} \in \mathbb{Z}_+$ for each $i = 1, \dots, n$ and $j = 1, \dots, m$ such that

$$(2.1) \quad \mathbf{s} = \sum_{i=1}^n y_i \mathbf{x}_i = \sum_{i=1}^n y_i \left(\sum_{j=1}^m \alpha_{ji} \mathbf{s}_j \right),$$

where $\sum_{i=1}^n y_i \neq 0$ and $\sum_{j=1}^m \alpha_{ji} \neq 0$ because each vector is not zero. Without loss of generality, we may assume that $\mathbf{s} = \mathbf{s}_1$ and $y_1 \neq 0$. Then (2.1) becomes

$$(2.2) \quad \mathbf{s}_1 = \left(\sum_{i=1}^n y_i \alpha_{1i} \right) \mathbf{s}_1 + \cdots + \left(\sum_{i=1}^n y_i \alpha_{mi} \right) \mathbf{s}_m,$$

equivalently

$$(2.3) \quad \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $y_1 \neq 0$, we have $\alpha_{i1} = 0$ for all $i = 2, \dots, m$ because $\alpha_{i1}y_1 = 0$ for each $i = 2, \dots, m$ by (2.3). It follows from $\sum_{j=1}^m \alpha_{j1} \neq 0$ that $\alpha_{11} \neq 0$.

Now, we will show that $y_k = 0$ for all $k = 2, \dots, n$. If $\alpha_{1k} \neq 0$, we obtain $y_k = 0$ because $\sum_{i=1}^n \alpha_{1i}y_i = 1$ and $\alpha_{11}y_1 \neq 0$. If $\alpha_{1k} = 0$, it follows from $\sum_{j=1}^m \alpha_{j1} \neq 0$ that $\alpha_{lk} \neq 0$ for some $l = 2, \dots, m$. Thus we have $y_k = 0$ because $\sum_{i=1}^n \alpha_{li}y_i = 0$ by (2.3). Therefore we have established

$$\mathbf{s}_1 = y_1 \mathbf{x}_1 = y_1 \alpha_{11} \mathbf{s}_1$$

by (2.1) and (2.2). It follows that $y_1 = \alpha_{11} = 1$, and hence $\mathbf{s}_1 = \mathbf{x}_1 \in X$. This implies $S \subseteq X$. A parallel argument shows that $S \supseteq X$. Therefore a basis of \mathbb{V} is uniquely determined. □

In Lemma 2.1, the cardinality of a basis of a finitely generated semidomain \mathbb{V} is called the *dimension* of \mathbb{V} , denoted by $\dim(\mathbb{V})$.

In contrast with vector spaces over fields, a semidomain \mathbb{V} over \mathbb{Z}_+ may have several sub-semidomains with the same dimension as \mathbb{V} . For example, let $\mathbf{x} = [0, 1, 2]^t$, $\mathbf{y} = [2, 1, 0]^t$ and $\mathbf{z} = [2, 2, 2]^t$. Let $\mathbb{V} = \text{span}\{\mathbf{x}, \mathbf{y}\}$. We then have that $\text{span}\{\mathbf{x}, \mathbf{z}\}$ and $\text{span}\{\mathbf{y}, \mathbf{z}\}$ are two-dimensional sub-semidomains of \mathbb{V} , neither of which equals \mathbb{V} .

Even more disconcerting, a semidomain \mathbb{V} can have sub-semidomains whose dimensions *exceed* $\dim(\mathbb{V})$. For example, let

$$S = \left\{ \begin{bmatrix} a_1 \\ b_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a_2 \\ 0 \\ b_2 \\ 0 \end{bmatrix}, \begin{bmatrix} a_3 \\ 0 \\ 0 \\ b_3 \end{bmatrix}, \begin{bmatrix} 0 \\ a_4 \\ b_4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ a_5 \\ 0 \\ b_5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ a_6 \\ b_6 \end{bmatrix} \right\}$$

be a subset of \mathbb{Z}_+^4 , where $a_i b_i \neq 0$ for each $i = 1, 2, \dots, 6$. If $\mathbb{V} = \text{span}(S)$, then $\dim(\mathbb{V}) = 6$ because S is linearly independent, even though \mathbb{V} is a sub-semidomain of the 4-dimensional semidomain \mathbb{Z}_+^4 .

If \mathbb{V} and \mathbb{W} are semidomains over \mathbb{Z}_+ , a mapping $T : \mathbb{V} \rightarrow \mathbb{W}$ is called a *linear transformation* if

$$T(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and for all $\alpha, \beta \in \mathbb{Z}_+$. If $\mathbb{V} = \mathbb{W}$, the word *operator* is used instead of transformation. If $T : \mathbb{V} \rightarrow \mathbb{W}$ is linear, then it is obvious that the behavior of T on basis of \mathbb{V} determines the behavior of T completely. This proves:

LEMMA 2.2. *For every linear transformation T on a finitely generated semidomain \mathbb{V} , we have $\dim(T(\mathbb{V})) \leq \dim(\mathbb{V})$.*

Let \mathbb{V} and \mathbb{W} be two semidomains over \mathbb{Z}_+ . A linear transformation $T : \mathbb{V} \rightarrow \mathbb{W}$ is called *injective* if $T(\mathbf{x}) = T(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. The map T is called *surjective* if $T(\mathbb{V}) = \mathbb{W}$. The map T is called *invertible* if it is both injective and surjective.

LEMMA 2.3. *Let \mathbb{V} and \mathbb{W} be finitely generated semidomains over \mathbb{Z}_+ . If $T : \mathbb{V} \rightarrow \mathbb{W}$ is an injective linear transformation, then*

$$\dim(T(\mathbb{V})) = \dim(\mathbb{V}).$$

PROOF. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be any basis of \mathbb{V} . Then

$$S = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$$

spans $T(\mathbb{V})$. Suppose that S is linearly dependent. Then there exists an index j in $\{1, \dots, n\}$ such that $T(\mathbf{v}_j) = \sum_{i=1, i \neq j}^n \alpha_i T(\mathbf{v}_i)$ with $\sum_{i=1, i \neq j}^n \alpha_i \neq 0$, where $\alpha_i \in \mathbb{Z}_+$. Since T is injective, we obtain $\mathbf{v}_j = \sum_{i=1, i \neq j}^n \alpha_i \mathbf{v}_i$, a contradiction to the fact that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is linearly independent. Therefore S is a basis of $T(\mathbb{V})$. \square

LEMMA 2.4. *Assume that $T : \mathbb{V} \rightarrow \mathbb{W}$ is a surjective linear transformation for finitely generated semidomains \mathbb{V} and \mathbb{W} over \mathbb{Z}_+ . Then T is invertible if and only if T preserves the dimension of every sub-semidomain of \mathbb{V} .*

PROOF. If T is invertible, then the conclusion follows from Lemma 2.3. Conversely, assume that T preserves the dimension of every sub-semidomain of \mathbb{V} . If T is not injective, then there exist distinct nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ such that $T(\mathbf{x}) = T(\mathbf{y})$. If \mathbf{x} and \mathbf{y} are linearly dependent, then $\mathbf{x} = \alpha\mathbf{y}$ or $\mathbf{y} = \beta\mathbf{x}$ for some nonzero scalars $\alpha, \beta \in \mathbb{Z}_+$. This

implies that $T(\mathbf{x}) = T(\mathbf{y}) = \mathbf{0}$ and hence T reduces the dimension of $\text{span}(\{\mathbf{x}\})$ or $\text{span}(\{\mathbf{y}\})$ \square

THEOREM 2.5. *If T is a surjective linear operator on a finitely generated semidomain \mathbb{V} over \mathbb{Z}_+ , then the following statements are equivalent:*

- (1) T is invertible;
- (2) T permutes the basis of \mathbb{V} ;
- (3) T preserves the dimension of every sub-semidomain of \mathbb{V} .

PROOF. The equivalence of (1) and (3) follows from Lemma 2.4. Clearly (2) implies (1). So, it suffices to show that (1) implies (2). If T is invertible and $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of \mathbb{V} , then $X = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is a basis of $T(\mathbb{V}) = \mathbb{V}$ by Lemma 2.3. Then we have $X = S$ from Lemma 2.1, and hence T permutes the basis of \mathbb{V} . \square

Let T be a linear operator on \mathbb{Z}_+^n defined by $T(X) = \alpha X$ for all $X \in \mathbb{Z}_+^n$, where $\alpha (\geq 2)$ is in \mathbb{Z}_+ . Then we can easily show that T is not surjective while T is injective. Therefore, for a linear operator T on a finitely generated semidomain over \mathbb{Z}_+ , the injectiveness and the surjectiveness of T are not equivalent.

3. Rank-1 matrices and their preservers

The set of all $m \times n$ matrices with entries in \mathbb{Z}_+ is denoted by $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Addition, multiplication by scalars, and the product of matrices over \mathbb{Z}_+ are similarly defined as over a field. We identify $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ with \mathbb{Z}_+^{mn} in the usual way when we discuss it as a semidomain and consider its sub-semidomains. Let

$$\mathbb{E}_{m,n} = \{E_{ij} \mid i = 1, \dots, m \text{ and } j = 1, \dots, n\},$$

where E_{ij} is the matrix whose $(i, j)^{\text{th}}$ entry is 1 and whose other entries are 0. We call each member of $\mathbb{E}_{m,n}$ a *cell*. Clearly, $\mathbb{E}_{m,n}$ is the *standard* basis of a semidomain $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ and thus $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ is the finitely generated semidomain.

PROPOSITION 3.1. *The only invertible matrices in $\mathbb{M}_{n \times n}(\mathbb{Z}_+)$ are permutation matrices.*

PROOF. Let $A = [a_{ij}]$ be an invertible matrix in $\mathbb{M}_{n \times n}(\mathbb{Z}_+)$. Define the $(0, 1)$ -matrix $B = [b_{ij}]$ by $b_{ij} = 1$ if and only if $a_{ij} > 0$ for all $i, j = 1, \dots, n$. Then B is an invertible Boolean matrix, and hence B

is a permutation matrix (See [1]). Therefore A is a permutation matrix because "1" is the only unit element of \mathbb{Z}_+ . \square

If \mathbf{a} and \mathbf{b} are nonzero vectors in \mathbb{Z}_+^n , we denote $\mathbf{a} \simeq \mathbf{b}$ if \mathbf{a} and \mathbf{b} have an irreducible common factor. That is, $\mathbf{a} \simeq \mathbf{b}$ if and only if there exists an irreducible vector \mathbf{p} in \mathbb{Z}_+^n such that $\mathbf{a} = \alpha\mathbf{p}$ and $\mathbf{b} = \beta\mathbf{p}$ for some nonzero integers α and β . Then we can easily show that \simeq is an equivalence relation in \mathbb{Z}_+^n .

PROPOSITION 3.2. *If $\mathbf{a} = [a_1, \dots, a_n]^t$ and $\mathbf{b} = [b_1, \dots, b_n]^t$ are nonzero vectors in \mathbb{Z}_+^n with $\alpha\mathbf{a} = \beta\mathbf{b}$ for some nonzero scalars $\alpha, \beta \in \mathbb{Z}_+$, then we have $\mathbf{a} \simeq \mathbf{b}$.*

PROOF. Let $\alpha' = \gcd(a_1, \dots, a_n)$. Then there exists an irreducible vector \mathbf{p} in \mathbb{Z}_+^n such that $\mathbf{a} = \alpha'\mathbf{p}$. Thus $\alpha\mathbf{a} = \beta\mathbf{b}$ becomes

$$(3.1) \quad \alpha\alpha'\mathbf{p} = \beta\mathbf{b}.$$

Let $\gamma = \gcd(\alpha\alpha', \beta)$, $\gamma_1 = \frac{\alpha\alpha'}{\gamma}$ and $\gamma_2 = \frac{\beta}{\gamma}$. Then γ_1 and γ_2 are nonzero in \mathbb{Z}_+ , and (3.1) becomes

$$(3.2) \quad \gamma_1\mathbf{p} = \gamma_2\mathbf{b}.$$

Therefore we have γ_1 divides every $\gamma_2 b_i$ for all $i = 1, \dots, n$. Since γ_1 is relatively prime to γ_2 , it follows that γ_1 divides every entry in \mathbf{b} . Thus we have $\mathbf{b} = \gamma_1\mathbf{c}$ for some nonzero vector \mathbf{c} in \mathbb{Z}_+^n . By the cancellation, (3.2) becomes $\mathbf{p} = \gamma_2\mathbf{c}$. Then γ_2 is a unit in \mathbb{Z}_+ because γ_2 divides every entry in the irreducible vector \mathbf{p} . That is, $\gamma_2 = 1$ so that $\mathbf{b} = \gamma_1\mathbf{p}$. Therefore \mathbf{a} and \mathbf{b} have an irreducible common factor \mathbf{p} , and thus $\mathbf{a} \simeq \mathbf{b}$. \square

The *rank* or *factor rank*, $r(A)$, of a nonzero matrix $A \in \mathbb{M}_{m \times n}(\mathbb{Z}_+)$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. It follows that for a nonzero matrix $A \in \mathbb{M}_{m \times n}(\mathbb{Z}_+)$, $r(A) = k$ if and only if k is the smallest positive integer for which there exist k rank-1 matrices whose sum is A . The rank of a zero matrix is zero.

It is obvious that for a matrix A in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, $r(A) = 1$ if and only if there exist two nonzero vectors $\mathbf{a} \in \mathbb{Z}_+^m$ and $\mathbf{b} \in \mathbb{Z}_+^n$ such that $A = \mathbf{a}\mathbf{b}^t$. We call \mathbf{a} the *left factor*, and \mathbf{b} the *right factor* of A . But these vectors \mathbf{a} and \mathbf{b} are not uniquely determined by A . For example,

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \left(2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

For any index $i \in \{1, \dots, n\}$, we denote $\mathbf{e}_i^{(n)}$ as the irreducible vector in \mathbb{Z}_+^n with “1” in i^{th} position and zero elsewhere. Thus for any cell E_{ij} in $\mathbb{E}_{m,n}$, we have $E_{ij} = \mathbf{e}_i^{(m)} \mathbf{e}_j^{(n)t}$.

LEMMA 3.3. *Let A and B be rank-1 matrices in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ with factorizations $A = \mathbf{a}\mathbf{x}^t$ and $B = \mathbf{b}\mathbf{y}^t$. Then $r(A + B) = 1$ if and only if $\mathbf{a} \simeq \mathbf{b}$ or $\mathbf{x} \simeq \mathbf{y}$.*

PROOF. Suppose that $r(A + B) = 1$. Let

$$A = \mathbf{a}\mathbf{x}^t = [x_1\mathbf{a}, \dots, x_n\mathbf{a}] = [a_1\mathbf{x}^t, \dots, a_m\mathbf{x}^t]^t$$

and

$$B = \mathbf{b}\mathbf{y}^t = [y_1\mathbf{b}, \dots, y_n\mathbf{b}] = [b_1\mathbf{y}^t, \dots, b_m\mathbf{y}^t]^t.$$

If $A + B$ has exactly one nonzero i^{th} row or exactly one nonzero j^{th} column, so do A and B . In this case, A and B have an irreducible common left factor $\mathbf{e}_i^{(m)}$ or an irreducible common right factor $\mathbf{e}_j^{(n)}$. Thus we have $\mathbf{a} \simeq \mathbf{b}$ or $\mathbf{x} \simeq \mathbf{y}$.

Now, assume that $A + B$ has at least two nonzero rows and at least two nonzero columns. Furthermore, without loss of generality, we may assume that columns of $A + B$ are all nonzero.

Case 1. $x_i y_i = 0$ for some $i = 1, \dots, n$. If $x_i = 0$, then $y_i \neq 0$ because $A + B$ has no zero column. Since A is not a zero matrix, there exists an index j different from i such that $x_j \neq 0$. Therefore, the i^{th} and j^{th} columns of $A + B$ are $y_i \mathbf{b}$ and $x_j \mathbf{a} + y_j \mathbf{b}$, respectively. Since $r(A + B) = 1$, there exist nonzero scalars α, β in \mathbb{Z}_+ such that $\alpha y_i \mathbf{b} = \beta(x_j \mathbf{a} + y_j \mathbf{b})$, equivalently $\beta x_j \mathbf{a} = |\alpha y_i - y_j| \mathbf{b}$. Since $\beta x_j \neq 0$, we have $|\alpha y_i - y_j| \neq 0$. It follows from Proposition 3.2 that $\mathbf{a} \simeq \mathbf{b}$. Similarly, a parallel argument holds if $y_i = 0$.

Case 2. $x_i y_i \neq 0$ for all $i = 1, \dots, n$. Consider any two i^{th} and j^{th} columns of $A + B$. Since $r(A + B) = 1$, there exist two nonzero scalars α and β in \mathbb{Z}_+ such that $\alpha(x_i \mathbf{a} + y_i \mathbf{b}) = \beta(x_j \mathbf{a} + y_j \mathbf{b})$, equivalently $|\alpha x_i - \beta x_j| \mathbf{a} = |\beta y_j - \alpha y_i| \mathbf{b}$. If $|\alpha x_i - \beta x_j| \neq 0$ for some i and j with $i \neq j$, then we have $|\beta y_j - \alpha y_i| \neq 0$. By Proposition 3.2, we have $\mathbf{a} \simeq \mathbf{b}$. Now, if $|\alpha x_i - \beta x_j| = 0$ for all distinct indices i and j , then $|\beta y_j - \alpha y_i| = 0$ for all i and j . Thus,

$$\alpha x_i = \beta x_j \quad \text{and} \quad \beta y_j = \alpha y_i.$$

This implies that $x_i y_j = x_j y_i$ for all $i, j = 1, \dots, n$. Thus there exist nonzero integers s and t such that $s x_i = t y_i$ for all $i = 1, \dots, n$. Therefore we have $s \mathbf{x} = t \mathbf{y}$. It follows from Proposition 3.2 that $\mathbf{x} \simeq \mathbf{y}$. Thus we have shown the sufficiency.

The necessity is an immediate consequence. □

We say that a linear operator T on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ is a *rank- k preserver* if $r(A) = k$ implies $r(T(A)) = k$ for all $A \in \mathbb{M}_{m \times n}(\mathbb{Z}_+)$.

Let $R_i = \sum_{j=1}^n E_{ij}$ for all $i = 1, \dots, m$ and $C_j = \sum_{i=1}^m E_{ij}$ for all $j = 1, \dots, n$. We call R_i an i^{th} *row matrix*, and C_j a j^{th} *column matrix*. A *line matrix* is a row matrix or a column matrix. Clearly, every line matrix has rank-1. Let $\mathbb{R} = \{R_i \mid 1 \leq i \leq m\}$ and $\mathbb{C} = \{C_j \mid 1 \leq j \leq n\}$.

LEMMA 3.4. *If T is an invertible rank-1 preserver on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, then T maps every line matrix into a line matrix.*

PROOF. Suppose that T does not map some line matrix into a line matrix. Without loss of generality, we may assume that T does not map i^{th} row matrix into a line matrix. Since T is invertible on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, it follows from Theorem 2.5 that T permutes $\mathbb{E}_{m,n}$. Thus, there exist two distinct cells E_{ij} and E_{ik} in $\mathbb{E}_{m,n}$ such that $T(E_{ij} + E_{ik}) = E_{pq} + E_{rs}$, where $p \neq r$ and $q \neq s$. But then we have $r(E_{ij} + E_{ik}) = 1$, while $r(T(E_{ij} + E_{ik})) = 2$, a contradiction. Therefore T maps every line matrix into a line matrix. □

COROLLARY 3.5. *Suppose that T be an invertible rank-1 preserver on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Then we have either*

- (1) T maps \mathbb{R} onto \mathbb{R} and maps \mathbb{C} onto \mathbb{C} , or
- (2) T maps \mathbb{R} onto \mathbb{C} and maps \mathbb{C} onto \mathbb{R} .

PROOF. By Lemma 3.4, T maps every line matrix into a line matrix. For the case of $m \neq n$, Lemma 3.4 ensures that (1) is satisfied because T is invertible. Thus we can assume that $m = n$. Let T map an i^{th} row matrix to an l^{th} row matrix. That is, $T(R_i) = R_l$. Suppose that for some row matrix R_j with $i \neq j$, $T(R_j) = C_k$ for some $k = 1, \dots, n$. Then we have $r(R_i + R_j) = 1$, while $r(T(R_i + R_j)) = 2$, a contradiction. Thus we have established that if T maps a row matrix to a row matrix, then T maps \mathbb{R} onto \mathbb{R} , and maps \mathbb{C} onto \mathbb{C} because T is invertible.

Similarly, if T maps a row matrix to a column matrix, then we obtain that T maps \mathbb{R} onto \mathbb{C} , and maps \mathbb{C} onto \mathbb{R} . □

We say that a linear operator T on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ is a (U, V) -operator if there exist invertible matrices $U \in \mathbb{M}_{m \times m}(\mathbb{Z}_+)$ and $V \in \mathbb{M}_{n \times n}(\mathbb{Z}_+)$ such that either $T(X) = UXV$ for all X in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, or $m = n$ and $T(X) = UX^tV$ for all X in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Evidently all (U, V) -operator T is invertible, and T and T^{-1} are rank-1 preservers.

PROPOSITION 3.6. *If T is a (U, V) -operator, then T preserves all ranks.*

PROOF. Let A be any nonzero matrix in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Then there exist $r(A)$ rank-1 matrices $A_1, \dots, A_{r(A)}$ such that $A = A_1 + \dots + A_{r(A)}$. Since T is a rank-1 preserver, we have $r(T(A)) \leq r(A)$. Also, there exist $r(T(A))$ rank-1 matrices $B_1, \dots, B_{r(T(A))}$ such that $T(A) = B_1 + \dots + B_{r(T(A))}$ so that $A = T^{-1}(B_1) + \dots + T^{-1}(B_{r(T(A))})$ because T^{-1} is linear. Thus we have $r(A) \leq r(T(A))$ because T^{-1} is a rank-1 preserver. That is, $r(A) = r(T(A))$ for all $A \in \mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Therefore T preserves all ranks. \square

We define a sub-semidomain of $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ whose nonzero members have rank-1 as a *rank-1 sub-semidomain*.

If \mathbb{F} is a field, then it can be shown that there are only two kinds of rank-1 subspaces in $\mathbb{M}_{m,n}(\mathbb{F})$. They are of the form $\{\mathbf{a}\mathbf{b}^t \mid \mathbf{a} \in X\}$ for some subspace X of \mathbb{F}^n or of the form $\{\mathbf{a}\mathbf{b}^t \mid \mathbf{b} \in Y\}$ for some subspace Y of \mathbb{F}^m . We call the former “*left factor spaces*” and the latter “*right factor spaces*”. Therefore, rank-1 spaces of matrices over a field are just the factor spaces. Lemma 3.3. shows that the same result is satisfied for a rank-1-sub semidomain over \mathbb{Z}_+ :

LEMMA 3.7. *For a sub-semidomain \mathbb{V} of $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, we have that \mathbb{V} is a rank-1-sub semidomain if and only if it is a factor space.*

Let \mathbb{M}^1 denote the set of all rank-1 matrices in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$.

LEMMA 3.8. *Suppose that T is a rank-1 preserver on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. If T preserves the dimension of all rank-1-sub semidomains, then the restriction of T to \mathbb{M}^1 is injective or T reduces the rank of some rank-2 matrix to 1.*

PROOF. For each $B \in \mathbb{M}^1$, define $\mathbb{W}_B = \text{span}\{X \in \mathbb{M}^1 \mid T(X) = T(B)\}$. Suppose that $\dim(\mathbb{W}_B) = 1$ for all matrix B in \mathbb{M}^1 . Then $\mathbb{W}_B = \text{span}\{B\}$ by Lemma 2.1. Therefore $T(X) = T(B)$ implies that $X = B$. That is, T is injective in \mathbb{M}^1 . If there exists a matrix B in \mathbb{M}^1 such that $\dim(\mathbb{W}_B) > 1$, then $r(X + Y) = 2$ for some $X, Y \in \mathbb{W}_B$. But we have $r(T(X + Y)) = r(2T(B)) = 1$ because T is a rank-1 preserver. Hence T reduces the rank of some rank-2 matrix to 1. \square

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, we shall use the notation $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i and j . This provides a reflexive, anti-symmetric and transitive relation. If A and B are matrices in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ with $A \geq B$, it follows from the linearity of T that $T(A) \geq T(B)$ for any linear operator T on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$.

THEOREM 3.9. *If T is a surjective linear operator on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$, then the following are equivalent:*

- (i) T is an invertible rank-1 preserver;
- (ii) T is a (U, V) -operator;
- (iii) T is a ranks 1 and 2 preserver, and preserves the dimension of all rank-1-sub semidomains.

PROOF. Assume (i). By Corollary 3.5, there are two cases: (a) T maps \mathbb{R} onto \mathbb{R} and maps \mathbb{C} onto \mathbb{C} , or (b) T maps \mathbb{R} onto \mathbb{C} and maps \mathbb{C} onto \mathbb{R} . Suppose that (a) is satisfied. We then have that $T(R_i) = R_{\alpha(i)}$ and $T(C_j) = C_{\beta(j)}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$, where α and β are some permutations of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Thus, for any cell $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = E_{\alpha(i)\beta(j)}$. Let U and V be the permutation matrices corresponding to α and β , respectively.

Then for any matrix $X = \sum_{i=1}^m \sum_{j=1}^n x_{ij} E_{ij} \in \mathbb{M}_{m \times n}(\mathbb{Z}_+)$, we have that

$$T(X) = \sum_{i=1}^m \sum_{j=1}^n x_{ij} E_{\alpha(i)\beta(j)} = UXV.$$

Thus T is a (U, V) -operator. A similar argument shows that if (b) is satisfied, then $m = n$ and there exist permutation matrices U and V such that $T(X) = UX^tV$ for all X in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Therefore T is a (U, V) -operator. This proves that (i) implies (ii). It follows directly from Lemma 2.4 and Proposition 3.6 that (ii) implies (iii) because a (U, V) -operator is invertible. So, it suffices to show that (iii) implies (i). Assume (iii). By Lemma 3.8, T is injective in \mathbb{M}^1 . Let E be an arbitrary cell in $\mathbb{E}_{m,n}$. Since T is surjective, there exists a matrix X in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ such that $T(X) = E$. Since $X \neq O$, we can choose a cell F in $\mathbb{E}_{m,n}$ such that $X \geq F$ so that $E \geq T(F)$. We then have $E = T(F)$ because T is injective in \mathbb{M}^1 . This implies that T is surjective on $\mathbb{E}_{m,n}$. Since T is injective in \mathbb{M}^1 and $\mathbb{E}_{m,n}$ is a finite set, we conclude that T permutes $\mathbb{E}_{m,n}$. By Theorem 2.5, T is invertible on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$. Thus (i) is satisfied. \square

4. Rank preservers

In this section, we characterize the linear operators that preserve rank of matrices over \mathbb{Z}_+ .

We say that a linear operator T on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ is a *rank preserver* if T preserves all ranks.

PROPOSITION 4.1. *Let A and B be matrices in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ with $\alpha A \neq \beta B$ for all nonzero scalars $\alpha, \beta \in \mathbb{Z}_+$. If $r(A) = r(B) = 1$, then there exists a rank-1 matrix C in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ such that $r(A + C) = 1$ and $r(B + C) = 2$.*

PROOF. Since $r(A) = r(B) = 1$, we have that either $r(A + B) = 2$ or $r(A + B) = 1$. For the case of $r(A + B) = 2$, the conclusion is satisfied by letting $C = A$. So we may assume that $r(A + B) = 1$. By Lemma 2.3, A and B have an irreducible common factor. If A and B have an irreducible common left factor, then we may write A and B as

$$A = \mathbf{a} \mathbf{x}^t = [x_1 \mathbf{a}, \dots, x_n \mathbf{a}] \quad \text{and} \quad B = \mathbf{a} \mathbf{y}^t = [y_1 \mathbf{a}, \dots, y_n \mathbf{a}],$$

where $\mathbf{a} = [a_1, \dots, a_n]^t$ is an irreducible vector. Then we have $\alpha \mathbf{x} \neq \beta \mathbf{y}$ for all nonzero scalars $\alpha, \beta \in \mathbb{Z}_+$ because $\alpha A \neq \beta B$. Since \mathbf{a} is not zero-vector, $a_i \neq 0$ for some $i = 1, \dots, m$. Let

$$C = \begin{cases} \mathbf{e}_j^{(m)} \mathbf{x}^t & \text{if } a_j = 0 \text{ for some } j \neq i, \\ \mathbf{e}_i^{(m)} \mathbf{x}^t & \text{otherwise} \end{cases}$$

be a matrix in $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ so that $r(C) = 1$. Then $r(A + C) = 1$ because A and C have a common right factor. But B and C have neither a common left factor nor a common right factor. It follows from Lemma 2.3 that $r(B + C) = 2$.

Similarly, a parallel argument holds if A and B have an irreducible common right factor. □

LEMMA 4.2. *Let T be a surjective rank-1 preserver on $\mathbb{M}_{m \times n}(\mathbb{Z}_+)$ with $\min(m, n) > 1$. If T is not injective, then T decreases the rank of some rank-2 matrix.*

PROOF. If T is injective in \mathbb{M}^1 , then the proof of Theorem 3.9 shows that T is surjective on $\mathbb{E}_{m,n}$ (i.e. T permutes $\mathbb{E}_{m,n}$), and thus T is invertible by Theorem 2.5, a contradiction. Therefore T is not injective in \mathbb{M}^1 . Thus there exist distinct rank-1 matrices X and Y such that $T(X) = T(Y)$. Suppose that there exist distinct nonzero scalars α and β in \mathbb{Z}_+ such that $\alpha X = \beta Y$. Then we have

$$\alpha T(X) = T(\alpha X) = T(\beta Y) = \beta T(Y) = \beta T(X).$$

Since \mathbb{Z}_+ has no zero divisors and $T(X) \neq O$, we have $\alpha = \beta$, a contradiction. So, we may assume that $\alpha X \neq \beta Y$ for all nonzero scalars $\alpha, \beta \in \mathbb{Z}_+$. By Proposition 4.1, there exists a rank-1 matrix C such that $r(X + C) = 1$ while $r(Y + C) = 2$. But $T(Y + C) = T(X + C)$ so that $r(T(Y + C)) = r(T(X + C)) = 1$ because T is a rank-1 preserver. Therefore T decreases the rank of rank-2 matrix $Y + C$. □

THEOREM 4.3. *Let T be a surjective linear operator on $M_{m \times n}(\mathbb{Z}_+)$ with $\min(m, n) > 1$. Then T is a rank preserver if and only if T is a (U, V) -operator.*

PROOF. By Lemma 4.2 and Theorem 3.9, we see that the necessity of the condition is satisfied. The sufficiency follows from Proposition 3.6. \square

THEOREM 4.4. *Let T be a surjective linear operator on $M_{m \times n}(\mathbb{Z}_+)$ with $\min(m, n) > 1$. Then T is a rank preserver if and only if T is a rank-1 and rank-2 preserver.*

PROOF. Suppose that T is a rank-1 and rank-2 preserver. Then T is invertible by Lemma 4.2. Therefore T is a rank preserver by Theorem 3.9 and Theorem 4.3. The converse is obvious. \square

Thus we have characterized the linear operators that preserve the rank of matrices over the nonnegative integers.

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