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ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF NA RANDOM VARIABLES †

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables which are dominated randomly by another random variable. We discuss the limit properties of weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ under some appropriate conditions, where $\{a_{ni}, 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants. As corollary, the results of Bai and Cheng (2000) and Sung (2001) are extended from the i.i.d. case to not necessarily identically distributed negatively associated setting. The corresponding results of Chow and Lai (1973) also are extended.

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1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of random variables ($r.v.$'s) with $EX_n = 0$, and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. There has been much literature on the limit properties of weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ when $\{X_n,\ n \geq 1\}$ are assumed to be independent and identically distributed ($i.i.d.$) $r.v.$'s (see Bai and Cheng (2000), Sung (2001), Cuzick (1995), Chow and Lai (1973) among others). In particular, Bai and Cheng (2000) complemented the results of Cuzick (1995) and obtained the following strong law of large numbers $\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0$ $a.s.$ when $\{X_n, n \geq 1\}$ is a sequence of $i.i.d.$ $r.v.$'s with $EX_1 = 0$ and

$$E[\exp(h|X_1|^{\gamma})] < \infty \quad \text{for some} \quad h > 0\ (\gamma > 0), \tag{1.1}$$

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and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying

$$A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \quad A_{\alpha,n} = (\sum_{i=1}^{n} |a_{ni}|^\alpha / n)^{1/\alpha} \tag{1.2}$$

for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\gamma(\alpha-1)/\alpha(1+\gamma)}$.

Sung (2001) extended this result and obtained another almost sure limiting law when condition (1.1) is replaced by stronger condition

$$E[\exp(h|X_1|^\gamma)] < \infty \quad \text{for any} \quad h > 0 \ (\gamma > 0). \tag{1.3}$$

In this case, $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ if $0 < \gamma \le 1$.

In this paper we shall study limit properties for weighted sums of negatively associated *r.v.*'s. In particular, we shall consider the case when $\{X_n, n \ge 1\}$ are negatively associated *r.v.*'s with $P(|X_i| > x) \le cP(|X| > x)$ for all i, $x \ge 0$ and some constant $c > 0$, *i.e.*, $\{X_i\}$ are dominated randomly by another random variable X. As corollary, the results of Bai and Cheng (2000) and Sung (2001) are extended from *i.i.d.* case to not necessarily identically distributed negatively associated setting. The corresponding results of Chow and Lai (1973) also are extended.
First we shall give the definition of negatively associated *r.v.*'s:

DEFINITION 1.1 (JOAG-DEV AND PROSCHAN (1983)). *A finite family of r.v.'s* $\{X_i, 1 \le i \le n\}$ *is said to be negatively associated (NA) if, for every pair of disjoint subsets* A *and* B *of* $\{1, 2, \cdots, n\}$, *we have*

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0,$$

whenever f_1 *and* f_2 *are coordinatewise increasing functions and the covariance exists. An infinite family of r.v.'s is NA if every finite subfamily is NA.*

The notion of negative association was first introduced by Alam and Saxena (1981). Joag-Dev and Proschan (1983) showed that many well known multivariate distributions possess the NA property. Because of its wide applications in multivariate statistical analysis and system reliability, the notion of negative association has received considerable attention recently. We refer to Joag-Dev and Proschan (1983) for fundamental properties, Matula (1992) for three series theorem, Roussas (1994) for the central limit theorem of random fields, Shao and Su (1999) for the law of the iterated logarithm, Wang *et.al.*(1998), Amini and Bozorgnia (2000), Baek, Kim and Liang (2003) and Liang (2000) for complete convergence.

Throughout this paper, $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, where $a_{ni}^{+} = \max(a_{ni}, 0)$, $a_{ni}^{-} = \max(-a_{ni}, 0)$, c_0 , c, c', and c'' denote positive constant whose values are unimportant and may vary at different place.

2. MAIN RESULTS

In this section we will deal with an almost sure convergence for weighted sums of NA case.

THEOREM 2.1. *Let $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$, and let $\{X_i,\ i \ge 1\}$ be a sequence of random variables with $P(|X_i| > x) \le cP(|X| > x)$ for all $i \ge 1$, $x \ge 0$ and some constant $c > 0$. Assume that $\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ is an array of constants, and that $X_{ni}^2 \sum_{i=1}^{n} a_{ni}^2 \le v_n |X_i|^{\delta} / \log n$ a.s. for some $\delta > 0$ and some sequence $\{v_n\}$ of constants such that $v_n \to 0$.*

(a) Let $|a_{ni} X_{ni} I(X_{ni} \ge 0)| \le c|X_i|^{\beta}/\log n$ a.s. for $1 \le i \le n,\ n \ge 1$, some $0 < \beta \le \gamma$ and some constant $c > 0$. If $E(e^{h|X|^{\gamma}}) < \infty$ for any $h > 0$ ($\gamma > 0$), then

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n} a_{ni} X_{ni}| > \varepsilon) < \infty \quad \textit{for any } \varepsilon > 0 \textit{ and } r \ge 2. \tag{2.1}$$

(b) Let $|a_{ni} X_{ni} I(X_{ni} \ge 0)| \le u_n |X_i|^{\beta}/\log n$ a.s. for $1 \le i \le n,\ n \ge 1$, some $0 < \beta \le \gamma$ and some sequence $\{u_n\}$ of constants such that $u_n \to 0$. If $E(e^{h|X|^{\gamma}}) < \infty$ for some $h > 0$ ($\gamma > 0$), then for every $\beta > 0$, (2.1) remains true.

Furthermore, both (a) and (b) imply $\sum_{i=1}^{n} a_{ni} X_{ni} \to 0$ a.s. as $n \longrightarrow \infty$.

THEOREM 2.2. *Let $0 < \gamma \le 1$, and let $\{X_n, n \ge 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $P(|X_i| > x) \le cP(|X| > x)$ for all $i \ge 1$, $x \ge 0$ and some constant $c > 0$. Assume that $\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ is an array of constants satisfying*

$$\sum_{i=1}^{n} a_{ni}^2 = o((\log n)^{(2/\gamma)-1}) \quad \text{and} \quad \limsup_{n\to\infty} \sup_{i} |a_{ni}| \le c_0 < \infty. \tag{2.2}$$

(a) *If* $E(e^{h|X|^\gamma}) < \infty$ *for any* $h > 0$, *then*

$$\sum_{i=1}^{n} a_{ni}X_i/(\log n)^{1/\gamma} \longrightarrow 0 \ \ a.s. \ \ as \ \ n \longrightarrow \infty.$$

(b) *If* $E(e^{h|X|^\gamma}) < \infty$ *for some* $h > 0$ *and* $\beta > 0$, *then*

$$\sum_{i=1}^{n} a_{ni}X_i/(\log n)^{(1/\gamma)+\beta} \longrightarrow 0 \ \ a.s. \ \ as \ \ n \longrightarrow \infty.$$

REMARK 2.1. *In Theorem 2.2, we list the condition* $\limsup_{n\to\infty}\sup_i |a_{ni}| \le c_0 < \infty$ *in (2.2). Actually, we use only* $\max_{1\le i\le n}|a_{ni}| \le c$ *for some constant* $c > 0$ *in our proof. This condition is mild, it includes many known weights, we can obtain the following corollaries by Theorem 2.2.*

COROLLARY 2.1. *Let* $0 < \gamma \le 1$, *and let* $\{X_n, n \ge 1\}$ *be a sequence of NA random variables with* $EX_n = 0$ *and* $P(|X_i| > x) \le cP(|X| > x)$ *for all* $i \ge 1$, $x \ge 0$ *and some constant* $c > 0$. *Assume that* $\{a_{ni}, 1 \le i \le n,\ n \ge 1\}$ *is an array of constants satisfying (1.2) for* $1 < \alpha \le 2$.

(a) *If* $E(e^{h|X|^\gamma}) < \infty$ *for any* $h > 0$ *and* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$, *then*

$$\sum_{i=1}^{n} a_{ni}X_i/b_n \longrightarrow 0 \ \ a.s. \ \ as \ \ n \longrightarrow \infty.$$

(b) *If* $E(e^{h|X|^\gamma}) < \infty$ *for some* $h > 0$ *and* $b_n = n^{1/\alpha}(\log n)^{(1/\gamma)+\beta}$ *for* $\beta > 0$, *then*

$$\sum_{i=1}^{n} a_{ni}X_i/b_n \longrightarrow 0 \ \ a.s. \ \ as \ \ n \longrightarrow \infty.$$

REMARK 2.2. *Suppose that* $\{X_n, n \ge 1\}$ *are i.i.d. r.v.'s with* $EX_1 = 0$.

(a) *Bai and Cheng (2000) proved Corollary 2.1 (b) and* $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\gamma(\alpha-1)/\alpha(1+\gamma)}$, *Sung (2001) sharped this result of Bai and Cheng (2000) and choosed* $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\beta}$ $(\beta > 0)$. *Clearly, Corollary 2.1 (b) extends their results from the i.i.d case to not necessarily identically distributed NA setting.*

(b) *Sung (2001) discussed Corollary 2.1(a). Obviously, Corollary 2.1(a) extends the result of Sung (2001) from i.i.d case to NA setting.*

COROLLARY 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $P(|X_i| > x) \leq cP(|X| > x)$ for all $i \geq 1$, $x \geq 0$ and some constant $c > 0$. Assume that $\{a_{ni}, 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying*

$$\limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni}^2 < \infty.$$

If $E(e^{h|X|}) < \infty$ for all $h > 0$, then

$$\sum_{i=1}^{n} a_{ni} X_i / \log n \longrightarrow 0 \quad a.s. \quad as \quad n \longrightarrow \infty.$$

REMARK 2.3. *If $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables, then Corollary 2.2 reduces to the corresponding result of Chow and Lai (1973, Theorem 10). Therefore, Corollary 2.2 extends the result of Chow and Lai (1973, Theorem 10) to not necessarily independent and identically distributed NA case. Obviously, Theorem 2.2 is a more general result than Corollary 2.2.*

By the similar arguments as used in Theorem 2.2, it is easy to show that

THEOREM 2.3. *Let $0 < \gamma \leq 1$, and let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $P(|X_i| > x) \leq cP(|X| > x)$ for all $i \geq 1$, $x \geq 0$ and some constant $c > 0$. Assume that $\{c_i \mid i \geq 1\}$ is a sequence of constants satisfying*

$$\sum_{i=1}^{n} c_i^2 = o((\log n)^{(2/\gamma)-1}) \quad \text{and} \quad \max_i |c_i| \leq c < \infty.$$

(a) If $E(e^{h|X|^\gamma}) < \infty$ for any $h > 0$, then

$$\sum_{i=1}^{n} c_{n-i} X_i / (\log n)^{1/\gamma} \longrightarrow 0 \quad a.s. \quad as \quad n \longrightarrow \infty.$$

(b) If $E(e^{h|X|^\gamma}) < \infty$ for some $h > 0$, then for every $\beta > 0$,

$$\sum_{i=1}^{n} c_{n-i} X_i / (\log n)^{(1/\gamma)+\beta} \longrightarrow 0 \quad a.s. \quad as \quad n \longrightarrow \infty.$$

COROLLARY 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $P(|X_i| > x) \leq cP(|X| > x)$ for all $i \geq 1$, $x \geq 0$ and some constant $c > 0$. Assume that $\{c_i \mid i \geq 1\}$ is a sequence of constants.*

(a) *If* $\sum_{i=0}^{\infty} c_i^2 < \infty$ *and* $E(e^{h|X|}) < \infty$ *for all* $h > 0$, *then*

$$\sum_{i=1}^{n} c_{n-i} X_i / \log n \longrightarrow 0 \quad a.s. \quad as \quad n \longrightarrow \infty.$$

(b) *If* $c_n = O(n^{-\xi})$ *for some* $\xi > 1/2$ *and* $E(e^{h|X|^\gamma}) < \infty$ *for all* $h > 0$ *and* $0 < \gamma < 1$, *then*

$$\sum_{i=1}^{n} c_{n-i} X_i / (\log n)^{1/\gamma} \longrightarrow 0 \quad a.s. \quad as \quad n \longrightarrow \infty.$$

REMARK 2.4. *When* $\{X_n, n \geq 1\}$ *is a sequence of i.i.d. random variables, Corollary 2.3 reduces to the corresponding results of Chow and Lai (1973, Theorems 2 and 4) and extends the results of Chow and Lai (1973) from the i.i.d. case to not necessarily identically distributed NA setting. Obviously, Theorem 3 is a more general result than Corollary 2.3.*

3. PROOF OF MAIN RESULTS

PROOF. We prove only (a), the proof of (b) is similar. It suffices to show that

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n} a_{ni}^{+} X_{ni}| > \varepsilon) < \infty \quad for\ any\ \varepsilon > 0, \tag{3.1}$$

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n} a_{ni}^{-} X_{ni}| > \varepsilon) < \infty \quad for\ any\ \varepsilon > 0. \tag{3.2}$$

We prove only (3.1), the proof of (3.2) is analogous. To prove (3.1), we need only to prove that

$$\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \varepsilon) < \infty \quad for\ any\ \varepsilon > 0, \tag{3.3}$$

$$\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} < -\varepsilon) < \infty \quad for\ any\ \varepsilon > 0. \tag{3.4}$$

We first prove (3.3). From the definition of NA variables, we know that $\{a_{ni}^{+} X_{ni}, 1 \leq i \leq n,\ n \geq 1\}$ is still an arrays of rowwise NA random variables. For all $x \in \mathbb{R}$, by putting $|x|^\delta \le O(e^{c|x|^\beta})$, $e^x \le 1 + x + \frac{1}{2}x^2 H(x)$, and $t = M \log n/\varepsilon$, where $H(x) = \max(1, e^x)$, M is large constant and will be specified later on, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} > \varepsilon) \\
&\le \sum_{n=1}^{\infty} n^{r-2} e^{-\varepsilon t} E e^{t \sum_{i=1}^{n} a_{ni}^{+} X_{ni}} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{n} E e^{t a_{ni}^{+} X_{ni}} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{n} \{1 + \frac{1}{2} t^2 (a_{ni}^{+})^2 E X_{ni}^2 H(t a_{ni}^{+} X_{ni})\} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{n} \{1 + c(\log n)^2 a_{ni}^2 [E X_{ni}^2 + E X_{ni}^2 I(X_{ni} \ge 0) e^{t a_{ni}^{+} X_{ni}}]\} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{n} \{1 + c v_n (\log n) \frac{a_{ni}^2}{\sum a_{ni}^2} [E(|X_i|^\delta) + E(|X_i|^\delta e^{c|X_i|^\beta})]\} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{n} \{1 + O(1) v_n (\log n) \frac{a_{ni}^2}{\sum a_{ni}^2} E(e^{c'|X_i|^\beta})\} \\
&\le \sum_{n=1}^{\infty} n^{r-2-M} \prod_{i=1}^{n} \{1 + c''(\log n) \frac{a_{ni}^2}{\sum a_{ni}^2}\} \\
&\le \sum_{n=1}^{\infty} n^{(r+c'')-(2+M)} < \infty
\end{aligned}$$

provided $M > (r + c'') - 1$. Thus, (3.3) is proved. By replacing X_{ni} by $-X_{ni}$ from the above statement and noticing $\{a_{ni}^{+}(-X_{ni}), 1 \le i \le n,\ n \ge 1\}$ is still an arrays of rowwise NA random variables, we know that

$$\sum_{n=1}^{\infty} n^{r-2} P(\sum_{i=1}^{n} a_{ni}^{+} X_{ni} < -\varepsilon) < \infty \quad \text{for any} \quad \varepsilon > 0.$$

Hence the result follows by (3.3) and (3.4). □

PROOF. We prove only (a), the proof of (b) is similar. Since

$$\sum_{i=1}^{n} a_{ni} X_i / (\log n)^{1/\gamma} = \sum_{i=1}^{n} a_{ni}^{+} X_i / (\log n)^{1/\gamma} - \sum_{i=1}^{n} a_{ni}^{-} X_i / (\log n)^{1/\gamma},$$

we may assume, without loss of generality, that $a_{ni} > 0$. Let $X'_{ni} = X_i I(X_i \leq (\log n)^{1/\gamma}) + (\log n)^{1/\gamma} I(X_i > (\log n)^{1/\gamma})$,
$X''_{ni} = X_i - X'_{ni}$ for $1 \leq i \leq n$ and $n \geq 1$. Then X''_{ni} and X'_{ni} are still NA random variables by the property of NA (*cf.* Joag and Proschan (1983)), so

$$|\sum_{i=1}^{n} a_{ni}X_i|/(\log n)^{1/\gamma} \leq |\sum_{i=1}^{n} a_{ni}X'_{ni}|/(\log n)^{1/\gamma} + |\sum_{i=1}^{n} a_{ni}X''_{ni}|/(\log n)^{1/\gamma}$$

$=: A_n + B_n$.
Note that $E(e^{|X|^\gamma}) < \infty$ implies

$$\sum_{n=1}^{\infty} P(|X_n| > (\log n)^{1/\gamma}) < \infty.$$

Hence, by the Borel-Cantelli Lemma,

$$\sum |X''_{ni}|$$

is bounded $a.s.$ it follows from (2.2) that

$$B_n = |\sum a_{ni}X''_{ni}|/(\log n)^{1/\gamma} \leq \max_{1\leq i\leq n} |a_{ni}| \sum_{i=1}^{n} |X''_{ni}|/(\log n)^{1/\gamma}$$

$$\leq c\sum_{i=1}^{n} |X''_{ni}|/(\log n)^{1/\gamma} \longrightarrow 0 \quad a.s.$$

as $n \longrightarrow \infty$. To prove

$$A_n = |\sum a_{ni}X'_{ni}|/(\log n)^{1/\gamma} \longrightarrow 0 \quad a.s.,$$

we will apply Theorem 1 to the random variables X'_{ni} and weight $(\log n)^{-1/\gamma}a_{ni}$.

Note that $EX'_{ni} + EX''_{ni} = EX_i = 0$ and $EX''_{ni} \geq 0$, so $EX'_{ni} \leq 0$. In addition, we observe that

$$\begin{aligned}
&|a_{ni}X'_{ni}I(X'_{ni} \geq 0)|/(\log n)^{1/\gamma} \\
&\quad < |a_{ni}X_i I(0 \leq X_i \leq (\log n)^{1/\gamma})|/(\log n)^{1/\gamma} + |a_{ni}I(X_i > (\log n)^{1/\gamma})| \\
&\quad < \max_{1\leq i\leq n} |a_{ni}|\{|X_i|I(|X_i| \leq (\log n)^{1/\gamma})/(\log n)^{1/\gamma} + I(X_i > (\log n)^{1/\gamma})\} \\
&\quad < c\{(\log n)^{(1-\gamma)/\gamma}|X_i|^\gamma/(\log n)^{1/\gamma} + |X_i|^\gamma/\log n\} \\
&\quad < c|X_i|^\gamma/\log n \text{ and } (X'_{ni})^2 \sum_{i=1}^{n} a_{ni}^2/(\log n)^{2/\gamma} \\
&\qquad\qquad \leq X_i^2 \sum_{i=1}^{n} a_{ni}^2/(\log n)^{2/\gamma} \leq o(1)X_i^2/\log n.
\end{aligned}$$

Hence, according to Theorem 2.1, we have

$$A_n = \sum_{i=1}^{n} a_{ni} X'_{ni} / (\log n)^{1/\gamma} \longrightarrow 0 \quad a.s.$$

as $n \longrightarrow \infty$. Hence the result of Theorem 2.2 follows by A_n and B_n. □

PROOF. Set $b_{ni} = a_{ni}/n^{1/\alpha}$. Since weights a_{ni} satisfy (1.2) and $1 < \alpha \le 2$,

$$|b_{ni}| = |a_{ni}|/n^{1/\alpha} \le (\sum_{i=1}^{n} |a_{ni}|^{\alpha})^{1/\alpha}/n^{1/\alpha} = A_{\alpha,n} \quad \text{and}$$

$$\sum_{i=1}^{n} b_{ni}^2 = \sum_{i=1}^{n} a_{ni}^2/n^{2/\alpha} \le (\sum_{i=1}^{n} |a_{ni}|^{\alpha})^{2/\alpha}/n^{2/\alpha} = A_{\alpha,n}^2 = o(\log^{2/\gamma - 1} n).$$

Hence, Theorem 2.2 yields Corollary 2.1. □

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