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AN ACTION OF A GALOIS GROUP ON A TENSOR PRODUCT

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ABSTRACT. Let K be a Galois extension of a field F with $G = Gal(K/F)$. Let L be an extension of F such that $K \otimes_F L = N_1 \oplus N_2 \oplus \cdots \oplus N_k$ with corresponding primitive idempotents $e_1, e_2, \cdots, e_k$, where N_i's are fields. Then G acts on $\{e_1, e_2, \cdots, e_k\}$ transitively and $Gal(N_1/K) \cong \{\sigma \in G \mid \sigma(e_1) = e_1\}$. And, let R be a commutative F-algebra, and let P be a prime ideal of R. Let $T = K \otimes_F R$, and suppose there are only finitely many prime ideals $Q_1, Q_2, \cdots, Q_k$ of T with $Q_i \cap R = P$. Then G acts transitively on $\{Q_1, Q_2, \cdots, Q_k\}$, and $Gal(qf(T/Q_1)/qf(R/P)) \cong \{\sigma \in G \mid \sigma(Q_1) = Q_1\}$ where $qf(T/Q_1)$ is the quotient field of T/Q_1.

Let L be an extension of a field F. If K is a finite separable extension of F, then $K \otimes_F L$ is a direct sum of finite separable extension fields of K(cf. [1, (1.50) p.25]). So, if K is a finite Galois extension of F with $G = Gal(K/F)$, then $K \otimes_F L = N_1 \oplus \cdots \oplus N_k$ where N_i's are finite separable extension fields of K. Let $e_1, \cdots, e_k$ be the corresponding primitive idempotents of $K \otimes_F L$.(i.e., $e_i \in N_i$, ${e_i}^2 = e_i$, $e_i e_j = 0$ for $i \neq j$, and $\sum_{i=1}^{k} e_i = 1$.) G acts on $K \otimes_F L$ by $\sigma \otimes id_L$ for $\sigma \in G$. Then $\{\sigma(e_1), \cdots, \sigma(e_k)\} = \{e_1, \cdots, e_k\}$ for all $\sigma \in G$. We will show in Theorem 2 that G acts transitively on $\{e_1, \cdots, e_k\}$(i.e., there exists $\sigma_i \in G$ with $\sigma_i(e_1) = e_i$ for each i, $1 \leq i \leq k$), and $Gal(N_1/L) \cong \{\sigma \in G \mid \sigma(e_1) = e_1\}$. In fact, we will show the above theorem for the more general case when K is a Galois extension of F(possibly $[K : F] = \infty$) such that $K \otimes_F L$ is a finite direct sum of fields. Now, we start with the case where $L = F(\alpha)$ for some $\alpha \in L$ with $[L : F] < \infty$.

THEOREM 1. *Let $K \supseteq F$ be a Galois extension of fields*(possibly $[K : F] = \infty$), *and let G=$Gal(K/F)$. Let $L = F(\alpha)$ be a field with $[L : F] < \infty$. Let f be the minimal polynomial of α over F. Let $f = g_1 \cdots g_k$ in $K[x]$, where the g_i's are irreducible in $K[x]$. Let $K \otimes_F L =$*

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$N_1 \oplus \cdots \oplus N_k \cong K[x]/(g_1) \oplus \cdots \oplus K[x]/(g_k)$ with corresponding primitive idempotents $e_1, \cdots, e_k$. Then,

(i) *G acts transitively on $\{e_1, \cdots, e_k\}$, where $\sigma \in G$ acts on $K \otimes_F L$ by $\sigma \otimes id_L$.*

(ii) *Let $H = \{\sigma \in G \mid \sigma(e_1) = e_1\}$.*

Then $H = \{\sigma \in G \mid \sigma(g_1) = g_1\}$ and $H \cong Gal(N_1/L)$. More exactly, view $N_1 = K[x]/(g_1)$ and $K \subseteq N_1$ and $L = F[x]/(f) \subseteq N_1$. Then the injection $Gal(N_1/L) \hookrightarrow Gal(K/F)$ given by $\tau \mapsto \tau|_K$ has the image H.

PROOF. (i) $K \otimes_F L \cong K \otimes_F (F[x]/(f)) \cong K[x]/fK[x] \cong K[x]/(g_1) \oplus \cdots \oplus K[x]/(g_k)$. Note that in $K[x]/fK[x]$, $\text{ann}_{K[x]/(f)}(e_i) = (g_i)/(f)$, so for $\sigma \in G$, $\sigma(e_i) = e_j$ if and only if $\sigma(g_i) = g_j$ in $K[x]$. Let K' be the splitting field of f over K, and let α_i be a root of g_i in K'. As f is the minimal polynomial of α and α_i, there is a τ in $Gal(K'/F)$ such that $\tau(\alpha) = \alpha_i$. Let $\sigma = \tau|_K \in G$. Then $\sigma(g_1) = \tau(g_1) = g_i$ as g_1 and g_i are the minimal polynomials of α and α_i respectively and $\tau(\alpha) = \alpha_i$. So $\sigma(e_1) = e_i$, and the action of G on $\{e_1, \cdots, e_k\}$ is transitive.
(ii) View $N_1 = K(\alpha) \supseteq L = F(\alpha)$, where $\min_{\alpha,K} = g_1$ and $\min_{\alpha,F} = f$. We have the canonical monomorphism $\theta : Gal(N_1/L) \longrightarrow Gal(K/F)$ given by $\tau \mapsto \tau|_K$. Let $\sigma = \theta(\tau) \in im(\theta)$. We have $0 = \tau(g_1(\alpha)) = \sigma(g_1)(\tau(\alpha)) = \sigma(g_1)(\alpha)$, so $\sigma(g_1) = g_1$. Hence $im(\theta) \subseteq H$. But, let $\rho \in H$. Since $\rho(g_1) = g_1$ and $N_1 = K(\alpha) \cong K[x]/(g_1)$, ρ extends to γ an automorphism of N_1 with $\gamma(\alpha) = \alpha$. So, $\gamma \in Gal(N_1/L)$ and $\gamma|_K = \rho$. So, $im(\theta) = H$. □

Now, We can extend the above theorem to the case where L is any extension of F such that $K \otimes_F L \cong N_1 \oplus \cdots \oplus N_k$, where N_i's are fields.

THEOREM 2. *Let $K \supseteq F$ be a Galois extension of fields(possibly $[K : F] = \infty$), and Let G=$Gal(K/F)$. Let L be a field extension of F such that $K \otimes_F L = N_1 \oplus \cdots \oplus N_k$ with corresponding primitive idempotents $e_1, \cdots, e_k$ and fields N_i. Then,*
(i) *G acts transitively on $\{e_1, \cdots, e_k\}$.*
(ii) *Let $H = \{\sigma \in G \mid \sigma(e_1) = e_1\}$. Then $H \cong Gal(N_1/L)$. More precisely, view $K \subseteq N_1$ via $K \longrightarrow K \otimes_F L \longrightarrow N_1$, and $L \subseteq N_1$ via $L \longrightarrow K \otimes_F L \longrightarrow N_1$. Then H is the image of $\theta : Gal(N_1/L) \longrightarrow Gal(K/F) = G$, a monomorphism given by $\tau \mapsto \tau|_K$.*

PROOF. Let L_0= the algebraic closure of F in L. Then $K \otimes_F L_0 \subseteq K \otimes_F L$, so $K \otimes_F L_0$ has only finitely many idempotents, and $K \otimes_F L_0 = M_1 \oplus \cdots \oplus M_m$ where each M_i is a field Galois over L_0. Then since L_0 is algebraically closed in L, each $M_i \otimes_{L_0} L$ is a field. So, $K \otimes_F L \cong$

$(K \otimes_F L_0) \otimes_{L_0} L \cong \bigoplus_{i=1}^m (M_i \otimes_F L)$=the direct sum of fields. This shows that all the primitive idempotents $e_i \in K \otimes_F L_0$. Likewise, if L_1=the separable closure of F in L_0, the same reasoning shows that all $e_i \in K \otimes_F L_1$. Then there is a finitely generated subextension L' of F in L_1 with all $e_i \in K \otimes_F L'$. By the theorem of primitive element $L' = F(\alpha)$ for some $\alpha \in L'$ as L' is finite separable over F. So the above theorem applies to $K \otimes_F L'$. Since G-action on $K \otimes_F L$ extends that on $K \otimes_F L'$, (i) follows from (ii) of Theorem 1. Also, if $K \otimes_F L' \cong T_1 \oplus \cdots \oplus T_k$, then $N_1 \oplus \cdots \oplus N_k = K \otimes_F L \cong (K \otimes_F L') \otimes_{L'} L \cong (T_1 \otimes_{L'} L) \oplus \cdots \oplus (T_k \otimes_{L'} L)$. So, since no new idempotents can be arise, each $T_i \otimes_{L'} L$ is a field and $T_i \otimes_{L'} L \cong N_i$ (The indices match because the summands are sorted out by the idempotents e_i). So, $Gal(N_1/L) \cong Gal(T_1/L') \cong H$ by Theorem 1 above. More exactly, when we $K \subseteq N_1$ by $K \to K \otimes_F L \to N_1$ and $L' \subseteq L \subseteq N_1$ by $L \to K \otimes_F L \to N_1$, and $T_1 = (K \otimes_F L')e_1 \subseteq (K \otimes_F L)e_1 = N_1$, then $T_1 = K \cdot L'$ and the isomorphism from $Gal(N_1/L)$ onto $Gal(T_1/L')$ is given by $\tau \mapsto \tau|_{T_1}$, and the isomorphism from $Gal(T_1/L')$ onto H is given by $\rho \mapsto \rho|_K$ by Theorem 1 above. So the composition $Gal(N_1/L) \to H$ is given by $\tau \mapsto \tau|_K$. □

REMARK 1. (i) Since K is Galois over F, for any field extension of F, $K \otimes_F L$ has no nonzero nilpotent elements as this is true for $K_0 \otimes_F L$ for every finitely generated subextension K_0 of F in K. In fact, as $K \otimes_F L = \varinjlim K_0 \otimes_F L$ over the finitely generated subextensions K_0 of F in K, either $K \otimes_F L$ is a finite direct sum of fields or it has infinitely many idempotents (in which case $K \otimes_F L$ is not Noetherian).
(ii) If L is a finitely generated field extensions of F, then the separable algebraic closure L' of F in L is also finitely generated over F, so $[L' : F] < \infty$. Since $K \otimes_F L'$ is a finite direct sum of fields, the arguments in the proof of Theorem 1 above shows that $K \otimes_F L$ is also a finite direct sum of fields of the same number of summands. So, the Theorem 2 applies.

Using Theorem 2, we now extend to the case where L is an F-algebra.

THEOREM 3. *Let $K \supseteq F$ be a Galois extension of fields*(possibly $[K : F] = \infty$), *and let G=$Gal(K/F)$. Let R be a commutative F-algebra and let P be a prime ideal of R. Let $T = K \otimes_F R$, and suppose there are only finitely many prime ideals $Q_1, \cdots, Q_k$ of T with $Q_i \cap R = P$*(Such Q_i always exists by [2, Th.5.9] as T is integral over R). Then,

(i) *G acts transitively on $\{Q_1, \cdots, Q_k\}$, where σ acts on T by $\sigma \otimes id_R$ for $\sigma \in G$.*

(ii) $K \otimes_F qf(R/P) \cong qf(T/Q_1) \oplus \cdots \oplus qf(T/Q_k)$, *where* $qf(R/P)$ *is the quotient field of* R/P.

(iii) *Each* $qf(T/Q_i)$ *is Galois over* R/P, *and* $Gal(qf(T/Q_1)/qf(R/P)) \cong H$, *where* $H = \{\sigma \in G \mid \sigma(Q_1) = Q_1\}$.

PROOF. We can, by localizing, assume that P is a maximal ideal of R.(Replace R by R_P and T by $T_P = R_P \otimes_R T = R_P \otimes_R (R \otimes_F K) = R_P \otimes_F K$.) Then, the Q_i's are all the prime ideals of T with $Q_i \supseteq PT$. Let $\bar{R} = R/P$, $\bar{T} = T/PT \cong (K \otimes_F R)/im(K \otimes_R P) \cong K \otimes_R \bar{R}$, and $\bar{Q}_i = Q_i/PT$. The $\bar{Q}_i$'s are all the prime ideals of $\bar{T}$, and all the maximal ideals of $\bar{T}$ by [2, Th.5.12] as $\bar{T}$ is integral over $\bar{R}$. Since $\bar{R}$ is a field and K is separable over F, $K \otimes_F \bar{R} \cong \bar{T}$ has no nilpotent elements. So, since $\bigcap_{i=1}^k \bar{Q}_i$ is the nilradical of $\bar{T}$, $\bigcap_{i=1}^k \bar{Q}_i = (0)$. By the Chinese Remainder Theorem, $K \otimes_F \bar{R} \cong \bar{T} \cong \bar{T}/\bar{Q}_1 \oplus \cdots \oplus \bar{T}/\bar{Q}_k = \bar{T}e_1 \oplus \cdots \oplus \bar{T}e_k \cdots (*)$, where $e_1, \cdots, e_k$ are the primitive idempotents of $\bar{T}$. So by Theorem 2 above, G acts on $\{e_1, \cdots, e_k\}$ transitively. Since $\bar{Q}_i = \text{ann}_{\bar{T}}(e_i)$, G has the same action on the $\bar{Q}_i$ as on the e_i, and this is clearly the same as the action of G on the Q_i. This gives (i). (ii) follows from $(*)$ above, since $\bar{T}/\bar{Q}_i \cong T/Q_i$ which is a field. (iii) follows from Theorem 2 above, since $H = \{\sigma \in G \mid \sigma(Q_1) = Q_1\} = \{\sigma \in G \mid \sigma(e_1) = e_1\}$. □

REMARK 2. If T is Noetherian, then the condition of only finitely many Q_i is satisfied, since the Q_i's are the prime ideals minimal over P.

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