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The Phase Sensitivity of the Coincidence Detection in one Output Port of a Mach-Zehnder Interferometer

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The phase sensitivity of the coincidence detection in one output port of a Mach-Zehnder interferometer is analysed for twin Fock state inputs. Firstly, the ideal detectors with quantum efficiency of unity are assumed for the detection of the output photons. The sensitivity is found out to be independent of the photon number of input light, which means that the Heisenberg limit cannot be reached in the coincidence detection even with ideal detectors. Secondly, the practical detectors with quantum efficiencies less than unity are discussed.

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I. INTRODUCTION

Considerable effort has been devoted to the precision measurement for the phase shift for a long time. by Caves to overcome the standard quantum limit $\Delta\theta = \frac{1}{\sqrt{n}}$ where n is the number of photons for the measurement time interval [1]. Holland and Burnett opened up the possibility of reaching the Heisenberg sensitivity, $\Delta\theta = \frac{1}{n}$ by proposing twin Fock state light inputs for a Mach-Zehnder Interferometer (MZI) [2]. Since then, more detailed methods have been studied to get precise interferometric measurements [3-10]. Photon number correlated light turns out to be an alternative to the twin Fock states, regardless of its photon statistics [4]. It has been shown that the phase sensitivity of a MZI is dependent on how the interferometer is measured even for the twin Fock states inputs [6]. Furthermore it was found out that it is almost impossible for the Heisenberg sensitivity to be achieved by the practical detectors with a quantum efficiency less than the unity [7]. A different scheme to circumvent the difficulties has been discussed, in which the Bose-Einstein condensates is used for the input of a MZI instead of the nonclassical lights [11].

Some other applications of a MZI with a small number of photons have been experiments with entangled photon pairs from the parametric down-conversion process or the experiments related to quantum information processing [12, 13]. In this regard, various kinds of measurements with a MZI have been carried out, in which one of them is the coincidence detection at one output port of the interferometer for the measurement of the photonic de Broglie wavelength [14]. Here we'd like to analyse the limit of phase resolution of the coincidence detection in an output port, and make a comparison with the results of the coincidence detection for the two output ports.

II. THE COINCIDENCE DETECTION OF ONE OUTPUT PORT OF A MZI WITH IDEAL DETECTORS

Let us suppose we are trying to measure the phase difference between the two paths of a MZI at an output denoted by $\hat{a}_5$ as shown in Fig 1. If the output is devided by a lossless 50-50 beam splitter BS III, the annihilation operators $\hat{a}_7$ and $\hat{a}_8$ for the two modes are related to the annihilation operator $\hat{a}_5$ as follows,

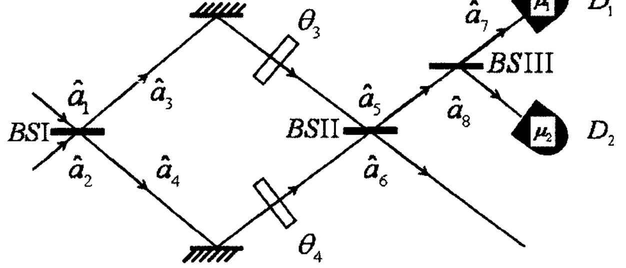

FIG. 1. The schematic diagram of a Mach-Zehnder interferometer for the coincidence detection at one output port.

$$\hat{a}_7 = \frac{1}{\sqrt{2}}\hat{a}_5 + \frac{i}{\sqrt{2}}\hat{v}$$
$$\hat{a}_8 = \frac{i}{\sqrt{2}}\hat{a}_5 + \frac{1}{\sqrt{2}}\hat{v} \tag{1}$$

where $\hat{v}$ is the photon annihilation operator for the vacuum state. When the coincidence is calculated for the vacuum states, the result can be described as

$$<\hat{n}'_c> \; = \; <:\hat{n}_7\hat{n}_8:> \; = \; \frac{1}{4} <\hat{a}_5^\dagger \hat{a}_5^\dagger \hat{a}_5 \hat{a}_5>. \tag{2}$$

The output mode $\hat{a}_5$ of the MZI is related to the input modes $\hat{a}_1$ and $\hat{a}_2$, like as

$$\begin{aligned}\hat{a}_5 &= \frac{i}{\sqrt{2}}\hat{a}_3 e^{i\theta_3} + \frac{1}{\sqrt{2}}\hat{a}_4 e^{i\theta_4} \\ &= \frac{1}{2}\left\{(i\hat{a}_1 - \hat{a}_2)e^{i\theta_3} + (i\hat{a}_1 - \hat{a}_2)e^{i\theta_4}\right\},\end{aligned} \tag{3}$$

where θ_3 and θ_4 denote phase in the path 2 and path 3, respectively. Since $\hat{a}_5^\dagger$ is given by

$$\hat{a}_5^\dagger = \frac{1}{2}\left\{(-i\hat{a}_1^\dagger - \hat{a}_2^\dagger)e^{-i\theta_3} + (i\hat{a}_1^\dagger + \hat{a}_2^\dagger)e^{-i\theta_4}\right\}, \tag{4}$$

we can have the coincidence measurement,

$$<\hat{n}_c> \; \equiv \; <(\hat{n}_5\hat{n}_5 - \hat{n}_5)>, \tag{5}$$

by the commutation relation $\left[\hat{a}_5, \hat{a}_5^\dagger\right] = 1$, neglecting the constant $\frac{1}{4}$ for the convenience. Here $\hat{a}_5$ represents the photon number operator for the mode 5. If we employ the Schwinger angular momentum operators for the interferometer [15],

$$\hat{J}_x = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2\right),$$
$$\hat{J}_z = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\right), \tag{6}$$

and total number operator $\hat{N}$,

$$\hat{N} = \frac{1}{2}\left(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2\right), \tag{7}$$

for the input photons, $\hat{a}_5$ gives

$$\hat{a}_5 = \hat{N} + \hat{J}_x\sin\theta - \hat{J}_z\cos\theta \tag{8}$$

for the phase difference between the two paths, i.e., $\theta = \theta_4 - \theta_3$. Because of commutation relations, $\left[\hat{N}, \hat{J}_x\right] = \left[\hat{N}, \hat{J}_z\right] = 0$, the coincidence measurement becomes,

$$\begin{aligned}<\hat{n}_c> = \; &<[(\hat{N}^2 - \hat{N}) + (\hat{N}\hat{J}_x - \hat{J}_x)\sin\theta \\ &-(2\hat{N}\hat{J}_z - \hat{J}_z)\cos\theta - (\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)\sin\theta\cos\theta \\ &+ \hat{J}_x^2\sin^2\theta + \hat{J}_z^2\cos^2\theta]>.\end{aligned} \tag{9}$$

For the general Fock state input to the MZI, it leads to

$$\begin{aligned}<n_1, n_2> = \; &\frac{1}{4}(n_1+n_2)^2 - \frac{1}{2}(n_1+n_2) \\ &-\frac{1}{2}(n_1-n_2)(n_1+n_2-1)\cos\theta \\ &+\frac{1}{4}(2n_1n_2+n_1+n_2) + \frac{1}{4}(n_1-n_2)^2\cos^2\theta\end{aligned} \tag{10}$$

If we consider two kinds of twin Fock state inputs, we get the following results

$$<1,1|\hat{n}_c|1,1> \; = \; \frac{1}{2}(1-\cos 2\theta) \tag{11}$$

and

$$<n,n|\hat{n}_c|n,n> \; \approx n^2[1+\frac{1}{4}(1-\cos 2\theta)] \tag{12}$$

for a large number $n(\gg 1)$. Likewise, we can perform a calculation of the expectation value for the general Fock states, which gives an expression for the variance as follows,

$$\begin{aligned}(\Delta n_c)^2 = \; &<\hat{n}_c^2> - <\hat{n}_c>^2 \\ = \; &\frac{1}{16}[n_2(n_2-1)(n_1+1)(n_1+2) \\ &+ n_1(n_1-1)(n_2+1)(n_2+2)]\sin^4\theta \\ &+ \frac{1}{8}(n_1+n_2-1)[n_1(n_2+1)(n_1-n_2-2) \\ &+ n_2(n_1+1)(n_1-n_2+2)]\sin^2\theta \\ &- \frac{1}{4}(n_1+n_2-1)[n_1(n_2+1) \\ &+ n_2(n_1+1)]\sin^2\theta + \frac{1}{16}[(n_1-n_2)^2 \\ &\times <n_1(n_2+1)+n_2(n_1+1)> \\ &+ 2(n_1-n_2)<n_2(n_1+1)(n_1-n_2+2) \\ &+ n_1(n_2+1)(n_1-n_2-2)> \\ &+ <n_1(n_2+1)(n_1-n_2-2)^2 + n_2(n_1+1) \\ &\times (n_1-n_2+2)>]\sin^2\theta\cos^2\theta\end{aligned} \tag{13}$$

Now, the square of the uncertainty in the phase difference can be found in some special cases such as

$$(\Delta\theta)^2 = \frac{(\Delta n_c)^2}{\left(\frac{\partial <\hat{n}_c>}{\partial\theta}\right)^2} = \frac{1}{4}\left(1+\frac{1}{\cos^2\theta}\right) \tag{14}$$

for the $n_1 = n_2 = 1$ state input, On the other hand, we have

$$(\Delta\theta)^2 \approx \frac{2}{\cos^2\theta} + \frac{1}{8}\tan^2\theta, \qquad (15)$$

for the large photon number state input ($n_1 = n_2 = n \gg 1$). It reveals that the phase uncertainty, in this case, is independent of the input photon number, with the minimum phase uncertainty value of $\Delta\theta = \sqrt{2}$ rad.

III. THE COINCIDENCE DETECTION WITH NONIDEAL DETECTORS

Let us suppose that we are trying to measure the coincidence detection with two detections D_1 and D_2 of quantum efficiencies μ_1 and μ_2, respectively. If we adopt the model that Yuen, Shapro and Yurke devised for practical detections [16,17], the photon annihilation operator $\hat{a}'$ for the detected fields mode is described by a photon annihilation opreators $\hat{a}$ for the input mode and $\hat{v}$ for the vacuum state mode as follows,

$$\hat{a}' = \sqrt{\mu}\,\hat{a} + \sqrt{1-\mu}\,\hat{v}. \qquad (16)$$

This allows us to lead to photon number operators such as

$$\begin{aligned} \hat{n}_7' &= \mu_1\hat{n}_7 + \sqrt{\mu_1(1-\mu_1)}\,(\hat{a}_7^\dagger \hat{v}_7 + \hat{v}_7^\dagger \hat{a}_7) \\ &\quad + (1-\mu_1)\hat{v}_7^\dagger\hat{v}_7 \\ \hat{n}_8' &= \mu_2\hat{n}_8 + \sqrt{\mu_2(1-\mu_2)}\,(\hat{a}_8^\dagger \hat{v}_8 + \hat{v}_8^\dagger \hat{a}_8) \\ &\quad + (1-\mu_2)\hat{v}_8^\dagger\hat{v}_8 \end{aligned} \qquad (17)$$

Considering the relations

$$\hat{v}|0 \geqq 0, <0|\hat{v}^\dagger = 0, <0|0> = 1 \qquad (18)$$

we can easily get the expectation values of the coincidence, and the squared coincidence

$$<\hat{n}_c'> = \mu_1\mu_2<\hat{n}_c>, \qquad (19)$$

and

$$<\hat{n}_c'^2> = \mu_1^2\mu_2^2<\hat{n}_7^2\hat{n}_8^2> + \mu_1^2\mu_2(1-\mu_2)<\hat{n}_7^2\hat{n}_8> \qquad (20)$$

respectively. Then the square of the uncertainty in the phase difference is finally given by

$$\begin{aligned} (\Delta\theta_c')^2 &= \frac{<\hat{n}_c'^2> - <\hat{n}_c'>^2}{[\frac{\partial<\hat{n}_c>}{\partial\theta}]^2} \frac{\mu_1^2\mu_2^2(\Delta\hat{n}_c)^2}{\mu_1^2\mu_2^2[\frac{\partial<\hat{n}_c>}{\partial\theta}]^2} \\ &+ \frac{\mu_1^2\mu_2(1-\mu_2)<\hat{n}_7^2\hat{n}_8>}{\mu_1^2\mu_2^2[\frac{\partial<\hat{n}_c>}{\partial\theta}]^2} \\ &+ \frac{\mu_2^2\mu_1(1-\mu_1)<\hat{n}_7\hat{n}_8^2>}{\mu_1^2\mu_2^2[\frac{\partial<\hat{n}_c>}{\partial\theta}]^2} \\ &+ \frac{\mu_1\mu_2(1-\mu_1)(1-\mu_2)<\hat{n}_7\hat{n}_8>}{\mu_1^2\mu_2^2[\frac{\partial<\hat{n}_c>}{\partial\theta}]^2} \end{aligned} \qquad (21)$$

When we apply the equation to some special case, we get the following results :

$$\begin{aligned} (\Delta\theta)^2 &= \frac{1}{4}(1+\frac{1}{\cos^2\theta}) + \frac{\mu_1+\mu_2-2\mu_1\mu_2}{4\mu_1\mu_2}\frac{1}{\cos^2\theta\sin^2\theta} \\ &+ \frac{(1-\mu_1)(1-\mu_2)}{4\mu_1\mu_2} \end{aligned} \qquad (22)$$

for the $n_1 = n_2 = 1$, and

$$(\Delta\theta)^2 \approx \frac{2}{\cos^2\theta} + \frac{1}{8}\tan^2\theta + \frac{\mu_1+\mu_2-2\mu_1\mu_2}{\mu_1\mu_2}\frac{1}{n(\sin^2\theta\cos^2\theta)} \qquad (23)$$

for $n_1 = n_2 = n \gg 1$. We can easily see that imperfect detectors with less than unit quantum efficiencies can degrade the sensitivity, as we expected. However the effect of the practical detectors is not critical because of the relatively large value of the phase uncertainty in the ideal case. Actually, the third term in the right side of Eq. (23), which is related to the practical detection, can be negligible for large photon numbers.

IV. CONCLUSION

When we compare the coincidence measurement at one output port with that at two output ports of a MZI, we find that the self coincidence detection case, in which the measurement accuracy scales as $\frac{1}{n}$. Contrary to coincidence detection for the two out put ports, coincidence detection of one output port cannot beat the standard limit of sensitivity, $\frac{1}{\sqrt{n}}$. However such a short-noise limit sensitivity can be reached by one Fock state input $|0, n>$, but not by the twin Fock state input $|n, n>$. As shown in the Eq. (15) for the ideal case of a large twin Fock state input, the minimum uncertainty in the phase difference is $\sqrt{2}$ rad for the zero phase difference, i.e., $\theta = 0$, while the value of $\Delta\theta$ has the sensitivity of $\frac{1}{\sqrt{2}}$ rad for the input the $|1, 1>$ state. It is interesting to know that the latter case is better than the former case, despite a large input photon number. We can conclude that exploiting the four ports fot MZI, ie., two input ports and two output ports can be more favorable in the point view sensitivity. These results can be applicable to boson interferometers using Bose-Einstein condensates.

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