

## NON-COMPACT DOUGLAS-PLATEAU PROBLEM

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ABSTRACT. In this article, we prove the existence of two embedded minimal annuli in a slab which are all bounded by a Jordan convex curve and a straight line.

### 1. Introduction

The classical Douglas-Plateau problem for two *compact* contour is to find a minimal annulus bounded by two disjoint Jordan curves. One classical result due to Douglas says that if  $A_1$  and  $A_2$  are the least area disks bounded by Jordan curves  $\gamma_1$  and  $\gamma_2$ , respectively, satisfying

$$\inf\{\text{Area}(S)\} < \text{Area}(A_1) + \text{Area}(A_2)$$

where the infimum is over all surfaces' areas of annular type bounded by  $\gamma_1$  and  $\gamma_2$ , then there is a minimal annulus with the boundary  $\gamma_1 \cup \gamma_2$ . Additionally, D. Hoffman and W. Meeks of [5], W. Meeks and B. White of [10], and Yi Fang and J-F. Hwang of [4] gave another kind of sufficient or necessary conditions for this problem. In general, the *non-compact* Douglas-Plateau problem becomes more difficult. There is a classical example bounded by two parallel straight lines, which is a piece of one of *Riemann's minimal examples*. Recall the one-parameter family of Riemann's minimal examples are the only complete minimal surfaces of  $\mathbf{R}^3$  foliated by circles and straight lines in parallel planes except planes, catenoids, and helicoids, see [13]. Recently, Yi Fang and J-F. Hwang[4] solved various *non-compact* problems for minimal annuli with two convex boundary curves lying on parallel planes. For example, they proved the existence of two embedded minimal annuli bounded by continuously embedded, proper, complete, non-compact, *non-flat* convex curves in

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parallel planes. They used the Jenkins and Serrin's minimal graphs of [7] as the barriers confining all of approximating surfaces.

In this paper, we consider minimal annuli in a slab, whose boundary consists of a planar convex Jordan curve and a straight line by that:

**THEOREM 1.** *Let  $\gamma \subset P_1$  be a convex Jordan curve and let  $\ell \subset P_0$  be a straight line, where  $P_t := \{(x, y, z) \in \mathbf{R}^3 \mid z = t\}$  is a horizontal plane at the height  $t \in \mathbf{R}$ . Denote  $D_\gamma$  by the compact planar disk in  $P_1$  with  $\partial D_\gamma = \gamma$  and let  $P_0^+$  be the half-plane of  $P_0$  bounded by  $\ell$ . Suppose that there is a (maybe branched) compact minimal surface  $\Sigma$  with  $\partial\Sigma \subset D_\gamma \cup P_0^+$ , then there are two embedded minimal annuli  $\mathcal{A}$  and  $\mathcal{B}$  such that*

- (1)  $\partial\mathcal{A} = \partial\mathcal{B} = \gamma \cup \ell$ .
- (2) For each  $t \in (0, 1)$ ,  $\mathcal{A} \cap P_t$  and  $\mathcal{B} \cap P_t$  are strictly convex Jordan curves.
- (3)  $\text{Int}(\mathcal{A}) \cap \text{Int}(\mathcal{B}) = \emptyset$ .
- (4) Now let  $N$  be a connected compact non-planar (maybe branched) minimal surface such that  $\partial N \subset \overline{D_\gamma} \cup \overline{P_0^+}$ , then

$$\text{Int}(\mathcal{A}) \cap \text{Int}(N) = \emptyset, \quad \mathcal{B} \cap N \neq \emptyset.$$

- (5)  $\mathcal{A}$  and  $\mathcal{B}$  have the same symmetry groups as that of  $\gamma \cup \ell$ .

The basic idea to prove this theorem is to approximate the straight line  $\ell \subset P_0$  with convex Jordan curves  $\gamma_n \subset P_0$ ,  $n = 1, 2, 3, \dots$ . Then, by Proposition 1 in the next section, we can get embedded minimal annuli  $A_n$  and  $B_n$  bounded by  $\gamma \cup \gamma_n$ . Take a Riemann's minimal example bounded by the straight line  $\ell$  and a circle lying on  $P_1$  containing  $\gamma$  in its interior, and use it as the barrier confining all of  $A_n$ 's and  $B_n$ 's. Then with the similar method of the proof of Theorem 3.1 in [4], we prove that there are subsequences of  $\{A_n\}$  and  $\{B_n\}$  converge to embedded minimal annuli  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, in the interior of the slab bounded by  $P_0$  and  $P_1$ . Moreover, since the boundary curves  $\gamma$  and  $\gamma_n$  are convex, M. Shiffman's first theorem in [14] shows that the intermediate curves  $A_n \cap P_t$  and  $B_n \cap P_t$ ,  $0 < t < 1$ , are all strictly convex Jordan curves. Therefore we can divide the approximating annulus  $A_n$  into two graphs over a vertical plane, each of which is simply connected. The same to  $B_n$ . Then we can use the Courant-Lebesgue lemma in [3] to prove that the convergence can be extended to their boundaries, respectively, and  $\partial\mathcal{A} = \partial\mathcal{B} = \gamma \cup \ell$ .

## 2. Preliminaries

Let us define a slab and a solid cylinder as followings:

$$S(a, b) = \{(x, y, z) \in \mathbf{R}^3 \mid a \leq z \leq b\}$$

$$C_r = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 \leq r^2\},$$

respectively, and  $K_M$  is the Gaussian curvature of a surface  $M \subset \mathbf{R}^3$ .

LEMMA 1 (COMPACTNESS LEMMA [1]). *Let  $\{M_i\}$ ,  $i = 1, 2, 3, \dots$ , be a sequence of minimal surfaces in a bounded domain  $\Omega$  of  $\mathbf{R}^3$ . Suppose there is a constant  $c > 0$  such that  $|K_{M_i}(x)| < c$  for all  $i$ . Then we have a subsequence of  $\{M_i\}$  converges smoothly to an immersed minimal surface  $M_\infty$  (with multiplicity) in  $\Omega$  and  $|K_{M_\infty}(x)| \leq c$ . If each  $M_i$  is embedded, then  $M_\infty$  is also embedded.*

LEMMA 2 (INTERIOR CURVATURE ESTIMATE [4]). *If  $A \subset S(t_1, t_2) \cap C_r$  is a minimal annulus such that  $\partial A = \gamma_1 \cup \gamma_2$ , where  $\gamma_i \subset P_{t_i}$   $i = 1, 2$ , are convex Jordan curves, then there is an absolute constant  $c_0 > 0$ , such that  $K(p) \leq \frac{c_0 r^2}{d^4}$ , where  $d = \text{dist}(p, \partial S(t_1, t_2))$ .*

With this lemma, Yi Fang and J-H. Hwang gave a generalization of a theory dealing with smooth convex boundary developed by D. Hoffman and W. Meeks in [5] and B. White in [15], to the continuous boundary case by following:

PROPOSITION 1. [4] *Let  $D_\alpha$  and  $D_\beta$  be two open disks lying on parallel planes, and their boundaries  $\alpha$  and  $\beta$  are continuous convex Jordan curves, respectively. Suppose that there is a (maybe branched) minimal surface  $\Sigma$  with  $\partial \Sigma \subset D_\alpha \cup D_\beta$ , then there exist at least two embedded minimal annuli  $A$  and  $B$  bounded by  $\alpha \cup \beta$  such that*

- (1)  *$A$  is stable,  $B$  is unstable. Recall a minimal surface is called stable if, with respect to any non-trivial normal variation that fixing the boundary, the second derivative of area functional is positive. If the second derivative of area is negative for some variation, then this surface is called unstable.*
- (2) *If  $M \neq A$  is a compact (maybe branched) minimal surface with  $\partial M \subset D_\alpha \cup D_\beta$ , then  $M$  is contained in the compact region of  $\mathbf{R}^3$  bounded by  $A \cup D_\alpha \cup D_\beta$  and*

$$\text{Int}(A) \cap \text{Int}(M) = \emptyset, \quad \text{Int}(B) \cap \text{Int}(M) \neq \emptyset.$$

- (3) *Both  $A$  and  $B$  have the same symmetry group of the boundary.*

Finally we state some general compactness arguments of minimal surfaces of A. Ros and J. Pérez in [12]. First we need to control uniformly the relative size of the domain that expresses locally a minimal surface as a graph over the tangent plane. Let  $M$  be a surface of  $\mathbf{R}^3$  with the tangent plane  $TM_p$ ,  $p \in M$ , and the Gauss map  $G : M \rightarrow S^2$ . Given  $p \in M$  and  $r > 0$ , we label the tangent disk of radius  $r$  by

$$D^T(p, r) := \{p + v \mid v \in TM_p, |v| < r\}.$$

Denote  $W(p, r, \epsilon)$  the compact slice of the solid cylinder of radius  $r$  around the affine normal line at  $p$ ,

$$W(p, r, \epsilon) := \{q + tG(q) \mid q \in D^T(p, r), |t| < \epsilon\}$$

Now we formulate the notion of convergence for minimal surfaces.

DEFINITION 1. [12] Let  $M$  and  $M_n$ ,  $n = 1, 2, 3, \dots$ , be properly embedded minimal surfaces lying on an open set  $O \subset \mathbf{R}^3$ . We say that the sequence  $\{M_n\}$  converges to  $M$  in  $O$  with finite multiplicity if  $M$  is the accumulation set of  $\{M_n\}$  and for all  $p \in M$  there exist  $r, \epsilon > 0$  with

- $M \cap W(p, r, \epsilon)$  can be expressed as the graph of a function

$$u : D^T(p, r) \rightarrow \mathbf{R}$$

- For all  $n$  large enough,  $M_n \cap W(p, r, \epsilon)$  consists of a finite number (independent of  $n$ ) of graphs over  $D^T(p, r)$  which converge to  $u$  in the  $C^m$ -topology, for each  $m \geq 0$ .

In the situation above, we define the *multiplicity* of a given  $p \in M$  as the number of graphs in  $M_n \cap W(p, r, \epsilon)$ , for  $n$  large enough. Clearly, this multiplicity remains constant on each connected component of  $M$ . Given a sequence of subsets  $\{F_n\}$  in the open domain  $O$ , its *accumulation set* is defined by  $\{p \in O \mid \text{There is } p_n \in F_n \text{ with } p_n \rightarrow p\}$ . Notice the Gaussian curvature bound also follows the compactness result in the following general settings:

LEMMA 3 (Bounded Area [12]). Let  $\{M_n\}$  be a sequence of properly embedded minimal surfaces contained in an open region  $O$ . Suppose that  $\{M_n\}$  has an accumulation point and that for any 3-ball  $B \subset O$  there exist positive constants  $c_i = c_i(B)$ ,  $i = 1, 2$ , with

$$\text{Area}(M_n \cap B) \leq c_1, \quad |K_{M_n \cap B}| \leq c_2$$

for all  $n$ . Then there exists a subsequence  $\{M_k\}$  of  $\{M_n\}$  and a properly embedded minimal surface  $M \subset O$  such that  $\{M_k\}$  converges to  $M$  in  $O$  with finite multiplicity.

LEMMA 4 (Unbounded Area [12]). Let  $\{M_n\}$  denote a sequence of properly embedded minimal surfaces contained in an open subset  $O$  of  $\mathbf{R}^3$ . Suppose that there is a sequence  $p_n \in M_n$  converging to a point  $p \in O$  and that for any 3-ball  $B \subset O$  there exist a positive constant  $c = c(B)$  with  $|K_{M_n \cap B}| \leq c$  for all  $n$ . Then there exists a subsequence  $\{M_k\}$  and a connected minimal surface  $M$  in  $O$  satisfying

- (1)  $M$  is contained in the accumulation set of  $\{M_k\}$ .
- (2)  $p \in M$  and  $K_M(p) = \lim_{k \rightarrow \infty} K_{M_k}(p_k)$ .
- (3)  $M$  is embedded in  $O$ , but not necessarily properly embedded.
- (4) Any divergent path in  $M$  either diverges in  $O$  or has infinite length.

Although both general compactness arguments hold in the sequence of properly embedded minimal surfaces *without boundary*, with suitable modifications using the method of B. White in [15], we may have the similar result of a sequence of properly embedded minimal surfaces *with boundary*.

### 3. The proof of the theorem

In this section, we will prove the theorem by several steps. Let  $\ell$  denote the  $y$ -axis, and let  $\{D_n\}$ ,  $n = 1, 2, \dots$ , be a sequence of disks in  $P_0$  such that

$$(D_n \cap \ell) \subset (D_{n+1} \cap \ell), \quad (D_n \cap P_0^+) \subset (D_{n+1} \cap P_0^+)$$

$$\lim_{n \rightarrow \infty} D_n = \ell, \quad \partial \Sigma \subset D_\gamma \cup (D_1 \cap P_0^+),$$

where  $P_0^+ := P_0 \cap \{x > 0\}$  is the half plane bounded by  $\ell$ . For example, if we take

$$r_n = a + n^2 + n + \frac{1}{n}, \quad a_n = a + n^2 + n$$

for a large  $a \geq 0$  satisfying the given conditions, then the disk

$$D_n := \{(x, y, 0) \mid (x - a_n)^2 + y^2 = r_n^2\}$$

satisfies the above all conditions. On the other hand, let us denote the continuous convex curve  $\gamma_n$  by the union of an circular arc of  $\partial D_n$  in the positive half-plane and a portion of  $\ell$  contained in  $\overline{D}_n$  such that

$$\gamma_n := (\partial D_n \cap P_0^+) \cup (\ell \cap \overline{D}_n).$$

If we denote the compact set  $\Gamma_n$  in the half-plane  $P_0^+$  with the boundary  $\partial \Gamma_n = \gamma_n$ , then

$$\Gamma_1 \subset \Gamma_2 \subset \dots \longrightarrow P_0^+.$$

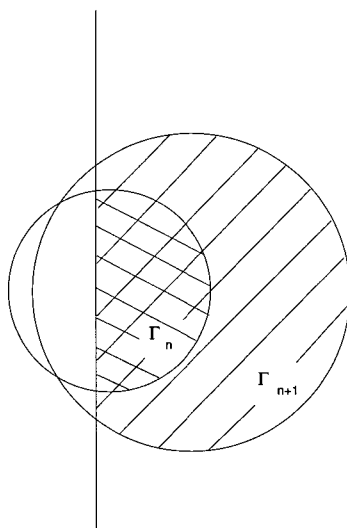


FIGURE 1.

Observe that for any  $r > 0$  there is an integer  $n_0 > 0$  such that

$$\gamma_n \cap C_r = \ell \cap C_r \quad \text{whenever } n > n_0.$$

Since both  $\gamma$  and  $\gamma_n$  are continuous convex curves in the parallel planes and  $\partial\Sigma \subset D_\gamma \cup \Gamma_n$ , by Proposition 1 there exist a stable embedded minimal annulus  $A_n$  and an unstable embedded minimal annulus  $B_n$  bounded by  $\gamma \cup \gamma_n$  for all  $n = 1, 2, \dots$ , respectively.

STEP 1. In this step, we will show that both sequences  $\{A_n\}$  and  $\{B_n\}$  converge in the interior of the slab  $S(0, 1)$ . Denote  $D \subset P_1$  by a disk whose boundary  $\partial D$  is a circle containing  $\gamma$  in its interior. Recall  $\partial\Sigma \subset D \cup D_n$  for all  $n$ , together with Proposition 1 again, it leads us to take a *stable* embedded minimal annulus  $R_n$  such that

$$\partial R_n = \partial D \cup \partial D_n.$$

CLAIM 1. *The sequence of Gaussian curvature  $\{K_{R_n}\}$  of  $R_n$  is uniformly bounded.*

*Proof.* Suppose not, then we have a point  $p_n \in R_n$  where  $K_{R_n}$  reaches its minimum, that is;

$$\begin{aligned} -C_n := K_{R_n}(p_n) &\leq K_{R_n}(p) \quad \text{for all } p \in R_n \\ C_n &\rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Denote by  $p_n = (x_n, y_n, z_n)$ ,  $0 \leq z_n \leq 1$ , and expanding homothetically each  $R_n$  such that

$$\tilde{R}_n := \sqrt{C_n}(R_n - p_n),$$

where  $R_n - p_n := \{p \in \mathbf{R}^3 \mid p + p_n \in R_n\}$ . This new minimal surface  $\tilde{R}_n$  passes through the origin and the Gaussian curvature attains its minimum at that point with value  $-1$  for all  $n = 1, 2, \dots$ . Moreover, the boundary of  $\tilde{R}_n$  is a planar convex curve and the spherical image of  $\tilde{R}_n$  is equal to that of  $R_n$ , so we have

$$\int_{\tilde{R}_n} K_{\tilde{R}_n} dA = \int_{R_n} K_{R_n} dA > -4\pi$$

since the Gauss map is one-to-one in  $R_n$ . Additionally,  $\{\tilde{R}_n\}$  is a sequence of properly embedded minimal surfaces, with uniform Gaussian curvature bound and an accumulation point at the origin. Under these conditions, with a suitable modification to deal with the boundary curves using the methods of [15], Lemma 4 implies, after passing to a subsequence, that there exists a connected embedded minimal surface  $\tilde{M}$  which is contained in the accumulation set of  $\{\tilde{R}_n\}$  and passes the origin with absolute Gaussian curvature one at that point. So  $\tilde{M}$  is not a plane, and every divergent path in  $\tilde{M}$  either diverges to  $\partial\tilde{M}$  or has infinite length. Notice  $\tilde{M}$  is produced by analytic continuation, starting at the accumulation point  $0 \in \mathbf{R}^3$ , of the limits of graph pieces of the surfaces  $\tilde{R}_n$ . Therefore,  $\tilde{M}$  has total curvature not less than  $-4\pi$ . If the boundary of  $\tilde{M}$  exists, it must be the limit of arbitrary large homothetic images  $\partial\tilde{R}_n \cap P_{t_0}$  where either  $t_0 := \sqrt{C_n}(1 - z_n)$  or  $t_0 := -\sqrt{C_n}z_n$ . It is clear that the radius of boundary circles of  $R_n$  are not less than the minimum  $m > 0$  of the radius of  $\partial D$  and that of  $\partial D_1$ . So the planar curvature of  $\partial\tilde{R}_n$  is bounded by

$$\frac{m}{\sqrt{C_n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty :$$

it means that the boundary curve  $\partial\tilde{M}$  must be a straight line. However, if we rotate  $\tilde{M}$  around it by  $\pi$  degrees, then we get a complete minimal surface without boundary with total curvature at least  $-8\pi$  containing a straight line, which is impossible, see [9]. We have shown that

$$\partial\tilde{M} = \emptyset.$$

As a result, the limit surface  $\tilde{M}$  is a minimal surface without boundary which has the total curvature not less than  $-4\pi$ . It means that  $\tilde{M}$  must be a catenoid.

Now take a disk  $\tilde{D} \subset P_0^+$  in the half-plane where  $\partial\Sigma \subset \partial D \cup \partial\tilde{D}$ , then, by Proposition 1 again, there is another Riemann's minimal example  $\tilde{R} \subset S(0, 1)$  bounded by  $\partial D \cup \partial\tilde{D}$ . We can choose a large number  $n_0 > 0$  satisfying

$$\tilde{D} \subset D_n \text{ for all } n \geq n_0.$$

Since every  $R_n$  is stable, by Proposition 1-2, if  $n \geq n_0$

$$(1) \quad \tilde{R} \subset V_{R_n},$$

where  $V_{R_n}$  is the solid of the slab  $S(0, 1)$  bounded by  $R_n$ , respectively. Let us denote  $r := \min_{0 \leq t \leq 1} |\tilde{R} \cap P_t| > 0$ , then we can say that

$$(2) \quad \begin{aligned} |\tilde{R}_n \cap P_0| &= \sqrt{C_n} |R_n \cap P_{z_n}| \\ &> \sqrt{C_n} |\tilde{R} \cap P_{z_n}| \\ &\geq r \sqrt{C_n} \longrightarrow \infty, \end{aligned}$$

which is contradict to that  $\tilde{M}$  is a catenoid. Note every intermediate circle of a catenoid must have the bounded length. So  $\{R_n\}$  has the uniform curvature bound, and the claim is proved.

CLAIM 2. Given  $p \in \mathbf{R}^3$ , there exists a positive number  $r_0 = r_0(p)$  and a positive uniform constant  $c = c(p)$  such that for all  $n$ ,

$$\text{Area}(R_n \cap B_r) \leq cr^2$$

if  $r > r_0$ , where  $B_r := B(p, r)$  is the ball of radius  $r$  with center  $p$ .

*Proof.* Consider the function  $|q - p|^2$ , where  $q \in R_n$ . Since  $R_n$  is minimal, we have

$$\Delta_{R_n} |q - p|^2 = 2 \operatorname{div}_{R_n} |q - p| = 4,$$

where  $\Delta_{R_n}$  and  $\operatorname{div}_{R_n}$  denote the Laplacian and the divergence of  $R_n$ , respectively. By Stokes' theorem integrating this gives

$$4\text{Area}(B_r \cap R_n) = \int_{B_r \cap R_n} \Delta_{R_n} |q - p|^2 = 2 \int_{\partial(B_r \cap R_n)} \langle q - p, \eta \rangle,$$

where  $\eta$  is the exterior conormal vector along the boundary curve. It is geometrically clear (and follows from the *co-area formula* as that of [2] or [8]) that

$$\int_{\partial B_r \cap R_n} \langle q - p, \eta \rangle \leq r \frac{d}{dr} (\text{Area}(B_r \cap R_n)).$$



Recall each end of Riemann’s minimal examples must be asymptotic to a horizontal plane. So for a sufficiently large  $r$ , the area of  $B_r \cap R_n$  has the linear growth with respect to  $r$ . Thus we have

$$\begin{aligned} &\text{Area}(B_r \cap R_n) \\ &= \frac{1}{2} \int_{\partial B_r \cap R_n} \langle q - p, \eta \rangle + \frac{1}{2} \int_{\partial D_n \cap B_r} \langle q - p, \eta \rangle + \frac{1}{2} \int_{\partial D} \langle q - p, \eta \rangle \\ &\leq \tilde{c}r^2 + \pi r^2 + \frac{1}{2} |\partial D| \cdot r \end{aligned}$$

for a constant  $\tilde{c} > 0$ , since  $\langle q - p, \eta \rangle \leq |q - p| \leq r$  on  $B_r$ .

We have shown the uniform curvature estimate and the uniform area bound condition of  $\{R_n\}$ . Therefore by Lemma 3, together with the arguments in [15] to deal of the boundary, there exists a (possibly non connected) complete embedded minimal surface  $\mathcal{R}$  such that for each component  $M$  of  $\mathcal{R}$  has the boundary  $\partial D \cup \ell$ , if it exists, at least one of the components has nonvoid boundary. Moreover, after passing to a subsequence, the surfaces  $R_n$ ’s converge uniformly on compact sets to  $\mathcal{R}$  in the interior of the slab  $S(0, 1)$  with finite multiplicity which depends on the connected component of  $\mathcal{R}$ . In particular, any component of  $\mathcal{R}$  must have genus zero and the total curvature is at least  $-4\pi$ , this control on the genus follows because each component is finitely covered by surfaces of genus zero. Suppose that  $\partial M = \partial D \cup \ell$ , then, by the second result of M. Shiffman of [14], each intermediate curve  $M \cap P_t$ ,  $0 < t < 1$ , is also a circle. The same holds with  $R_n \cap P_t$ ,  $0 < t < 1$ , so the multiplicity of the limit must be one. Since the sub-annulus  $R_n \cap S(1 - t, 1)$  can be chosen arbitrarily close to  $\bar{M} \cap S(1 - t, 1)$  for sufficiently small  $t > 0$ , there is no other component with nonempty boundary. Let  $M'$  be a component of  $\tilde{\mathcal{R}}$  has no boundary, then it would be a horizontal plane by the Halfspace theorem in [6]. In particular  $M'$  must intersect  $M$ , which is a contradiction. As consequence, the limit surface  $\mathcal{R}$  is just the above component  $M$ . Observe that

$$\partial \mathcal{R} = \partial D \cup \ell$$

and it can not be a plane, a catenoid, nor a helicoid, such that Therefore,  $\mathcal{R}$  must be a piece of a Riemann’s minimal example, see [13].

In the above processing, the stability of all of  $R_n$ ’s is crucial to prove the uniform curvature estimate of the sequence. In particular, see the previous arguments (1) and (2). Unfortunately, because that each  $B_n$  is not stable, we cannot use the same arguments for it. If  $m \geq n$  then

$$\partial A_n = \partial B_n = \gamma \cup \gamma_n \subset D \cup D_m.$$

By Proposition 1-2, since  $R_m$  is stable, we have

$$\text{Int}(A_n) \cap \text{Int}(R_m) = \emptyset, \quad \text{Int}(B_n) \cap \text{Int}(R_m) = \emptyset$$

for all  $m \geq n$ . Let  $V$  and  $V_m$  be the solids in  $\mathbf{R}^3$  such that

$$\partial V = \mathcal{R} \cup \overline{P_0^+} \cup \overline{D}, \quad \partial V_m = R_m \cup D_m \cup D$$

then  $A_n, B_n \subset V_m$  for all  $m \geq n$ , clearly, and  $V_m$  tends to  $V$ . Thus

$$A_n, B_n \subset V.$$

Recall  $\partial V$  meets the intermediate plane along the circles, so for given  $0 < t < 1$  we can choose a positive number  $r(t)$  such that

$$V \cap P_t \subset C_{r(t)}.$$

From now on, we have shown that for all  $n$ ,

$$A_n \cap S(t, 1) \subset C_{r(t)}, \quad B_n \cap S(t, 1) \subset C_{r(t)},$$

where  $0 < t < 1$ , respectively. Now we can use Lemma 2, and then Lemma 1, to prove that for any  $0 < t_m < 1$  there are subsequence of  $\{A_n \cap S(t_m, 1)\}$  and  $\{B_n \cap S(t_m, 1)\}$  converging to embedded compact minimal annuli

$$A_{t_m}, B_{t_m} \subset S(t_m, 1),$$

respectively, where  $t_m \rightarrow 0$  as  $m \rightarrow \infty$ . By a diagonal argument, we see that these subsequences converge to embedded minimal surfaces  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, in the interior of  $S(0, 1)$ . Now for each  $s \in (t_m, 1)$ , the intermediate curves  $A_n \cap P_s$  and  $B_n \cap P_s$  are convex Jordan curves and both convergences are smooth, so  $\mathcal{A} \cap P_s$  and  $\mathcal{B} \cap P_s$  are strictly convex. Hence  $\mathcal{A}$  and  $\mathcal{B}$  are minimal annuli.

STEP 2. In this step, we extend the above convergence result to the boundary. Take a vertical plane  $\Pi$ , we may let it be the  $xz$ -plane, and denote the projection to it by  $\text{Proj}_\Pi(x, y, z) = (x, 0, z)$ . Let us define

$$\Omega_\Pi^n := \text{Int}(\text{Proj}_\Pi(A_n)),$$

which is bounded by two line segments  $\ell_1 = \text{Proj}_\Pi(\gamma)$ ,  $\ell_2^n = \text{Proj}_\Pi(\gamma_n)$ , and two curves  $\alpha_n, \beta_n$  connecting a point of  $\ell_1$  and the other point of  $\ell_2^n$ . By Shiffman's first theorem in [14], every intermediate curve  $A_n \cap P_t$ ,  $0 < t < 1$ , is a strictly convex Jordan curve. Thus each  $\text{Proj}_\Pi(A_n \cap P_t)$  is also a line segment and  $A_n \cap P_t$  consists of two graphs over  $\Omega_\Pi^n$ . In this way, we may say that  $\alpha_n$  is the left side boundary and  $\beta_n$  is the right side boundary of  $\Omega_\Pi^n$ . Recall, in the proof of Lemma 1 in [4], we

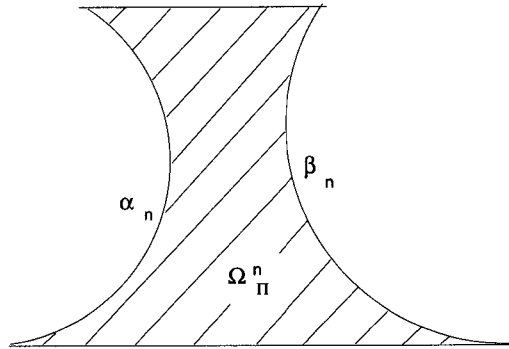


FIGURE 2.

have known that both  $\alpha_n$  and  $\beta_n$  are smooth convex curves. Moreover,  $\Omega_\Pi^n$  is a domain with piecewise smooth boundary by

$$\partial\Omega_\Pi^n = \ell_1 \cup \ell_2^n \cup \alpha_n \cup \beta_n$$

with only four corner points. Notice  $A_n$  consists of two subsets  $A_n^+$  and  $A_n^-$  where  $\text{Int}(A_n^+)$  and  $\text{Int}(A_n^-)$  are minimal graphs over  $\Omega_\Pi^n$ . Let us denote  $\gamma^\pm = \gamma \cap \partial A_n^\pm$  and  $\gamma_n^\pm = \gamma_n \cap \partial A_n^\pm$ , then we have

$$\partial A_n^\pm = \alpha_n \cup \beta_n \cup \gamma^\pm \cup \gamma_n^\pm.$$

Additionally, the limit surface  $\mathcal{A}$  also consists of two simply connected minimal surfaces  $\mathcal{A}^+$  and  $\mathcal{A}^-$ , whose interiors are minimal graphs over a domain in the vertical plane  $\Pi$ , and  $A_n^+$  converges to  $\mathcal{A}^+$  as  $n \rightarrow \infty$ . On the other hand, we choose a solid cylinder  $C_r$  for large  $r > 0$  which contains  $\gamma$  in its interior. In the previous argument, we have proved that both the left side boundary  $\alpha_n$  and the right side boundary  $\beta_n$  of  $\Omega_\Pi^n$  are convex for all vertical planes  $\Pi$ , so  $A_n \setminus C_r$  becomes a graph over a domain in the horizontal plane  $P_0$ , and  $A_n \cap \partial C_r$  is a simple curve. Let  $\tilde{\alpha}_n^+$  and  $\tilde{\beta}_n^+$  be the parts of  $\partial A_n^+$  such that  $\text{Proj}_\Pi(\tilde{\alpha}_n^+) = \alpha_n$  and  $\text{Proj}_\Pi(\tilde{\beta}_n^+) = \beta_n$ , then we have

$$\partial(A_n^+ \cap C_r) = (\tilde{\alpha}_n^+ \cap C_r) \cup (\tilde{\beta}_n^+ \cap C_r) \cup \gamma^+ \cup (\gamma_n^+ \cap C_r) \cup (A_n^+ \cap \partial C_r),$$

which is a connected simple closed curve. Therefore,  $A_n^+ \cap C_r$  is a simply connected minimal surface. Now we can take a closed unit disk  $\Delta$  in the

plane and a conformal embedding

$$X_n : \Delta \rightarrow \mathbf{R}^3$$

of  $A_n^+ \cap C_r$  such that for three fixed points  $p_i \in \partial\Delta$ , we have  $X_n(p_i) = q_i$  where  $q_i \in \partial A_n^+ \cap (\gamma \cup \ell) \cap C_r$ ,  $i = 1, 2, 3$ . Since  $\partial A_n \rightarrow \gamma \cup \ell$  and

$$(\gamma \cup \gamma_n) \cap C_r = (\gamma \cup \ell) \cap C_r$$

for all sufficiently large  $n$ , this is always possible. Observe all of the Dirichlet integrals  $\int_{\Delta} |DX_n|^2$  are uniformly bounded, since each  $X_n$  is conformal and  $\mathcal{A}^+ \cap C_r$  has a bounded area. Then by the well-known Courant-Lebesgue lemma, see [3], we can say that  $A_n^+ \cap C_r = X_n(\Delta)$  converges to  $(\mathcal{A}^+ \cup \gamma \cup \ell) \cap C_r$  and is continuous up to boundary. Similar argument for  $\mathcal{A}^-$  also holds. Thus we see that

$$\partial(\mathcal{A} \cap C_r) \cap (P_0 \cup P_1) = (\gamma \cup \ell) \cap C_r$$

for all  $r > 0$  large enough. Moreover, it is clear that  $\partial\mathcal{A} \subset P_0 \cup P_1$ . Therefore  $\mathcal{A}$  has the boundary  $\gamma \cup \ell$ . With the similar method to  $\mathcal{B}$ ,

$$\partial\mathcal{A} = \partial\mathcal{B} = \gamma \cup \ell$$

and they are continuous up to boundary.

STEP 3. Finally, let  $N$  be a connected non-planar compact (maybe branched) minimal surface such that  $\partial N \subset \overline{D_\gamma} \cup \overline{P_0^+}$ . Let  $W_n$  be the solid bounded by  $A_n \cup D_\gamma \cup \Gamma_n$ , and  $W$  be the solid bounded by  $\mathcal{A} \cup \overline{D_\gamma} \cup \overline{P_0^+}$ . By lemma 1,  $N \subset W_n$  for all  $n$ , and  $\text{Int}(A_n) \cap \text{Int}(N) = \emptyset$  as well as  $N \subset W$  since  $W_n \rightarrow W$ . Note  $\mathcal{A} \neq N$  since  $N$  is compact and  $\mathcal{A}$  is not compact. By the comparison principle for minimal surfaces, we have

$$\text{Int}(\mathcal{A}) \cap \text{Int}(N) = \emptyset.$$

By Proposition 1-2 again, for all  $n$  we have  $B_n \cap N \neq \emptyset$  using the fact that  $B_n$  is unstable. Therefore

$$\mathcal{B} \cap N \neq \emptyset.$$

In particular, it follows that  $\mathcal{A} \neq \mathcal{B}$ . Now let  $W'_n$  be the solid bounded by  $B_n \cup D_\gamma \cup \Gamma_n$ , and  $W'$  be the solid bounded by  $\mathcal{B} \cup \overline{D_\gamma} \cup \overline{P_0^+}$ . Then since  $W'_n \subset W_n$ ,  $\lim_{n \rightarrow \infty} W_n = W$ , and  $\lim_{n \rightarrow \infty} W'_n = W'$ , we have  $W' \subset W$ . By the comparison principle for minimal surfaces, we have

$$\text{Int}(\mathcal{A}) \cap \text{Int}(\mathcal{B}) = \emptyset.$$

Finally, since all of  $A_n$ 's and  $B_n$ 's have the same symmetry group as that of boundary by Proposition 1-3, the same holds for the limits  $\mathcal{A}$  and  $\mathcal{B}$ .

## References

- [1] M. Anderson, *Curvature estimates for minimal surfaces in 3-manifolds*, Ann. Sci. École. Norm. Sup. (4) **18** (1985), 89–105.
- [2] T. H. Colding and W. P. Minicozzi, *Minimal surfaces*, Courant Lect. Notes Math. **4** (1999).
- [3] J. Douglas, *The problem of Plateau for two contours*, J. Math. Phys. Massachusetts Inst. of Tech. **10** (1930/31), 315–359.
- [4] Yi Fang and Jenn-Fang Hwang, *Curvature estimates for minimal annuli and non-compact Douglas-Plateau problem*, Comm. Anal. Geom. **8** (2000), no. 4, 871–904.
- [5] D. Hoffman and W. Meeks, *A variational approach to the existence of complete embedded minimal surfaces*, Duke. Math. J. **57** (1988), 877–893.
- [6] ———, *The strong halfspace theorem for minimal surface*, Invent. Math. **101** (1990), 373–377.
- [7] H. Jenkins and J. Serrin, *Variational problems of minimal surface type, 2, Boundary value problems for the minimal surface equation*, Arch. Ration. Mech. Anal. **21** (1966), 321–342.
- [8] N. Korevaar, R. Kusner, and B. Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), 465–503.
- [9] F. Lopez, *The classification of complete minimal surfaces with total curvature greater than  $-12\pi$* , Trans. Amer. Math. Soc. **334** (1992), 49–74.
- [10] W. Meeks and B. White, *Minimal surfaces bounded by convex curves in parallel planes*, Comm. Math. Helv. **66** (1991), 263–278.
- [11] J. Perez and A. Ros, *Properly embedded minimal annuli bounded by a convex curve*, J. Inst. Math. Jussieu **1(2)** (2002), 293–305.
- [12] ———, *Properly embedded minimal surfaces with finite total curvature, in the global theory of minimal surfaces in flat spaces*, Lecture Notes in Math. **1775** (2002), 15–66.
- [13] B. Riemann, *Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung*, Abh. Königl. d. Wiss. Göttingen, Mathem. Cl. **13** (1867), 3–52.
- [14] M. Shiffman, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes*, Ann. of Math. **63** (1956), 77–90.
- [15] B. White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), 243–256.

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