퍼지 및 지능시스템학회 논문지 2005, Vol. 15, No. 7, pp. 789-793

구간치 퍼지수 상의 쇼케이 거리측도에 관한 성질

Some properties of Choquet distance measures for interval-valued fuzzy numbers

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요 약

구간치 퍼지집합은 Gorzalczang(1983)과 Turken(1986)에 의해 처음 제의되었다. 이를 토대로 Wang과 Li는 구간치 퍼지수에 관한 연산으로 일반화 하여 연구하였다. 최근에 홍(2002)는 왕과 리의 이론을 리만적분에 의해 구간치 퍼지집합상의 거리측도에 관한 연구를 하였다. 본 논문에서 우리는 일반측도와 관련된 리만적분 대신에 퍼지측도와 관련된 쇼케이적분을 이용한 구간치 퍼지수 상의 쇼케이 거리측도를 정의하고 이와 관련된 성질들을 조사하였다.

Abstract

Interval-valued fuzzy sets were suggested for the first time by Gorzalczang(1983) and Turken(1986). Based on this, Wang and Li extended their operations on interval-valued fuzzy numbers. Recently, Hong(2002) generalized results of Wang and Li and extended to interval-valued fuzzy sets with Riemann integral. In this paper, using Choquet integrals with respect to a fuzzy measure instead of Riemann integrals with respect to a classical measure, we define a Choquet distance measure for interval-valued fuzzy numbers and investigate its properties.

Key words : Interval-valued fuzzy number, Distance measure, Choquet integral.

1. Introduction

Interval-valued fuzzy sets were suggested for the first time by Gorzalczany[4] and Turksen[12]. Based on this Wang and Li definded fuzzy numbers and gave their extended operations. Recently Hong generalized results of Wang and Li and extended to interval-valued fuzzy sets with Riemann integral. In this paper, we propose the concept of Choquet integral with respect to a fuzzy measure instead of Riemann integral with a classical measure and we define a Choquet distance measure for interval-valued fuzzy numbers using Choquet integral with respect to a fuzzy measure.

In section 2, we give preliminary definitions which are required in the following discussion. In section 3, we deal some properties of interval-valued fuzzy numbers. In section 4, using the Choquet integral with respect to a fuzzy measure, we define a Choquet distance measure for interval-valued fuzzy numbers and investigate its properties(see [1,2,5,6,7,8]).

접수일자 : 2005년 10월 21일
완료일자 : 2005년 12월 5일

2. Definitions and Preliminaries

At first, we introduce interval numbers and their basic operations (see [3,4,5,6, 7,12,13]). Throughout this paper, R^+ will denote the interval $[0, \infty)$,

$$[I]=\{[a, b] \mid a, b \in R^+ \text{ and } a \leq b\}.$$

Then an element in $[I]$ is called an interval number.

Definition 2.1. If $a_t \in I,\ t \in T$, then we define

$$\vee_{t \in T} a_t = sup\{a_t : t \in T\},$$
$$\wedge_{t \in T} a_t = inf\{a_t : t \in T\}.$$

We also define for $[a_t, b_t] \in [I],\ t \in T$,

$$\vee_{t \in T}[a_t, b_t] = [\vee_{t \in T} a_t, \vee_{t \in T} b_t],$$
$$\wedge_{t \in T}[a_t, b_t] = [\wedge_{t \in T} a_t, \wedge_{t \in T} b_t].$$

Definition 2.2. Let $[a_1, b_1], [a_2, b_2] \in [I]$ and $k \in R^+$. We define

$$[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$$
$$[a_1, b_1] \cdot [a_2, b_2] = [a_1 \cdot a_2, b_1 \cdot b_2]$$
$$k[a_1, b_1] = [ka_1, kb_1],$$

$[a_1, b_1] \leq [a_2, b_2]$ if and only if

$$a_1 \le a_2 \text{ and } b_1 \le b_2,$$
$$[a_1, b_1] < [a_2, b_2] \text{ if and only if } [a_1, b_1] \le [a_2, b_2] \text{ but } [a_1, b_1] \ne [a_2, b_2].$$

Definition 2.3. Let X be an ordinary nonempty set, then the mapping $A: X \to [I]$ is called an interval-valued fuzzy set on X. All interval-valued fuzzy set on X are denoted by $IF(X)$.

Definition 2.4. Let $A \in IF(X)$, let $A(x) = [A_-(x), A^-(x)]$, where $x \in X$. Then two ordinary fuzzy set $A_-: X \to [I]$ and $A^-: X \to [I]$ are called lower fuzzy set and upper fuzzy set of A, respectively, simply write $A = [A^-, A^+]$.

Definition 2.5. Let $A \in IF(X)$, $[\lambda_1, \lambda_2] \in [I]$. Then we call

$$A_{[\lambda_1, \lambda_2]} = \{x \in X \mid A_-(x) \ge \lambda_1, A^-(x) \ge \lambda_2\}$$

and

$$A_{(\lambda_1, \lambda_2)} = \{x \in X \mid A_-(x) > \lambda_1, A^-(x) > \lambda_2\}$$

the $[\lambda_1, \lambda_2]$-level set of A and the (λ_1, λ_2)-level set of A, respectively. And let $A_{\lambda^-}(x) = \{x \in X \mid A_-(x) > \lambda\}$ and $A_{\lambda^+}(x) = \{x \in X \mid A^-(x) > \lambda\}$.

Definition 2.6. Let $A \in IF(X)$, $[\lambda_1, \lambda_2] \in [I]$. We define

$$([\lambda_1, \lambda_2]A)(x) = [\lambda_1, \lambda_2] \wedge [A_-(x), A^-(x)].$$

Definition 2.7. Let $A \in IF(X)$, i.e., $A: X \to [I]$. Assume that following conditions are satisfied;

(1) A is normal, i.e., there exists $x_0 \in X$ s.t. $A(x_0) = 1$.

(2) For arbitrary $[\lambda_1, \lambda_2] \in [I] - \{0\}$, $A_{[\lambda_1, \lambda_2]}$ is closed bounded interval.

Then we call A an interval-valued fuzzy number on X.

Let $IF^*(X)$ denoted the set of all interval-valued fuzzy numbers on X, and we write $[I]^+ = [I] - \{0\}$.

Definition 2.8. Let $A \in IF^*(X)$. Then A is called an interval convex fuzzy set, if for any $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$A(\lambda x + (1 - \lambda) y) \ge A(x) \wedge A(y).$$

Definition 2.9. Let $A, B \in IF^*(X)$ and $\bullet \in \{+, -, \cdot, \div\}$. We define their extended operations to

$$(A \bullet B)(z) = \vee_{z = x \bullet y} (A(x) \wedge B(y)).$$

For each $[\lambda_1, \lambda_2] \in [I]^+$, we write

$$A_{[\lambda_1, \lambda_2]} \bullet B_{[\lambda_1, \lambda_2]} = \{x \bullet y : x \in A_{[\lambda_1, \lambda_2]}, y \in B_{[\lambda_1, \lambda_2]}\}$$

Definition 2.10. Let $A \in IF^*(X)$. Then A is called a positive interval-valued fuzzy number, if $A(x) = \bar{0}$ whenever $x \le 0$; an A is called a negative interval-valued fuzzy number, if $A(x) = \bar{0}$ whenever $x \ge 0$.

All positive interval-valued fuzzy numbers and all negative interval-valued fuzzy numbers are denoted by $IF_+^*(X)$ and $IF_-^*(X)$, respectively.

Definition 2.11([8,9,10,11]) (1) A fuzzy measure on a measurable space (X, Ω) is an extended real-valued function $\mu : \Omega \to [0, \infty]$ satisfying

(i) $\mu(\phi) = 0, \mu(X) = 1$

(ii) whenever $A, B \in \Omega, A \subset B$,

then $\mu(A) \le \mu(B)$.

(2) μ is said to be continuous from below it for every increasing sequence $\{A_n\} \subset \Omega$ of measurable sets, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

(3) μ is said to be continuous from above it for every decreasing sequence $\{A_n\} \subset \Omega$ of measurable sets, we have

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

(4) if μ is said to be continuous from above and continuous from below, it is said to be continuous.

Recall that a function $f: X \to [0, \infty]$ is said to be measurable if $\{x | f(x) > \alpha\} \in \Omega$ for all $\alpha \in (-\infty, \infty)$.

Definition 2.12([8,9,10,11]) (1) The Choquet integral of nonnegative measurable function f is defined by

$$(C)\int f d\mu = \int_0^{\infty} \mu_f(r)\, dr$$

where $\mu_f(r) = \mu\{x \mid f(x) > r\}$ and the integral on the right-hand side is an ordinary one.

(2) A nonnegative measurable function f is called integrable if the choquet integral of f can be defined and its value is finite

We note that "$x \in X$ $\mu - a.e.$" stands for "$x \in X$ μ-almost everywhere". The property $P(x)$ holds for

$x \in X$ $\mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $P(x)$ holds for all $x \in A^c$, where A^c is the complement of A.

Definition 2.13 [11] (1) A sequence $\{f_n\}$ of nonnegative measurable functions is said to converge to f in measure, in symbols $f_n \rightarrow_M f$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x| \ |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

(2) A sequence $\{f_n\}$ of nonnegative measurable functions is said to converge to f in distribution, in symbols $f_n \rightarrow_D f$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu_{f_n}(r) = \mu_f(r) \quad a.e.,$$

where $\mu_f(r) = \mu(\{x|f(x) > r\})$.

Definition 2.14([8,9,10,11]) Let f, g be nonnegative measurable functions. We say that f and g are comonotonic, in symbol $f \sim g$ if and only if

$$f(x) < f(x') \Rightarrow g(x) \leq g(x') \text{ for all } x, x' \in X.$$

Theorem 2.15([8,9,10,11]) Let f, g, h be nonnegative measurable functions. Then we have

(1) $f \sim f$,
(2) $f \sim g \Rightarrow g \sim f$,
(3) $f \sim a$ for all $a \in R^+$,
(4) $f \sim g$ and $f \sim h \Rightarrow f \sim (g+h)$.

Theorem 2.16([8,9,10,11]) Let f, g be nonnegative measurable functions.

(1) If $f \leq g$, then $(C)\int f d\mu \leq (C)\int g d\mu$.

(2) If $f \sim g$ and $a, b \in R^+$, then
$$(C)\int (af + bg) d\mu = a(C)\int f d\mu + b(C)\int g d\mu.$$

(3) If $f \vee g$, then
$$(C)\int f \vee g \, d\mu \geq (C)\int f d\mu \vee (C)\int g d\mu.$$

Theorem 2.17([8,9]) (1) If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions, then we have

$$(C)\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} (C)\int f_n d\mu.$$

(2) If $\{f_n\}$ is a decreasing sequence of nonnegative measurable functions and f_1 is Choquet integrable, then we have

$$(C)\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} (C)\int f_n d\mu.$$

3. Interval-valued fuzzy numbers

Lemma 3.1. Let $A \in IF^*(X)$. Then A is normal if and only if A_- and A^- are normal fuzzy sets.

Lemma 3.2. Let $A \in IF^*(X)$. Then Definition 2.7(2) is equivalent to saying that for arbitrary $\alpha \in [0,1] - \{0\}$, $A_{-\alpha}$ and A_α^- are closed bounded intervals.

Lemma 3.2. Let $A \in IF^*(X)$. Then A is an interval convex fuzzy set if and only if A_- and A^- are convex fuzzy set.

It is noted that A_- (A^-) is convex if and only if for arbitrary $\alpha \in [0,1]$, $A_{-\alpha}$ (A_α^-) is convex set. Hence the following result which is a generalization of convex sets is convex.

Theorem 3.1. Let $A \in IF^*(X)$. Then A is an interval convex fuzzy set if and only if for any $[\lambda_1, \lambda_2] \in [I]^+$, $A_{[\lambda_1, \lambda_2]}$ is a convex set.

Theorem 3.2. Let $A, B \in IF^*(X)$ and $\bullet \in \{+, -, \cdot, \div\}$. Then we have

$$(A \bullet B)(z) = [(A_- \bullet B_-)(z), (A^- \bullet B^-)(z)].$$

Corollary 3.3. [3,12] Let $A, B \in IF^*(X)$. Then $A+B$, $A-B$, $A \cdot B \in IF^*(X)$. Especially, $A \div B \in IF^*(X)$ whenever $B \in IF_-^*(X) \cup IF_+^*(X)$.

The following theorems are important results immediate as an application of Theorem 3.2 and commutatively and associativity of fuzzy numbers under $+$ and $\bullet$ (see[3, 12])

Theorem 3.5. Let $A, B \in IF^*(X)$. Then $A \bullet B = B \bullet A$ where $\bullet \in \{+, \cdot\}$ and $A, B \in IF_-^*(X)$ or $A, B \in IF_+^*(X)$ whenever $\bullet$ choose.

Theorem 3.6. Let $A, B, C \in IF^*(X)$. Then $(A \bullet B) \bullet C = A \bullet (B \bullet C)$ where $\bullet \in \{+, \cdot\}$ and $A, B, C \in IF_-^*(X)$ or $A, B, C \in IF_+^*(X)$ whenever $\bullet$ choose.

4. Main Results

In this section, we define a Choquet distance measure between interval-valued fuzzy numbers.

Definition 4.1. For arbitrary interval-valued fuzzy numbers $A, B \in IF^*(X)$, the quantity

$$D_c(A, B) = (C)\int d_H(A(x), B(x))d\mu(x)$$
$$= \int_0^\infty \mu\{x \mid d_H(A(x), B(x)) > \alpha\}d\alpha$$

is the Choquet distance measure between A and B, where d_H is the Hausdorff metric between $A(x)$ and $B(x)$ which is defined as

$$d_H(A(x), B(x)) = d_H(A_-(x), B_-(x)) \vee d_H(A^-(x), B^-(x))$$

since $A_-(x)$ and $A^-(x)$ the lower and the upper end-point of $A(x)$,

$$A(x)=[A_-(x), A^-(x)].$$

Theorem 4.2. Let $A, B \in IF^*(X)$. Then we have

$$D_C: IF^*(X) \times IF^*(X) \to [0, \infty)$$

is pseudo-metric.
proof. It is clear.

Theorem 4.3. Let $A, B \in IF^*(X)$ and A, B are continuous. Then $D_C(A, B) = 0$ if and only if $A = B$ $\mu - a.e.$
proof. $(\Rightarrow)$

$$D_c(A, B) = (C)\int d_H(A(x), B(x))d\mu(x)$$
$$= \int_0^\infty \mu\{x \mid d_H(A(x), B(x)) > \alpha\}d\alpha$$
$$=0$$

Thus, for arbitrary n,

$$\mu\{x \mid d_H(A(x), B(x)) > \frac{1}{n}\}=0.$$

If we put $G_n = \{x \mid d_H(A(x), B(x)) > \frac{1}{n}\}$ for all n, then

$$G = \{x \mid d_H(A(x), B(x)) \geq 0\}$$
$$= \cup_{n=1}^{\infty}\{x \mid d_H(A(x), B(x)) > \frac{1}{n}\}$$
$$= \cup_{n=1}^{\infty} G_n$$

and $G_n \leq G$. Thus, we have

$$\mu(G) = \lim_{n\to\infty} \mu(G_n) = 0.$$

That is,

$$A = B \ \mu - a.e.$$

$(\Leftarrow)$ If $A = B$ $\mu - a.e.$, then

$$d_H(A(x), B(x)) = 0 \quad x \in R \ \mu - a.e..$$

Clearly, we have

$$\mu\{x \mid d_H(A(x), B(x)) > \alpha\} = 0.$$

Theorem 4.4 Let $\{A_n\}$ is an increasing sequence of interval-valued fuzzy numbers and for each $x \in X$, $d_H - \lim_{n\to\infty} A_n(x) = A(x)$. Then we have

$$\lim_{n\to\infty} D_C(A_n, A) = 0.$$

proof. Since $d_H - \lim_{n\to\infty} A(x) = A(x)$ for each $x \in R$, we have

$$\lim_{n\to\infty} d_H(A_n(x), A(x)) = 0.$$

We note that and $\{d_H(A_n(x), A(x))\}$ is a decreasing sequence and

$$(C)\int d_H(A_1(x), A(x))d\mu(x)$$

is finite, by Theorem 2.17 (2),

$$\lim_{n\to\infty} D_C(A_n, A)$$
$$= \lim_{n\to\infty}(C)\int d_H(A_n(x), A(x))d\mu(x)$$
$$= (C)\int \lim_{n\to\infty} d_H(A_n(x), A(x))d\mu(x) = 0$$

Theorem 4.5 Let $A, B \in IF^*(X)$. Then

$$D_C(\vee A_n, B) \geq \vee D_C(A_n, B).$$

proof. Using Theorem 2.16(3), we can prove as in the proof of Theorem 4.4.

5. References

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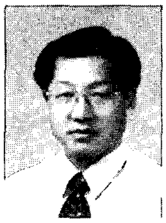
저 자 소 개



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