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CONDITIONAL FIRST VARIATION OVER WIENER PATHS IN ABSTRACT WIENER SPACE

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Abstract. In this paper, we define the conditional first variation over Wiener paths in abstract Wiener space and investigate its properties. Using these properties, we also investigate relationships among first variation, conditional first variation, Fourier-Feynman transform and conditional Fourier-Feynman transforms of functions in a Banach algebra which is equivalent to the Fresnel class. Finally, we provide another method evaluating the Fourier-Feynman transform for the product of a function in the Banach algebra with n linear factors.

1. Introduction and preliminaries

Let $C_0[0,T]$ be the space of all continuous paths x on $[0,T]$ with $x(0)=0$ which is known as the classical Wiener space. The concept of an L_1 analytic Fourier-Feynman transform for functions on this space was introduced by Brue in [1]. In [3], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform, and in [12] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theories for $1 \le p \le 2$ that extended the results in [1, 3] and gave various relationships between the L_1 and L_2 theories.

On the other hand, in [2], Cameron obtained the Wiener integral of first variation of a function F in terms of the Wiener integral of the product F with a linear factor. In [16], Park, Skoug and Storvick found the Fourier-Feynman transform of the product of a function with n linear factors from the Banach algebra $\mathcal{S}$ which was introduced by Cameron and Storvick in [4]. In [9], Chang, Song and Yoo expressed analytic Feynman integral of the first variation of a function F in terms

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of analytic Feynman integral of the product of F with a linear factor on abstract Wiener space. And then, they derived the Fourier-Feynman transform for the product of a function in the Fresnel class with n linear factors.

In [7], Chang, Cho and Yoo introduced a concept conditional analytic Feynman integral over Wiener paths in abstract Wiener space. And also, in [10], Cho introduced the Banach algebra $\mathcal{F}(C_0(\mathbb{B});u)$ which is equivalent to the Fresnel class, and he evaluated various conditional analytic Feynman integrals of functions in certain classes which are all equivalent to the Fresnel class, and then, investigated relationships between analytic Feynman integral and conditional analytic Feynman integral of functions in $\mathcal{F}(C_0(\mathbb{B});u)$.

In this paper, we define the conditional first variation over Wiener paths in abstract Wiener space and investigate its properties. Using these properties, we also investigate relationships among first variation, conditional first variation, Fourier-Feynman transform and conditional Fourier-Feynman transforms of functions in the Banach algebra $\mathcal{F}(C_0(\mathbb{B});u)$. Finally, we provide another method evaluating the Fourier-Feynman transform for the product of a function in the Banach algebra with n linear factors. This result over Wiener paths in abstract Wiener space extends the result in [9].

Let $(\Omega,\mathcal{A},P)$ be a probability space, let B be a real normed linear space and let $\mathcal{B}(B)$ be the Borel σ-field on B. Let $X:(\Omega,\mathcal{A},P)\to(B,\mathcal{B}(B))$ be a random variable and let $F:\Omega\to\mathbb{C}$ be an integrable function. Let P_X be the probability distribution of X on $(B,\mathcal{B}(B))$ and let $\mathcal{D}$ be the σ-field $\{X^{-1}(A):A\in\mathcal{B}(B)\}$. Let $P_{\mathcal{D}}$ be the probability measure induced by P, that is, $P_{\mathcal{D}}(E)=P(E)$ for $E\in\mathcal{D}$. By the definition of conditional expectation there exists a $\mathcal{D}$-measurable function $E[F|X]$(the conditional expectation of F given X) defined on Ω such that the relation

$$\int_E E[F|X](\omega)\,dP_{\mathcal{D}}(\omega)=\int_E F(\omega)\,dP(\omega)$$

holds for every $E\in\mathcal{D}$. But there exists a P_X-integrable function ψ defined on B which is unique up to P_X-a.e. such that $E[F|X](\omega)=(\psi\circ X)(\omega)$ for $P_{\mathcal{D}}$-a.e. ω in Ω. ψ is also called the conditional expectation of F given X and without loss of generality, it is denoted by $E[F|X](\xi)$ for $\xi\in B$. Throughout this paper, we will consider the function ψ as the conditional expectation of F given X.

2. Wiener paths in abstract Wiener space

Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space([15]). Let $\{e_j : j \geq 1\}$ be a complete orthonormal set in the real separable Hilbert space $\mathcal{H}$ such that e_j's are in $\mathbb{B}^*$, the dual space of real separable Banach space $\mathbb{B}$. For each $h \in \mathcal{H}$ and $y \in \mathbb{B}$, define the stochastic inner product $(h, y)^\sim$ of h and y by

$$(h, y)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (y, e_j), & \text{if the limit exists;} \\ 0, & \text{otherwise,} \end{cases}$$

where $(\cdot,\cdot)$ denotes the dual pairing between $\mathbb{B}$ and $\mathbb{B}^*$([13]). Note that for each $h(\neq 0)$ in $\mathcal{H}$, $(h,\cdot)^\sim$ is a Gaussian random variable on $\mathbb{B}$ with mean zero, variance $|h|^2$; also $(h, y)^\sim$ is essentially independent of choice of the complete orthonormal set used in its definition and further, $(h, \lambda y)^\sim = (\lambda h, y)^\sim = \lambda(h, y)^\sim$ for all $\lambda \in \mathbb{R}$. It is well-known that if $\{h_1, h_2, \cdots, h_n\}$ is an orthogonal set in $\mathcal{H}$, then the random variables $(h_j, \cdot)^\sim$'s are independent. Moreover, if both h and y are in $\mathcal{H}$, then $(h, y)^\sim = \langle h, y\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the inner product of h and y.

Let $C_0(\mathbb{B})$ denote the space of all continuous functions on $[0, T]$ into $\mathbb{B}$ which vanish at 0. Then $C_0(\mathbb{B})$ is a real separable Banach space with the norm $\|x\|_{C_0(\mathbb{B})} \equiv \sup_{0\leq t\leq T} \|x(t)\|_{\mathbb{B}}$. The minimal σ-field making the mapping $x \to x(t)$ measurable is $\mathcal{B}(C_0(\mathbb{B}))$, the Borel σ-field on $C_0(\mathbb{B})$. Further, Brownian motion in $\mathbb{B}$ induces a probability measure $m_{\mathbb{B}}$ on $(C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})))$ which is mean-zero Gaussian([17]). We can find a concrete form of $m_{\mathbb{B}}$ as follows. Let $\vec{t} = (t_1, t_2, \cdots, t_k)$ be given with $0 = t_0 < t_1 < t_2 < \cdots < t_k \leq T$. Let $T_{\vec{t}} : \mathbb{B}^k \to \mathbb{B}^k$ be given by

$$\begin{aligned} &T_{\vec{t}}(x_1, x_2, \cdots, x_k) \\ &= \left(\sqrt{t_1 - t_0}\,x_1, \sqrt{t_1 - t_0}\,x_1 + \sqrt{t_2 - t_1}\,x_2, \cdots, \sum_{j=1}^{k} \sqrt{t_j - t_{j-1}}\,x_j \right). \end{aligned}$$

We define a set function $\nu_{\vec{t}}$ on $\mathcal{B}(\mathbb{B}^k)$ by

$$\nu_{\vec{t}}(B) = \left(\prod_{1}^{k} m\right)\left(T_{\vec{t}}^{-1}(B)\right)$$

for $B \in \mathcal{B}(\mathbb{B}^k)$. Then $\nu_{\vec{t}}$ is a Borel measure. Let $f_{\vec{t}} : C_0(\mathbb{B}) \to \mathbb{B}^k$ be the function defined by

$$f_{\vec{t}}(x) = (x(t_1), x(t_2), \cdots, x(t_k)).$$

For Borel subsets $B_1, B_2, \cdots, B_k$ of $\mathbb{B}$, $f_{\vec{t}}^{-1}(\prod_{j=1}^{k} B_j)$ is called the I-set with respect to $B_1, B_2, \cdots, B_k$. Then the collection $\mathcal{I}$ of all I-sets is a semi-algebra. We define a set function $m_{\mathbb{B}}$ on $\mathcal{I}$ by

$$m_{\mathbb{B}}\left(f_{\vec{t}}^{-1}\left(\prod_{j=1}^{k} B_j\right)\right) = \nu_{\vec{t}}\left(\prod_{j=1}^{k} B_j\right).$$

Then $m_{\mathbb{B}}$ is well-defined and countably additive on $\mathcal{I}$. Using Carathéodory extension process, we have a Borel measure $m_{\mathbb{B}}$ on $\mathcal{B}(C_0(\mathbb{B}))$.

A complex-valued measurable function defined on $C_0(\mathbb{B})$ is said to be Wiener measurable and a Wiener measurable function is said to be Wiener integrable if it is integrable.

DEFINITION 1. Let $F : C_0(\mathbb{B}) \to \mathbb{C}$ be Wiener integrable and let $X : (C_0(\mathbb{B}), \mathcal{B}(C_0(\mathbb{B})), m_{\mathbb{B}}) \to (B, \mathcal{B}(B))$ be a random variable, where B is a real normed linear space with the Borel σ-field $\mathcal{B}(B)$. The conditional expectation $E[F|X]$ of F given X defined on B is called the conditional Wiener integral of F given X.

Now, we introduce Wiener integration theorem without proof. For the proof see [17].

THEOREM 2 (Wiener integration theorem). *Let $\vec{t} = (t_1, t_2, \cdots, t_k)$ be given with $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_k \le T$ and let $f : \mathbb{B}^k \to \mathbb{C}$ be a Borel measurable function. Then*

$$\int_{C_0(\mathbb{B})} f(x(t_1), x(t_2), \cdots, x(t_k))\, dm_{\mathbb{B}}(x)$$

$$\stackrel{*}{=} \int_{\mathbb{B}^k} (f \circ T_{\vec{t}})(x_1, x_2, \cdots, x_k)\, d\left(\prod_{j=1}^{k} m\right)(x_1, x_2, \cdots, x_k),$$

where by $\stackrel{*}{=}$ we mean that if either side exists, then both sides exist and they are equal.

A subset E of $C_0(\mathbb{B})$ is called a scale-invariant null set if $m_{\mathbb{B}}(\lambda E) = 0$ for any $\lambda > 0$. A property is said to hold scale-invariant almost everywhere (in abbreviation, s-a.e.) if it holds except for a scale-invariant null set. Let F be defined on $C_0(\mathbb{B})$ and let $F^{\lambda}(x) = F(\lambda^{-\frac{1}{2}} x)$ for $\lambda > 0$. Suppose that $E[F^{\lambda}]$ exists for any $\lambda > 0$ and it has the analytic extension $J_{\lambda}^{*}(F)$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C} : \mathrm{Re}\, \lambda > 0\}$. Then we call $J_{\lambda}^{*}(F)$ the analytic Wiener integral of F over $C_0(\mathbb{B})$ with parameter λ and it is denoted by

$$E^{anw_{\lambda}}[F] = J_{\lambda}^{*}(F).$$

Moreover, if for non-zero real q, $E^{anw_\lambda}[F]$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic Feynman integral of F over $C_0(\mathbb{B})$ with parameter q and denoted by

$$\int_{C_0(\mathbb{B})}^{anf_q} F(x)dm_{\mathbb{B}}(x) = E^{anf_q}[F] = \lim_{\lambda\to -iq} E^{anw_\lambda}[F].$$

Let $\tau : 0 = t_0 < t_1 < \cdots < t_k = T$ be a partition of $[0,T]$ and let x be in $C_0(\mathbb{B})$. Define the polygonal function $[x]$ of x on $[0,T]$ by

$$(1)\quad [x](t) = \sum_{j=1}^{k} \chi_{(t_{j-1},t_j]}(t)\left[x(t_{j-1}) + \frac{t-t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1}))\right]$$

for $t \in [0,T]$. For each $\vec{\xi} = (\xi_1, \cdots, \xi_k) \in \mathbb{B}^k$, let $[\vec{\xi}]$ be the polygonal function of $\vec{\xi}$ on $[0,T]$ given by (1) with replacing $x(t_j)$ by ξ_j for $j = 0, 1, \cdots, k$ ($\xi_0 = 0$). Note that both $[x] : [0,T] \to \mathbb{B}$ and $[\vec{\xi}] : [0,T] \to \mathbb{B}$ are in $C_0(\mathbb{B})$.

The following lemma is useful to define the conditional analytic Wiener and Feynman integrals. For detailed proof, see [7].

LEMMA 3. *Let F be defined and integrable on $C_0(\mathbb{B})$. Let $X_\tau : C_0(\mathbb{B}) \to \mathbb{B}^k$ be a random variable given by $X_\tau(x) = (x(t_1), \cdots, x(t_k))$. Then for every Borel measurable subset B of $\mathbb{B}^k$, we have*

$$\int_{X_\tau^{-1}(B)} F(x)\, dm_{\mathbb{B}}(x) = \int_B E[F(x - [x] + [\vec{\xi}])]\, dP_{X_\tau}(\vec{\xi}),$$

where P_{X_τ} is the probability distribution of X_τ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$.

REMARK 1. Throughout this paper, unless otherwise specified, we will denote P_{X_τ} as the probability distribution of X_τ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}))$.

By the definition of conditional Wiener integral (Definition 1) and Lemma 3, we have

$$(2)\qquad E[F|X_\tau](\vec{\xi}) = E[F(x - [x] + [\vec{\xi}])] \quad \text{for } P_{X_\tau}\text{- a.e. } \vec{\xi}.$$

For $\lambda > 0$ let $X_\tau^\lambda(x) = X_\tau(\lambda^{-\frac{1}{2}}x)$ and for $\vec{\xi} \in \mathbb{B}^k$ suppose $E[F^\lambda|X_\tau^\lambda](\vec{\xi})$ exists. From (2) we have

$$E[F^\lambda|X_\tau^\lambda](\vec{\xi}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$$

for $P_{X_\tau^\lambda}$-a.e. $\vec{\xi} \in \mathbb{B}^k$ where $P_{X_\tau^\lambda}$ is the probability distribution of X_τ^λ on $(\mathbb{B}^k, \mathcal{B}(\mathbb{B}^k))$. If $E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}])]$ has the analytic extension $J_\lambda^*(F)(\vec{\xi})$ on $\mathbb{C}_+$, then we write

$$E^{anw_\lambda}[F|X_\tau](\vec{\xi}) = J_\lambda^*(F)(\vec{\xi})$$

for $\lambda \in \mathbb{C}_+$. $E^{anw_\lambda}[F|X_\tau]$ is a version of conditional analytic Wiener integral. For non-zero real q, if the limit

$$\lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_\tau](\vec{\xi})$$

exists, where λ approaches to $-iq$ through $\mathbb{C}_+$, then we write

$$E^{anf_q}[F|X_\tau](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F|X_\tau](\vec{\xi}).$$

$E^{anf_q}[F|X_\tau]$ is a version of conditional analytic Feynman integral.

3. First and conditional first variation over paths in abstract Wiener space

In this section, we define first and conditional first variation over Wiener paths in abstract Wiener space. And then, we investigate their properties and relationships with Fourier-Feynman transform and conditional Fourier-Feynman transform.

DEFINITION 4. Let F be a Wiener measurable function defined on $C_0(\mathbb{B})$ and let $w \in C_0(\mathbb{B})$. The derivative

$$\frac{\partial}{\partial t} F(x + tw)|_{t=0}$$

for $x \in C_0(\mathbb{B})$ if it exists, is called the first variation of F at x in the direction of w and denoted by

$$\delta_w F(x) = \frac{\partial}{\partial t} F(x + tw)|_{t=0}.$$

DEFINITION 5. Let F be a Wiener measurable function defined on $C_0(\mathbb{B})$ and let $F(\cdot + x)$ be integrable for $x \in C_0(\mathbb{B})$. Let $w \in C_0(\mathbb{B})$, let B be a real linear normed space and let $X : C_0(\mathbb{B}) \to B$ be a random variable. Let P_X be the probability distribution of X on $(B, \mathcal{B}(B))$. For $x \in C_0(\mathbb{B})$, if the derivative

$$\frac{\partial}{\partial t} E[F(\cdot + x + tw)|X](\xi)|_{t=0}$$

exists for P_X-a.e. $\xi \in B$, then it is called the conditional first variation of F given X at x in the direction of w and is denoted by

$$\delta_w E[F|X](x, \xi) = \frac{\partial}{\partial t} E[F(\cdot + x + tw)|X](\xi)|_{t=0}.$$

Suppose that, for some $\epsilon > 0$, $\delta_w F(x+tw)$ and $\delta_w E[F|X](x+tw,\xi)$ exist for $|t| < \epsilon$. Then we have

$$\begin{aligned} \delta_w F(x+tw) &= \frac{\partial}{\partial \alpha} F(x+tw+\alpha w)|_{\alpha=0} \\ &= \frac{\partial}{\partial \mu} F(x+\mu w)|_{\mu=t} \\ &= \frac{\partial}{\partial t} F(x+tw) \end{aligned} \tag{3}$$

and, similarly, we have

$$\delta_w E[F|X](x+tw,\xi) = \frac{\partial}{\partial t} E[F(\cdot + x + tw)|X](\xi). \tag{4}$$

The following theorem shows that the conditional first variation of a function is essentially the conditional Wiener integral of the first variation of the function under suitable conditions.

THEOREM 6. *Let $y, w \in C_0(\mathbb{B})$, let F be a Wiener measurable function defined on $C_0(\mathbb{B})$ and let $F(\cdot + y)$ be integrable. Let $X : C_0(\mathbb{B}) \to B$ be a random variable, where B is a real normed linear space, and let F have the first variation $\delta_w F(x)$ for $x \in C_0(\mathbb{B})$. Suppose that there exists an integrable function $G_{y,w}$ such that, for some $\epsilon > 0$,*

$$\sup_{|t|<\epsilon} |\delta_w F(x+y+tw)| \le G_{y,w}(x) \tag{5}$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$. Then, for $|t| < \epsilon$, both $E[\delta_w F(\cdot + y + tw)|X](\xi)$ and $E[F(\cdot + y + tw)|X](\xi)$ exist for P_X-a.e. $\xi \in B$, where P_X is the probability distribution of X on $(B, \mathcal{B}(B))$.

Moreover, suppose that there exist a P_X-integrable function $H_{y,w}$ on B such that

$$\sup_{|t|<\epsilon} |\delta_w E[F|X](y+tw,\xi)| \le H_{y,w}(\xi) \tag{6}$$

with the existence of $\delta_w E[F|X](y+tw,\xi)$ for P_X-a.e. $\xi \in B$. Then, for $|t| < \epsilon$, we have

$$\begin{aligned} &\int_A \delta_w E[F|X](y+tw,\xi)\,dP_X(\xi) \\ &= \int_A E[\delta_w F(\cdot + y + tw)|X](\xi)\,dP_X(\xi) \end{aligned} \tag{7}$$

for any A in $\mathcal{B}(B)$ and hence

$$\delta_w E[F|X](y+tw,\xi) = E[\delta_w F(\cdot + y + tw)|X](\xi) \tag{8}$$

for P_X-a.e. $\xi \in B$.

Proof. By the mean value theorem and (3), we have, for some t_1 with $|t_1| \le |t|$,

$$F(x+y+tw) = F(x+y) + t\delta_w F(x+y+t_1 w)$$

if $|t| < \epsilon$, so that $F(x+y+tw)$ is an integrable function of x by (5) and $E[F(\cdot+y+tw)|X](\xi)$ exists for P_X-a.e. $\xi \in B$. Further, suppose that (6) holds. Let $m_{\mathbb{B}}|_{\mathcal{D}}$ be the restriction of $m_{\mathbb{B}}$ on $\mathcal{D} \equiv \{X^{-1}(A)|A \in \mathcal{B}(B)\}$. Note that $H_{y,w}(X(x))$ is an integrable function of x by the change of variable theorem. Then, for $A \in \mathcal{B}(B)$ and $|t| < \epsilon$, we have

$$\begin{aligned}
&\int_A \delta_w E[F|X](y+tw,\xi)dP_X(\xi) \\
&= \int_{X^{-1}(A)} \frac{\partial}{\partial t} E[F(\cdot+y+tw)|X](X(x))dm_{\mathbb{B}}|_{\mathcal{D}}(x) \\
&= \frac{\partial}{\partial t}\int_{X^{-1}(A)} E[F(\cdot+y+tw)|X](X(x))dm_{\mathbb{B}}|_{\mathcal{D}}(x) \\
&= \frac{\partial}{\partial t}\int_{X^{-1}(A)} F(x+y+tw)dm_{\mathbb{B}}(x) \\
&= \int_{X^{-1}(A)} \frac{\partial}{\partial t} F(x+y+tw)dm_{\mathbb{B}}(x) \\
&= \int_{X^{-1}(A)} \delta_w F(x+y+tw)dm_{\mathbb{B}}(x) \\
&= \int_{X^{-1}(A)} E[\delta_w F(\cdot+y+tw)|X](X(x))dm_{\mathbb{B}}|_{\mathcal{D}}(x) \\
&= \int_A E[\delta_w F(\cdot+y+tw)|X](\xi)dP_X(\xi)
\end{aligned}$$

by (3), (4), (5), (6), [11, Theorem 2.27] and the change of variable theorem. Hence we have (7). □

The following corollary shows that (8) holds under more weak conditions if we replace the random variable X by X_τ which is given as in Lemma 3.

Corollary 7. *Let $y, w \in C_0(\mathbb{B})$, let F be a Wiener measurable function defined on $C_0(\mathbb{B})$. Let X_τ be given as in* Lemma 3 *and F have the first variation $\delta_w F(x)$ for $x \in C_0(\mathbb{B})$. Suppose that, for some $\epsilon > 0$, both $F(\cdot+y)$ and $\delta_w F(\cdot+y+tw)$ are integrable for $|t| < \epsilon$. Moreover, assume that, for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, there exists an integrable function*

$G_{\vec{\xi},y,w}$ *on* $C_0(\mathbb{B})$ *such that*

$$\sup_{|t|<\epsilon} |\delta_w F(x-[x]+[\vec{\xi}]+y+tw)| \le G_{\vec{\xi},y,w}(x) \tag{9}$$

for $m_{\mathbb{B}}$*-a.e.* $x \in C_0(\mathbb{B})$. *Then,* $\delta_w E[F|X_\tau](y+tw,\vec{\xi})$ *exists for* $|t|<\epsilon$ *and it is given by*

$$\delta_w E[F|X_\tau](y+tw,\vec{\xi}) = E[\delta_w F(\cdot+y+tw)|X_\tau](\vec{\xi})$$

for P_{X_τ}*-a.e.* $\vec{\xi} \in \mathbb{B}^k$.

Proof. Since $F(\cdot+y)$ is integrable, $E[F(\cdot+y)|X_\tau](\vec{\xi})$ exists for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$ and $F(x-[x]+[\vec{\xi}]+y)$ is integrable as a function of x by Lemma 3. An application of the mean value theorem shows that $F(x-[x]+[\vec{\xi}]+y+tw)$ is integrable for $|t|<\epsilon$. Similarly, $F(x+y+tw)$ is an integrable function of x. Thus we have

$$\begin{aligned}
&E[\delta_w F(\cdot+y+tw)|X_\tau](\vec{\xi})\\
&= \int_{C_0(\mathbb{B})} \delta_w F(x-[x]+[\vec{\xi}]+y+tw)\,dm_{\mathbb{B}}(x)\\
&= \int_{C_0(\mathbb{B})} \frac{\partial}{\partial t} F(x-[x]+[\vec{\xi}]+y+tw)\,dm_{\mathbb{B}}(x)\\
&= \frac{\partial}{\partial t}\int_{C_0(\mathbb{B})} F(x-[x]+[\vec{\xi}]+y+tw)\,dm_{\mathbb{B}}(x)\\
&= \frac{\partial}{\partial t} E[F(\cdot+y+tw)|X_\tau](\vec{\xi}),
\end{aligned}$$

where the third equality follows from (9) and [11, Theorem 2.27]. □

Corollary 8. *Let* $y,w \in C_0(\mathbb{B})$, *let* F_1, F_2 *be Wiener measurable functions defined on* $C_0(\mathbb{B})$ *and let* X_τ *be given as in* Lemma 3. *Let* F_1, F_2 *have the first variations* $\delta_w F_1(x), \delta_w F_2(x)$, *respectively, for* $x \in C_0(\mathbb{B})$. *Let* $F_1(\cdot+y)F_2(\cdot+y)$ *be integrable and suppose that, for some* $\epsilon>0$, *both* $F_1(\cdot+y+tw)\delta_w F_2(\cdot+y+tw)$ *and* $F_2(\cdot+y+tw)\delta_w F_1(\cdot+y+tw)$ *are integrable for* $|t|<\epsilon$. *Moreover, suppose that, for* P_{X_τ}*-a.e.* $\vec{\xi} \in \mathbb{B}^k$, *there exist integrable functions* $G^1_{\vec{\xi},y,w}, G^2_{\vec{\xi},y,w}$ *on* $C_0(\mathbb{B})$ *such that*

$$\begin{aligned}
&\sup_{|t|<\epsilon} |F_1(x-[x]+[\vec{\xi}]+y+tw)\delta_w F_2(x-[x]+[\vec{\xi}]+y+tw)|\\
&\le G^1_{\vec{\xi},y,w}(x)
\end{aligned} \tag{10}$$

and

$$(11)\qquad \begin{aligned}&\sup_{|t|<\epsilon}|F_2(x-[x]+[\vec{\xi}]+y+tw)\delta_w F_1(x-[x]+[\vec{\xi}]+y+tw)|\\ &\le G^2_{\vec{\xi},y,w}(x)\end{aligned}$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$. Then $\delta_w[F_1F_2|X_\tau](y+tw,\vec{\xi})$ exists for $|t|<\epsilon$ and it is given by the formula

$$\begin{aligned}&\delta_w E[F_1F_2|X_\tau](y+tw,\vec{\xi})\\ &= E[\delta_w[F_1(\cdot+y+tw)F_2(\cdot+y+tw)]|X_\tau](\vec{\xi})\\ &= E[F_1(\cdot+y+tw)\delta_w F_2(\cdot+y+tw)|X_\tau](\vec{\xi})\\ &\quad + E[F_2(\cdot+y+tw)\delta_w F_1(\cdot+y+tw)|X_\tau](\vec{\xi})\end{aligned}$$

for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$.

Proof. Let $F = F_1F_2$. By (10) and (11), the inequality (9) is satisfied for F. Now, the results follow from Corollary 7, immediately. □

The following lemma is useful to derive Theorem 10. For the proof of this lemma, see [6].

Lemma 9. *Let $0 < u \le T$ and let $w \in C_0(\mathbb{B})$ with $w(u) \in \mathcal{H}$. Let F be defined on $C_0(\mathbb{B})$ and it can be expressed by $F(x) = f(x(u))$ for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$ where f is a measurable function defined on $\mathbb{B}$. Let F be integrable and have the first variation $\delta_w F(x)$ for $x \in C_0(\mathbb{B})$. Suppose that there exists an integrable function G defined on $C_0(\mathbb{B})$ such that, for some $\epsilon > 0$,*

$$(12)\qquad \sup_{|t|<\epsilon}|\delta_w F(x+tw)| \le G(x)$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$, where G also can be expressed by $G(x) = g(x(u))$ for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$ with g being defined on $\mathbb{B}$. Then, we have

$$\int_{C_0(\mathbb{B})} \delta_w F(x)dm_{\mathbb{B}}(x) = \frac{1}{u}\int_{C_0(\mathbb{B})} (w(u),x(u))^\sim F(x)dm_{\mathbb{B}}(x)$$

with the existences of the both sides of the equation.

Theorem 10. *Let $0 < u \le T$, let $w \in C_0(\mathbb{B})$ with $w(u) \in \mathcal{H}$ and let X_τ be given as in* Lemma 3. *Let F be defined on $C_0(\mathbb{B})$ and, for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$ and P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, let $E[F(\cdot+x)|X_\tau](\vec{\xi})$ exist and be integrable as a function of x. For some $\epsilon > 0$, suppose that $\delta_w E[F|X_\tau](x+tw,\vec{\xi})(|t|<\epsilon)$ exists for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$ and $m_{\mathbb{B}}$-a.e.*

$x \in C_0(\mathbb{B})$. *Also, suppose that, for* P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, *there exists an integrable function* $G_{\vec{\xi}}$ *on* $C_0(\mathbb{B})$ *with*

$$\sup_{|t|<\epsilon} |\delta_w E[F|X_\tau](x+tw,\vec{\xi})| \le G_{\vec{\xi}}(x)$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$. *Moreover, suppose that, for* P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, *both* $E[F(\cdot + x + tw)|X_\tau](\vec{\xi})$ *and* $G_{\vec{\xi}}(x)$ *can be expressed by*

$$E[F(\cdot + x + tw)|X_\tau](\vec{\xi}) = f(x(u) + tw(u), \vec{\xi})$$

for $|t| < \epsilon$ *and*

$$G_{\vec{\xi}}(x) = g(x(u), \vec{\xi})$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$, *where* f, g *are measurable functions defined on* $\mathbb{B} \times \mathbb{B}^k$. *Then we have*

$$\begin{aligned}&\int_{C_0(\mathbb{B})} \delta_w E[F|X_\tau](x,\vec{\xi})dm_{\mathbb{B}}(x)\\ &= \frac{1}{u}\int_{C_0(\mathbb{B})} (w(u), x(u))^\sim E[F(\cdot + x)|X_\tau](\vec{\xi})dm_{\mathbb{B}}(x)\end{aligned}$$

for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$ *with the existence of the right-hand side of the equation.*

Proof. For $\vec{\xi} \in \mathbb{B}^k$, let $f_{\vec{\xi}}(x) = f(x(u), \vec{\xi})$ and $g_{\vec{\xi}}(x) = g(x(u), \vec{\xi})$ for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$. By Lemma 9, we have the result with replacing f, g by $f_{\vec{\xi}}$, $g_{\vec{\xi}}$, respectively. □

Now, we obtain a variety of integration by parts formula from Corollary 8 and Theorem 10.

COROLLARY 11. *Let* $0 < u \le T$, *let* $w \in C_0(\mathbb{B})$ *with* $w(u) \in \mathcal{H}$ *and let* X_τ *be given as in* Lemma 3. *Let* F_1, F_2 *be defined on* $C_0(\mathbb{B})$ *and, for* $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$ *and* P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, *let* $E[F_1(\cdot + x)F_2(\cdot + x)|X_\tau](\vec{\xi})$ *exist and be integrable as a function of* x. *For some* $\epsilon > 0$, *suppose that* $\delta_w E[F_1F_2|X_\tau](x + tw, \vec{\xi})(|t| < \epsilon)$ *exists for* P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$ *and* $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$. *Also, suppose that, for* P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, *there exists an integrable function* $G_{\vec{\xi}}$ *on* $C_0(\mathbb{B})$ *with*

$$\sup_{|t|<\epsilon} |\delta_w E[F_1F_2|X_\tau](x+tw,\vec{\xi})| \le G_{\vec{\xi}}(x)$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$. Moreover, suppose that, for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$, both $E[F_1(\cdot+x+tw)F_2(\cdot+x+tw)|X_\tau](\vec{\xi})$ and $G_{\vec{\xi}}(x)$ can be expressed by

$$E[F_1(\cdot+x+tw)F_2(\cdot+x+tw)|X_\tau](\vec{\xi}) = f(x(u)+tw(u),\vec{\xi}) \text{ for } |t|<\epsilon$$

and

$$G_{\vec{\xi}}(x) = g(x(u),\vec{\xi})$$

for $m_{\mathbb{B}}$-a.e. $x \in C_0(\mathbb{B})$, where f, g are measurable functions defined on $\mathbb{B}\times\mathbb{B}^k$. Then we have

$$\begin{aligned} &\frac{1}{u}\int_{C_0(\mathbb{B})}(w(u),x(u))^\sim E[F_1(\cdot+x)F_2(\cdot+x)|X_\tau](\vec{\xi})dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})}\delta_w E[F_1F_2|X_\tau](x,\vec{\xi})dm_{\mathbb{B}}(x) \end{aligned}$$

for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$ with the existence of the right-hand side of the equation.

In addition, suppose that the assumptions in Corollary 8 *hold and both $E[F_1(\cdot+x)\delta_w F_2(\cdot+x)|X_\tau](\vec{\xi})$ and $E[F_2(\cdot+x)\delta_w F_1(\cdot+x)|X_\tau](\vec{\xi})$ are integrable as functions of x for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$. Then we have*

$$\begin{aligned} &\frac{1}{u}\int_{C_0(\mathbb{B})}(w(u),x(u))^\sim E[F_1(\cdot+x)F_2(\cdot+x)|X_\tau](\vec{\xi})dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})}E[\delta_w[F_1(\cdot+x)F_2(\cdot+x)]|X_\tau](\vec{\xi})dm_{\mathbb{B}}(x) \\ &= \int_{C_0(\mathbb{B})}E[F_1(\cdot+x)\delta_w F_2(\cdot+x)|X_\tau](\vec{\xi})dm_{\mathbb{B}}(x) \\ &\quad + \int_{C_0(\mathbb{B})}E[F_2(\cdot+x)\delta_w F_1(\cdot+x)|X_\tau](\vec{\xi})dm_{\mathbb{B}}(x) \end{aligned}$$

for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$.

4. Transforms of the functions in Banach algebra $\mathcal{F}(C_0(\mathbb{B});u)$

For a given extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $\frac{1}{p}+\frac{1}{p'}=1$(possibly $p'=1$ if $p=\infty$). Let G_n and G be measurable functions such that, for each $\gamma>0$,

$$\lim_{n\to\infty}\int_{C_0(\mathbb{B})}|G_n(\gamma x)-G(\gamma x)|^{p'}dm_{\mathbb{B}}(x)=0.$$

Then we write

$$\underset{n\to\infty}{\text{l.i.m.}}(w_s^{p'})(G_n) \approx G$$

and call G the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by a continuously varying parameter.

Now, we define Fourier-Feynman transform and conditional Fourier-Feynman transform of functions on $C_0(\mathbb{B})$.

DEFINITION 12. Let F be defined on $C_0(\mathbb{B})$ and for $\lambda \in \mathbb{C}_+$ let

$$T_\lambda(F)(y) = E^{anw_\lambda}[F(\cdot + y)]$$

for s-a.e. $y \in C_0(\mathbb{B})$ if it exists. For a non-zero real q, we define the L_1 Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$T_q^{(1)}(F)(y) = E^{anf_q}[F(\cdot + y)]$$

if it exists for s-a.e. $y \in C_0(\mathbb{B})$ and for $1 < p \leq \infty$ we define the L_p Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula

$$T_q^{(p)}(F) \approx \underset{\lambda\to -iq}{\text{l.i.m.}}(w_s^{p'})(T_\lambda(F)),$$

where λ approaches to $-iq$ through $\mathbb{C}_+$.

DEFINITION 13. Let F be defined on $C_0(\mathbb{B})$ and let X_τ be given as in Lemma 3. For $\lambda \in \mathbb{C}_+$ and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$ let

$$T_\lambda[F|X_\tau](y, \vec{\xi}) = E^{anw_\lambda}[F(y + \cdot)|X_\tau](\vec{\xi})$$

for s-a.e. $y \in C_0(\mathbb{B})$ if it exists. For non-zero real q and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$, we define the L_1 conditional Fourier-Feynman transform $T_q^{(1)}[F|X_\tau]$ of F given X_τ by the formula

$$T_q^{(1)}[F|X_\tau](y, \vec{\xi}) = \lim_{\lambda\to -iq} T_\lambda[F|X_\tau](y, \vec{\xi})$$

if it exists for s-a.e. $y \in C_0(\mathbb{B})$ and for $1 < p \leq \infty$ we define the L_p conditional Fourier-Feynman transform $T_q^{(p)}[F|X_\tau]$ of F given X_τ by the formula

$$T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}) \approx \underset{\lambda\to -iq}{\text{l.i.m.}}(w_s^{p'})(T_\lambda[F|X_\tau](\cdot, \vec{\xi})),$$

where λ approaches to $-iq$ through $\mathbb{C}_+$.

Let $\mathcal{H}$ be an infinite dimensional real separable Hilbert space and let $\mathcal{M}(\mathcal{H})$ be the class of all $\mathbb{C}$-valued Borel measures on $\mathcal{H}$ with bounded variation. Let $0 < u \le T$ be fixed, but arbitrarily, and let $\mathcal{F}(C_0(\mathbb{B}); u)$ be the space of all equivalence classes of functions F which, for $\sigma \in \mathcal{M}(\mathcal{H})$, have the form

$$F(x) = \int_{\mathcal{H}} \exp\{i(h, x(u))^{\sim}\} d\sigma(h) \tag{13}$$

for s-a.e. $x \in C_0(\mathbb{B})$. It can be shown that the class $\mathcal{F}(C_0(\mathbb{B}); u)$ is a Banach algebra and it is isomorphic to $\mathcal{M}(\mathcal{H})$ as Banach algebras ([10]).

Now we introduce a useful integral, which appears in the proof of several results. The proof follows from the fact that the random variable $(h, \cdot)^{\sim}$ is normally distributed with mean 0 and variance $|h|^2$ if $h \neq 0$.

LEMMA 14. *Let $(\mathcal{H}, \mathbb{B}, m)$ be an abstract Wiener space and let $h \in \mathcal{H}$. Then we have*

$$\int_{\mathbb{B}} \exp\{i(h, x_1)^{\sim}\} dm(x_1) = \exp\left\{-\frac{|h|^2}{2}\right\}.$$

THEOREM 15. *Let F be given by (13), let X_τ be given as in* Lemma 3 *and let $w \in C_0(\mathbb{B})$. Choose u such that $t_{p^*-1} < u \le t_{p^*}$ for some $p^* \in \{1, \cdots, k\}$ and let*

$$\Gamma = \frac{(t_{p^*} - u)(u - t_{p^*-1})}{t_{p^*} - t_{p^*-1}}. \tag{14}$$

Moreover, suppose that

$$\int_{\mathcal{H}} |(h, w(u))^{\sim}| \, d|\sigma|(h) < \infty. \tag{15}$$

Then, for s-a.e. $y \in C_0(\mathbb{B})$, $\delta_w E[F|X_\tau](y, \vec{\xi})$ exists for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$ and it is given by the formula

$$\begin{aligned} &\delta_w E[F|X_\tau](y, \vec{\xi}) \\ &= \int_{\mathcal{H}} i(h, w(u))^{\sim} \exp\{i(h, [\vec{\xi}](u) + y(u))^{\sim}\} \exp\left\{-\frac{\Gamma}{2}|h|^2\right\} d\sigma(h). \end{aligned} \tag{16}$$

Proof. By (2) and Fubini's theorem, we have for $t \in \mathbb{R}$

$$\begin{aligned} &E[F(\cdot + y + tw)|X_\tau](\vec{\xi}) \\ &= \int_{\mathcal{H}} \int_{C_0(\mathbb{B})} \exp\{i(h, x(u) - [x](u) + [\vec{\xi}](u) + y(u) + tw(u))^{\sim}\} \\ &\quad dm_{\mathbb{B}}(x) d\sigma(h) \end{aligned}$$

$$= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u) + y(u) + tw(u))^{\sim}\}$$
$$\times \int_{C_0(\mathbb{B})} \exp\bigg\{i\bigg(h, x(u) - x(t_{p^*-1})$$
$$- \frac{u - t_{p^*-1}}{t_{p^*} - t_{p^*-1}}(x(t_{p^*}) - x(t_{p^*-1}))\bigg)^{\sim}\bigg\} dm_{\mathbb{B}}(x) d\sigma(h)$$

for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$.

Let $\alpha = \frac{(t_{p^*}-u)(u-t_{p^*-1})^{\frac{1}{2}}}{t_{p^*}-t_{p^*-1}}$ and $\beta = -\frac{(u-t_{p^*-1})(t_{p^*}-u)^{\frac{1}{2}}}{t_{p^*}-t_{p^*-1}}$. By Theorem 2, we have

$$E[F(\cdot + y + tw)|X_\tau](\vec{\xi})$$
$$= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u) + y(u) + tw(u))^{\sim}\} \int_{\mathbb{B}^2} \exp\{i[\alpha(h, x_1)^{\sim}$$
$$+ \beta(h, x_2)^{\sim}]\} dm^2(x_1, x_2) d\sigma(h)$$
$$= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u) + y(u) + tw(u))^{\sim}\} \exp\bigg\{-\frac{\alpha^2 + \beta^2}{2}|h|^2\bigg\} d\sigma(h)$$
$$= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u) + y(u) + tw(u))^{\sim}\} \exp\bigg\{-\frac{\Gamma}{2}|h|^2\bigg\} d\sigma(h),$$

where Γ is given by (14) and the second equality follows from Lemma 14. By (15) and [11, Theorem 2.27], we have

$$\delta_w E[F|X_\tau](y, \vec{\xi})$$
$$= \frac{\partial}{\partial t} E[F(\cdot + y + tw)|X_\tau](\vec{\xi})|_{t=0}$$
$$= \int_{\mathcal{H}} i(h, w(u))^{\sim} \exp\{i(h, [\vec{\xi}](u) + y(u))^{\sim}\} \exp\bigg\{-\frac{\Gamma}{2}|h|^2\bigg\} d\sigma(h)$$

for P_{X_τ}-a.e. $\vec{\xi} \in \mathbb{B}^k$. □

REMARK 2. It is not difficult to show that under the condition (15) we have, for s-a.e. $x \in C_0(\mathbb{B})$,

$$\delta_w F(x) = \int_{\mathcal{H}} i(h, w(u))^{\sim} \exp\{i(h, x(u))^{\sim}\} d\sigma(h).$$

Let $u = T$ and suppose that (16) holds for $\vec{\xi} \equiv \vec{0} \in \mathbb{B}^k$. Then, for $F \in \mathcal{F}(C_0(\mathbb{B}); T)$, we have $\delta_w E[F|X_\tau](x, \vec{0}) = \delta_w F(x)$ by Theorem 15. In this case, as a special case of the result of Theorem 15, (3.34) in [9] can be obtained, too.

THEOREM 16. *Let F be given by (13) and let X_τ be given as in Lemma 3. Let $1 \le p \le \infty$ and let q be a non-zero real number. Then, for s-a.e. $\vec{\xi} \in \mathbb{B}^k$, $T_q^{(p)}[F|X_\tau](y,\vec{\xi})$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and it is given by*

$$\begin{aligned} & T_q^{(p)}[F|X_\tau](y,\vec{\xi}) \\ (17) \qquad &= \int_{\mathcal{H}} \exp\{i(h,[\vec{\xi}](u)+y(u))^\sim\}\exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\}d\sigma(h), \end{aligned}$$

where Γ is given by (14).

Proof. For $\lambda > 0$ and s-a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$\begin{aligned} & T_\lambda[F|X_\tau](y,\vec{\xi}) \\ &= \int_{C_0(\mathbb{B})} F(\lambda^{-\frac{1}{2}}(x-[x])+[\vec{\xi}]+y)dm_{\mathbb{B}}(x) \\ &= \int_{\mathcal{H}} \exp\{i(h,[\vec{\xi}](u)+y(u))^\sim\}\int_{C_0(\mathbb{B})}\exp\{i(h,\lambda^{-\frac{1}{2}}(x(u)-[x](u)))^\sim\} \\ &\quad dm_{\mathbb{B}}(x)d\sigma(h) \\ &= \int_{\mathcal{H}} \exp\{i(h,[\vec{\xi}](u)+y(u))^\sim\}\exp\left\{-\frac{\Gamma}{2\lambda}|h|^2\right\}d\sigma(h) \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$ using a similar method in the proof of Theorem 15. By Morera's theorem and the dominated convergence theorem, we have the result for $\lambda \in \mathbb{C}_+$. Let $1 \le p \le \infty$ and let $T_q^{(p)}[F|X_\tau](y,\vec{\xi})$ be given by (17). For $p=1$ we have

$$\begin{aligned} & |T_\lambda[F|X_\tau](y,\vec{\xi}) - T_q^{(1)}[F|X_\tau](y,\vec{\xi})| \\ &\le \int_{\mathcal{H}}\left|\exp\left\{-\frac{\Gamma}{2\lambda}|h|^2\right\}-\exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\}\right|d|\sigma|(h) \end{aligned}$$

and for $1 < p \le \infty (\frac{1}{p}+\frac{1}{p'}=1)$ we have

$$\begin{aligned} & \int_{C_0(\mathbb{B})} |T_\lambda[F|X_\tau](\gamma y,\vec{\xi}) - T_q^{(p)}[F|X_\tau](\gamma y,\vec{\xi})|^{p'}dm_{\mathbb{B}}(y) \\ &\le \left[\int_{\mathcal{H}}\left|\exp\left\{-\frac{\Gamma}{2\lambda}|h|^2\right\}-\exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\}\right|d|\sigma|(h)\right]^{p'} \end{aligned}$$

for $\gamma > 0$. Letting $\lambda \to -iq$ through $\mathbb{C}_+$, by the dominated convergence theorem, we have (17) as the L_p conditional Fourier-Feynman transform of F. □

Using the same method in the proof of Theorem 16, we have the following corollary.

COROLLARY 17. *Let F be given by (13). Then $T_\lambda(F)$ exists for $\lambda \in \mathbb{C}_+$ and, for $1 \le p \le \infty$ and for non-zero real q, $T_q^{(p)}(F)$ exists. Moreover, they are given by*

$$T_\lambda(F)(y) = \int_{\mathcal{H}} \exp\{i(h, y(u))^\sim\} \exp\Big\{-\frac{u}{2\lambda}|h|^2\Big\} d\sigma(h) \tag{18}$$

and

$$T_q^{(p)}(F)(y) = \int_{\mathcal{H}} \exp\{i(h, y(u))^\sim\} \exp\Big\{-\frac{iu}{2q}|h|^2\Big\} d\sigma(h) \tag{19}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

REMARK 3. For any Borel subset H of $\mathcal{H}$, let

$$\sigma_{q,\vec{\xi}}(H) = \int_H \exp\{i(h, [\vec{\xi}](u))^\sim\} \exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\} \exp\left\{\frac{iu}{2q}|h|^2\right\} d\sigma(h)$$

and let $F_{q,\vec{\xi}}$ be given by (13) with replacing σ by $\sigma_{q,\vec{\xi}}$. Then, for s-a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$\begin{aligned} &T_q^{(p)}[F|X_\tau](y, \vec{\xi}) \\ &= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}](u) + y(u))^\sim\} \exp\Big\{-\frac{i\Gamma}{2q}|h|^2\Big\} d\sigma(h) \\ &= T_q^{(p)}(F_{q,\vec{\xi}})(y) \end{aligned} \tag{20}$$

and

$$T_{-q}^{(p)}[T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi})|X_\tau](y, -\vec{\xi}) = F(y) \tag{21}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

THEOREM 18. *Let F be given by (13), let $w \in C_0(\mathbb{B})$ and let X_τ be given as in* Lemma 3. *Let $1 \le p \le \infty$ and let q be a non-zero real number. Suppose that* (15) *holds. Then we have, for s-a.e. $\vec{\xi}_1, \vec{\xi}_2 \in \mathbb{B}^k$,*

$$\begin{aligned} T_q^{(p)}[\delta_w E[F|X\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) &= \delta_w E[T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}_2)|X_\tau](y, \vec{\xi}_1) \\ &= \delta_w E[T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) \end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$.

Proof. For $H \in \mathcal{B}(\mathcal{H})$, let

$$\sigma_{w,\vec{\xi}}(H) = \int_H i(h, w(u))^\sim \exp\{i(h, [\vec{\xi}](u))^\sim\} \times \exp\left\{-\frac{\Gamma}{2}|h|^2\right\} d\sigma(h)$$

and

$$\sigma_{q,\vec{\xi}}(H) = \int_H \exp\{i(h, [\vec{\xi}](u))^\sim\} \exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\} d\sigma(h) \quad \text{for } \vec{\xi} \in \mathbb{B}^k,$$

where Γ is given by (14).

Then, by Theorems 15 and 16, both $\delta_w E[F|X\tau](\cdot, \vec{\xi})$ and $T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi})$ are elements of $\mathcal{F}(C_0(\mathbb{B}); u)$. Therefore, for s-a.e. $\vec{\xi}_1, \vec{\xi}_2 \in \mathbb{B}^k$, we have

$$\begin{aligned}
&T_q^{(p)}[\delta_w E[F|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) \\
&= \int_{\mathcal{H}} \exp\{i(h, [\vec{\xi}_2](u) + y(u))^\sim\} \exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\} d\sigma_{w,\vec{\xi}_1}(h) \\
&= \int_{\mathcal{H}} i(h, w(u))^\sim \exp\{i(h, [\vec{\xi}_1](u) + [\vec{\xi}_2](u) + y(u))^\sim\} \\
&\quad \times \exp\left\{-\frac{i\Gamma}{2q}|h|^2\right\} \exp\left\{-\frac{\Gamma}{2}|h|^2\right\} d\sigma(h) \\
&= \int_{\mathcal{H}} i(h, w(u))^\sim \exp\{i(h, [\vec{\xi}_1](u) + y(u))^\sim\} \exp\left\{-\frac{\Gamma}{2}|h|^2\right\} d\sigma_{q,\vec{\xi}_2}(h) \\
&= \delta_w E[T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}_2)|X_\tau](y, \vec{\xi}_1).
\end{aligned}$$

Similarly, we have

$$T_q^{(p)}[\delta_w E[F|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2) = \delta_w E[T_q^{(p)}[F|X_\tau](\cdot, \vec{\xi}_1)|X_\tau](y, \vec{\xi}_2). \qquad \square$$

The following theorem shows that the Fourier-Feynman transform of conditional first variation of functions in $\mathcal{F}(C_0(\mathbb{B}); u)$ is essentially same to the conditional first variation of Fourier-Feynman transform of the functions.

THEOREM 19. *Let F be given by* (13) *and let $1 \le p \le \infty$. Under the assumptions in* Theorem 15, *we have, for non-zero real q and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$,*

$$T_q^{(p)}(\delta_w E[F|X_\tau](\cdot, \vec{\xi}))(y) = \delta_w E[T_q^{(p)}(F)|X_\tau](y, \vec{\xi})$$

for s-a.e. $y \in C_0(\mathbb{B})$.

Proof. For any Borel subset H of $\mathcal{H}$, let $\sigma_{q,u}(H) = \int_H \exp\{-\frac{iu}{2q}|h|^2\} d\sigma(h)$ and let $\sigma_{w,\vec{\xi}}$ be given as in the proof of Theorem 18. By Theorem

15 and Corollary 17, for s-a.e. $\vec{\xi} \in \mathbb{B}^k$, we have

$$
\begin{aligned}
&T_q^{(p)}(\delta_w E[F|X_\tau](\cdot,\vec{\xi}))(y) \\
&= \int_{\mathcal{H}} \exp\{i(h,y(u))^\sim\} \exp\left\{-\frac{iu}{2q}|h|^2\right\} d\sigma_{w,\vec{\xi}}(h) \\
&= \int_{\mathcal{H}} i(h,w(u))^\sim \exp\{i(h,[\vec{\xi}](u)+y(u))^\sim\} \exp\left\{-\left(\frac{\Gamma}{2}+\frac{iu}{2q}\right)|h|^2\right\} d\sigma(h) \\
&= \int_{\mathcal{H}} i(h,w(u))^\sim \exp\{i(h,[\vec{\xi}](u)+y(u))^\sim\} \exp\left\{-\frac{\Gamma}{2}|h|^2\right\} d\sigma_{q,u}(h) \\
&= \delta_w E[T_q^{(p)}(F)|X_\tau](y,\vec{\xi})
\end{aligned}
$$

for s-a.e. $y \in C_0(\mathbb{B})$ and hence the proof is completed. □

Using Remark 2 and Theorem 16, we have the following corollary.

COROLLARY 20. *Let F be given by (13), let $w \in C_0(\mathbb{B})$ and let X_τ be given as in* Lemma 3. *Let $1 \le p \le \infty$ and suppose that* (15) *holds. Then we have, for any non-zero real q and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$,*

$$
T_q^{(p)}[\delta_w F|X_\tau](y,\vec{\xi}) = \delta_w(T_q^{(p)}[F|X_\tau](\cdot,\vec{\xi}))(y)
$$

for s-a.e. $y \in C_0(\mathbb{B})$.

THEOREM 21. *Let F be given by* (13), *let $w \in C_0(\mathbb{B})$ with $w(u) \in \mathcal{H}$ and let X_τ be given as in* Lemma 3. *Let $1 \le p \le \infty$ and suppose that* (15) *holds. Then we have, for non-zero real q and for s-a.e. $\vec{\xi} \in \mathbb{B}^k$,*

$$
\begin{aligned}
&\int_{C_0(\mathbb{B})}^{anf_q} T_q^{(p)}[\delta_w F|X_\tau](x,\vec{\xi})\,dm_{\mathbb{B}}(x) \\
&= \int_{C_0(\mathbb{B})}^{anf_q} \delta_w(T_q^{(p)}[F|X_\tau](\cdot,\vec{\xi}))(x)\,dm_{\mathbb{B}}(x) \\
&= \frac{-iq}{u} \int_{C_0(\mathbb{B})}^{anf_q} (w(u),x(u))^\sim T_q^{(p)}[F|X_\tau](x,\vec{\xi})\,dm_{\mathbb{B}}(x)
\end{aligned}
$$

with the existences of both sides of the equality.

Proof. For $\rho > 0$ let $F_\rho(x) = F(\rho x)$ for $x \in C_0(\mathbb{B})$ and for $\gamma > 0$ let $z = \gamma w$. Then, by the definition of conditional Fourier-Feynman transform and Theorem 10, we have

$$
\begin{aligned}
&\int_{C_0(\mathbb{B})} \delta_{\rho z}(T_{\rho^{-2}}[F|X_\tau](\cdot,\vec{\xi}))(\rho x)\,dm_{\mathbb{B}}(x)\\
&= \int_{C_0(\mathbb{B})} \frac{\partial}{\partial t} T_{\rho^{-2}}[F|X_\tau](\rho x + t\rho z, \vec{\xi})|_{t=0}\,dm_{\mathbb{B}}(x)\\
&= \int_{C_0(\mathbb{B})} \frac{\partial}{\partial t} E[F(\rho\,\cdot + \rho x + t\rho z)|X_\tau(\rho\,\cdot)](\vec{\xi})|_{t=0}\,dm_{\mathbb{B}}(x)\\
&= \int_{C_0(\mathbb{B})} \frac{\partial}{\partial t} E[F_\rho(\,\cdot + x + tz)|X_\tau](\rho^{-1}\vec{\xi})|_{t=0}\,dm_{\mathbb{B}}(x)\\
&= \int_{C_0(\mathbb{B})} \delta_z E[F_\rho|X_\tau](x, \rho^{-1}\vec{\xi})\,dm_{\mathbb{B}}(x)\\
&= \frac{1}{u}\int_{C_0(\mathbb{B})} (z(u), x(u))^\sim E[F_\rho(\,\cdot + x)|X_\tau](\rho^{-1}\vec{\xi})\,dm_{\mathbb{B}}(x)\\
&= \frac{1}{u}\int_{C_0(\mathbb{B})} (z(u), x(u))^\sim E[F(\rho\,\cdot + \rho x)|X_\tau(\rho\,\cdot)](\vec{\xi})\,dm_{\mathbb{B}}(x)\\
&= \frac{1}{u}\int_{C_0(\mathbb{B})} (z(u), x(u))^\sim T_{\rho^{-2}}[F|X_\tau](\rho x, \vec{\xi})\,dm_{\mathbb{B}}(x)
\end{aligned}
$$

for s-a.e. $\vec{\xi} \in \mathbb{B}^k$. For $\lambda > 0$ let $\rho = \lambda^{-\frac{1}{2}}$ and $\gamma = \lambda^{\frac{1}{2}}$. Then we have

$$
\begin{aligned}
&\int_{C_0(\mathbb{B})} \delta_w(T_\lambda[F|X_\tau](\cdot,\vec{\xi}))(\lambda^{-\frac{1}{2}}x)\,dm_{\mathbb{B}}(x)\\
&= \frac{\lambda}{u}\int_{C_0(\mathbb{B})} (w(u), \lambda^{-\frac{1}{2}}x(u))^\sim T_\lambda[F|X_\tau](\lambda^{-\frac{1}{2}}x, \vec{\xi})\,dm_{\mathbb{B}}(x).
\end{aligned}
$$

By Morera's theorem we have the last equality for $\lambda \in \mathbb{C}_+$ and, letting $\lambda \to -iq$ through $\mathbb{C}_+$, we have the result by Theorem 16 and Corollary 20. □

Now we introduce a kind of integration by parts formula for conditional Fourier-Feynman transform of functions in $\mathcal{F}(C_0(\mathbb{B}); u)$. The proof follows from Theorem 21, immediately.

COROLLARY 22. *Let F_1, F_2 be given by* (13) *with replacing σ by σ_1, σ_2, respectively, let $w \in C_0(\mathbb{B})$ with $w(u) \in \mathcal{H}$ and let X_τ be given as in* Lemma 3. *Let $1 \le p \le \infty$ and let q be a non-zero real number. Suppose that*

$$\int_{\mathcal{H}} |\langle h, w(u)\rangle|\,d(|\sigma_1| + |\sigma_2|)(h) < \infty.$$

Then we have, for s-a.e. $\vec{\xi} \in \mathbb{B}^k$,

$$\begin{aligned}
&\frac{-iq}{u} \int_{C_0(\mathbb{B})}^{anf_q} (w(u), x(u))^{\sim} T_q^{(p)}[F_1F_2|X_\tau](x, \vec{\xi}) dm_{\mathbb{B}}(x) \\
&= \int_{C_0(\mathbb{B})}^{anf_q} T_q^{(p)}[\delta_w(F_1F_2)|X_\tau](x, \vec{\xi}) dm_{\mathbb{B}}(x) \\
&= \int_{C_0(\mathbb{B})}^{anf_q} T_q^{(p)}[F_2\delta_w F_1|X_\tau](x, \vec{\xi}) dm_{\mathbb{B}}(x) \\
&\quad + \int_{C_0(\mathbb{B})}^{anf_q} T_q^{(p)}[F_1\delta_w F_2|X_\tau](x, \vec{\xi}) dm_{\mathbb{B}}(x)
\end{aligned}$$

with the existence of each analytic Feynman integral.

COROLLARY 23. *Under the assumptions in* Theorem 21, *we have*

$$\int_{C_0(\mathbb{B})}^{anf_q} \delta_w F(x) dm_{\mathbb{B}}(x) = \frac{-iq}{u} \int_{C_0(\mathbb{B})}^{anf_q} (w(u), x(u))^{\sim} F(x) dm_{\mathbb{B}}(x). \tag{22}$$

Proof. The result immediately follows from Theorem 21 and the equation (21). □

REMARK 4. The equation (22) is a special case of the equation (2.5) in [6]. Note that we can also derive this equation using Corollary 2.4 in [6], directly.

5. Transforms of functions in $\mathcal{F}(C_0(\mathbb{B}); u)$ with n linear factors

Let $0 < u \le T$ be fixed, but arbitrarily, let F be defined on $C_0(\mathbb{B})$ and for any given fixed $n \in \mathbb{N}$ let

$$F_j(x) = F(x) \prod_{l=1}^{j} (w_l(u), x(u))^{\sim}$$

for s-a.e. $x \in C_0(\mathbb{B})$ where $w_j \in C_0(\mathbb{B})$ with $w_j(u) \in \mathcal{H}$ for $j = 1, \cdots, n$. For convenience, let $F_0 = F$.

Our first theorem gives a recurrence relation in which we express the transform of F_j in terms of the transforms and variation of F_{j-1} under suitable conditions.

THEOREM 24. *For* $1 \le p \le \infty$, *for a non-zero real* q *and for* $j = 1, \cdots, n$, *assume that both* $T_q^{(p)}(\delta_{w_j} F_{j-1})(y)$ *and* $T_q^{(p)}(F_{j-1})(y)$ *can be* *expressible as analytic Feynman integrals for s-a.e. $y \in C_0(\mathbb{B})$. Moreover, suppose that (22) holds with replacing w, F by w_j, $F_{j-1}(\cdot + y)$, respectively. Then $T_q^{(p)}(F_j)(y)$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and is given by the recurrence relation*

$$T_q^{(p)}(F_j)(y) = \frac{iu}{q} T_q^{(p)}(\delta_{w_j} F_{j-1})(y) + (w_j(u), y(u))^{\sim} T_q^{(p)}(F_{j-1})(y).$$

Proof. Since $T_q^{(p)}(\delta_{w_j} F_{j-1})(y)$ exists, we know that $\delta_{w_j} F_{j-1}(\lambda^{-\frac{1}{2}} x + y)$ is Wiener integrable for each $\lambda > 0$ and, for s-a.e. $y \in C_0(\mathbb{B})$, we have

$$\begin{aligned}
&T_q^{(p)}(\delta_{w_j} F_{j-1})(y) \\
&= \int_{C_0(\mathbb{B})}^{anf_q} \delta_{w_j} F_{j-1}(x+y)\, dm_{\mathbb{B}}(x) \\
&= -\frac{iq}{u} \int_{C_0(\mathbb{B})}^{anf_q} F_{j-1}(x+y)(w_j(u), x(u) + y(u))^{\sim} dm_{\mathbb{B}}(x) \\
&\quad + \frac{iq}{u} \int_{C_0(\mathbb{B})}^{anf_q} F_{j-1}(x+y)(w_j(u), y(u))^{\sim} dm_{\mathbb{B}}(x) \\
&= -\frac{iq}{u} \int_{C_0(\mathbb{B})}^{anf_q} F_j(x+y)\, dm_{\mathbb{B}}(x) \\
&\quad + \frac{iq}{u}(w_j(u), y(u))^{\sim} \int_{C_0(\mathbb{B})}^{anf_q} F_{j-1}(x+y)\, dm_{\mathbb{B}}(x) \\
&= -\frac{iq}{u} T_q^{(p)}(F_j)(y) + \frac{iq}{u}(w_j(u), y(u))^{\sim} T_q^{(p)}(F_{j-1})(y),
\end{aligned}$$

where the second equality follows from (22). Therefore, we have the result. □

The following theorem is an immediate result of Theorem 24.

THEOREM 25. *For $1 \le p \le \infty$, for a non-zero real q and for $j = 1, \cdots, n-1$, assume that*

$$T_q^{(p)}(\delta_{w_{j+1}} F_j)(y) = \delta_{w_{j+1}}(T_q^{(p)}(F_j))(y) \tag{23}$$

and

$$T_q^{(p)}(\delta_{w_{j+1}} F_{j-1})(y) = \delta_{w_{j+1}}(T_q^{(p)}(F_{j-1}))(y) \tag{24}$$

hold for s-a.e. $y \in C_0(\mathbb{B})$ with the existences of Fourier-Feynman transforms and first variations. Moreover, for some $\epsilon > 0$, both $T_q^{(p)}(\delta_{w_j} F_{j-1})$ $(y + tw_{j+1})$ and $T_q^{(p)}(F_{j-1})(y + tw_{j+1})$ can be expressible as analytic

Feynman integrals, and $F_{j-1}(\cdot + y + tw_{j+1})$ satisfies (22) with replacing w by w_j for $|t| < \epsilon$. Then, for s-a.e. $y \in C_0(\mathbb{B})$, we have the recurrence relation

$$\begin{aligned}&T_q^{(p)}(\delta_{w_{j+1}}F_j)(y)\\&= \frac{iu}{q}\delta_{w_{j+1}}(T_q^{(p)}(\delta_{w_j}F_{j-1}))(y) + \langle w_j(u), w_{j+1}(u)\rangle\\&\quad\times\ T_q^{(p)}(F_{j-1})(y) + (w_j(u), y(u))^{\sim}T_q^{(p)}(\delta_{w_{j+1}}F_{j-1})(y).\end{aligned}$$

Proof. For t in $\mathbb{R}$ with $|t| < \epsilon$, by Theorem 24, we have

$$\begin{aligned}&T_q^{(p)}(F_j)(y + tw_{j+1})\\&= \frac{iu}{q}T_q^{(p)}(\delta_{w_j}F_{j-1})(y + tw_{j+1})\\&\quad+\ (w_j(u), y(u) + tw_{j+1}(u))^{\sim}T_q^{(p)}(F_{j-1})(y + tw_{j+1}).\end{aligned}$$

Differentiating both sides of the last equation and letting $t = 0$, we have the result by (23) and (24). □

Now, let $F_0 = F$ where F is given by (13) and suppose that

$$\int_{\mathcal{H}} |h|^n d|\sigma|(h) < \infty.$$

Then, we have $\int_{\mathcal{H}} |h|^j d|\sigma|(h) < \infty$ for $j = 1, \cdots, n-1$ and

$$\delta_{w_1}(T_q^{(p)}(F_0))(y) = \int_{\mathcal{H}} i\langle h, w_1(u)\rangle \exp\left\{i(h, y(u))^{\sim} - \frac{iu}{2q}|h|^2\right\}d\sigma(h).$$

Hence we have

$$\begin{aligned}&\delta_{w_2}(\delta_{w_1}(T_q^{(p)}(F_0)))(y)\\&= -\int_{\mathcal{H}} \langle h, w_1(u)\rangle\langle h, w_2(u)\rangle \exp\{i(h, y(u))^{\sim}\} \times \exp\left\{-\frac{iu}{2q}|h|^2\right\}d\sigma(h)\end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$, which implies $\delta_{w_2}(\delta_{w_1}(T_q^{(p)}(F_0))) \in \mathcal{F}(C_0(\mathbb{B}); u)$. By Corollary 23 and Theorem 24, we have

$$T_q^{(p)}(F_1)(y) = \frac{iu}{q}\delta_{w_1}(T_q^{(p)}(F_0))(y) + (w_1(u), y(u))^{\sim}T_q^{(p)}(F_0)(y)$$

for s-a.e. $y \in C_0(\mathbb{B})$. Also, we have

$$\begin{aligned}\delta_{w_2}F_1(y) &= \frac{\partial}{\partial t}[F_0(y + tw_2)(w_1(u), y(u) + tw_2(u))^{\sim}]|_{t=0}\\&= \delta_{w_2}F_0(y)(w_1(u), y(u))^{\sim} + F_0(y)\langle w_1(u), w_2(u)\rangle\end{aligned}$$

and hence

$$\begin{aligned}
&T_q^{(p)}(\delta_{w_2}F_1)(y)\\
&= \frac{iu}{q}\delta_{w_1}(T_q^{(p)}(\delta_{w_2}F_0))(y) + (w_1(u), y(u))^{\sim}T_q^{(p)}(\delta_{w_2}F_0)(y)\\
&\quad+ \langle w_1(u), w_2(u)\rangle T_q^{(p)}(F_0)(y)\\
&= \delta_{w_2}(T_q^{(p)}(F_1))(y)
\end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$. Again, by Theorem 24 with $j = 2$, we see that

$$T_q^{(p)}(F_2)(y) = \frac{iu}{q}\delta_{w_2}(T_q^{(p)}(F_1))(y) + (w_2(u), y(u))^{\sim}T_q^{(p)}(F_1)(y)$$

for s-a.e. $y \in C_0(\mathbb{B})$. Continuing this process, we have

$$\begin{aligned}
&\delta_{w_n}(\cdots\delta_{w_1}(T_q^{(p)}(F_0))\cdots)(y)\\
&= \int_{\mathcal{H}}\left(\prod_{j=1}^{n} i\langle h, w_j(u)\rangle\right)\exp\left\{i(h, y(u))^{\sim} - \frac{iu}{2q}|h|^2\right\}d\sigma(h)
\end{aligned}$$

for s-a.e. $y \in C_0(\mathbb{B})$, which implies

$$\delta_{w_n}(\cdots\delta_{w_1}(T_q^{(p)}(F_0))\cdots) \in \mathcal{F}(C_0(\mathbb{B}); u).$$

Hence $\delta_{w_n}(T_q^{(p)}(F_{n-1}))(y)$ exists for s-a.e. $y \in C_0(\mathbb{B})$ and, using Theorem 25, it is given by

$$\begin{aligned}
&\delta_{w_n}(T_q^{(p)}(F_{n-1}))(y)\\
&= T_q^{(p)}(\delta_{w_n}F_{n-1})(y)\\
&= \left(\frac{iu}{q}\right)^0[\langle w_{n-1}(u), w_n(u)\rangle T_q^{(p)}(F_{n-2})(y) + (w_{n-1}(u), y(u))^{\sim}\\
&\quad\times \delta_{w_n}(T_q^{(p)}(F_{n-2}))(y)] + \left(\frac{iu}{q}\right)^1[\langle w_{n-2}(u), w_{n-1}(u)\rangle\\
&\quad\times \delta_{w_n}(T_q^{(p)}(F_{n-3}))(y) + \langle w_{n-2}(u), w_n(u)\rangle\delta_{w_{n-1}}(T_q^{(p)}(F_{n-3}))(y)\\
&\quad+ (w_{n-2}(u), y(u))^{\sim}\delta_{w_n}(\delta_{w_{n-1}}(T_q^{(p)}(F_{n-3})))(y)] + \left(\frac{iu}{q}\right)^2[\langle w_{n-3}(u),\\
&\qquad w_{n-2}(u)\rangle\delta_{w_n}(\delta_{w_{n-1}}(T_q^{(p)}(F_{n-4})))(y) + \langle w_{n-3}(u), w_{n-1}(u)\rangle\\
&\quad\times \delta_{w_n}(\delta_{w_{n-2}}(T_q^{(p)}(F_{n-4})))(y) + \langle w_{n-3}(u), w_n(u)\rangle\\
&\quad\times \delta_{w_{n-1}}(\delta_{w_{n-2}}(T_q^{(p)}(F_{n-4})))(y) + (w_{n-3}(u), y(u))^{\sim}
\end{aligned}$$

$$\times\,\delta_{w_n}(\delta_{w_{n-1}}(\delta_{w_{n-2}}(T_q^{(p)}(F_{n-4}))))(y)]+\cdots+\left(\frac{iu}{q}\right)^{n-2}[\langle w_1(u),$$

$$w_2(u)\rangle\delta_{w_n}(\delta_{w_{n-1}}(\cdots(\delta_{w_4}(\delta_{w_3}(T_q^{(p)}(F_0))))\cdots))(y)+\langle w_1(u),w_3(u)\rangle$$

$$\times\,\delta_{w_n}(\delta_{w_{n-1}}(\cdots(\delta_{w_4}(\delta_{w_3}(T_q^{(p)}(F_0))))\cdots))(y)+\cdots+\langle w_1(u),w_n(u)\rangle$$

$$\times\,\delta_{w_{n-1}}(\delta_{w_{n-2}}(\cdots(\delta_{w_3}(\delta_{w_2}(T_q^{(p)}(F_0))))\cdots))(y)+(w_1(u),y(u))^{\sim}$$

$$\times\,\delta_{w_n}(\delta_{w_{n-1}}(\cdots(\delta_{w_3}(\delta_{w_2}(T_q^{(p)}(F_0))))\cdots))(y)]+\left(\frac{iu}{q}\right)^{n-1}$$

$$\times\,\delta_{w_n}(\delta_{w_{n-1}}(\cdots(\delta_{w_2}(\delta_{w_1}(T_q^{(p)}(F_0))))\cdots))(y).$$

Thus we obtain

$$T_q^{(p)}(F_n)(y)=\frac{iu}{q}\delta_{w_n}(T_q^{(p)}(F_{n-1}))(y)+(w_n(u),y(u))^{\sim}T_q^{(p)}(F_{n-1})(y)$$

for s -a.e. $y\in C_0(\mathbb{B})$ by Theorem 24 and hence we have the following theorem.

THEOREM 26. *Under the above assumptions, for $k=1,\cdots,n$, we have*

$$T_q^{(p)}(F_k)(y)=\frac{iu}{q}\sum_{j=0}^{k-1}\left[\delta_{w_{j+1}}(T_q^{(p)}(F_j))(y)\left(\prod_{l=j+2}^{k}(w_l(u),y(u))^{\sim}\right)\right]$$

$$+\;T_q^{(p)}(F)(y)\left(\prod_{j=1}^{k}(w_j(u),y(u))^{\sim}\right)$$

for s-a.e. $y\in C_0(\mathbb{B})$.

REMARK 5. For $k=1$, setting $y=0$, we obtain the following Feynman integral formula.

$$\int_{C_0(\mathbb{B})}^{anf_q}F(x)(w_1(u),x(u))^{\sim}dm_{\mathbb{B}}(x)$$

$$=\frac{iu}{q}\int_{\mathcal{H}}i\langle h,w_1(u)\rangle\exp\left\{-\frac{iu}{2q}|h|^2\right\}d\sigma(h).$$

REMARK 6. Let $w_{\mathcal{H}}\in\mathcal{H}$ and let $w\in C_0(\mathbb{B})$ with $w(u)=w_{\mathcal{H}}$. With these settings, we can obtain the results in [9] as special cases of this paper.

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