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AVERAGING PROPERTIES AND SPREADING MODELS

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Abstract. In this paper, we study averaging properties in Banach spaces using the Brunel-Sucheston's spreading model. We show that the Schlumprecht space S does not have the Banach-Saks property and l_1 is finitely representable in the Schlumprecht space S using the spreading model properties.

1. Introduction

We shall state some fundamental properties about Brunel-Sucheston's spreading model.

Let (x_n) be a bounded sequence with no norm Cauchy subsequence in a Banach space X. Suppose that the limit

$$\lim_{\substack{m\to\infty \\ m\le n_1<\cdots<n_k}} \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\|$$

exists for each $(a_i)_{i=1}^k \in S_0$, where S_0 is the space of sequences $a = (a_i),\ i = 1, 2, \cdots$ of real numbers such that only finite many of a_i are different from zero. We shall call such a sequence (x_n) a good sequence. Then we can define the nonnegative function Ψ on S_0 by

$$\Psi((a_i)_{i=1}^k) = \lim_{\substack{m\to\infty \\ m\le n_1<\cdots<n_k}} \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\|.$$

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Clearly Ψ defines a seminorm on S_0. Furthermore, since (x_n) is assumed to have no norm Cauchy subsequence, Ψ indeed defines a norm on S_0 [2]. Hence we shall write $\left|\sum_{i=1}^{k} a_i e_i\right|$ in place of $\Psi((a_i)_{i=1}^{k})$ for each $(a_i)_{i=1}^{k} \in S_0$. Let F be the completion of $[\Psi(S_0), |\cdot|]$. We shall call $[F, (e_n)]$ the spreading model of a good sequence (x_n) and (e_n) the fundamental sequence of the spreading model. Then (x_n) and $[F, (e_n)]$ have the following properties :

(1) The norm $|\cdot|$ for F is invariant under spreading in the sense that

$$\left|\sum_{i=1}^{k} a_i e_i\right| = \left|\sum_{i=1}^{k} a_i e_{n_i}\right|$$

(2)

$$\lim_{\substack{m\to\infty \\ m\le n_1<\cdots<n_k}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| = \left|\sum_{i=1}^{k} a_i e_i\right| \quad \text{for every } (a_i)_{i=1}^{k} \in S_0.$$

A Banach space X is said to have the Banach-Saks property if every bounded sequence in X admits a subsequence whose arithmetic means converge in norm. In 1938, S. Kakutani[5] showed that if X is uniformly convex, then X has the Banach-Saks property. In 1963, T. Nishiura and D. Waterman[7] proved that if a Banach space X has the Banach-Saks property, then X is reflexive.

A Banach space X is said to have the weak Banach-Saks property if every weakly convergent sequence has a subsequence whose arithmetic means converge in norm. It is obvious that the Banach-Saks property implies the weak Banach-Saks property. Indeed, if a sequence (x_n) in a Banach space X converges weakly to x in X, the arithmetic means of any subsequence of (x_n) also converge weakly to x, thus, the strong limit of the arithmetic means, if it exists, must be x. For this reason, we can translate any weakly convergent sequence to a weakly null sequence and look for the arithmetic means to converge to zero in norm.

To end with this introduction, let us mention the following lemmas.

Lemma 1. [4] *If (x_n) is a sequence in a Banach space, then there exists a subsequence (x_{n_k}) of (x_n) such that either no subsequence of (x_{n_k}) has convergent arithmetic means or every subsequence of (x_{n_k}) has arithmetic means converging to the same limit.*

LEMMA 2. [8, Theorem 1] *Let* (x_n) *be a good sequence in a Banach space* X *and* $[F,(e_n)]$ *be its spreading model. Then for any* $\epsilon > 0$ *and integer* $t \geq 2$ *one can select a subsequence* (x'_n) *of* (x_n) *with the following property :*

For every k, $n_i \in \mathbb{N}$ $(i = 1, 2, \cdots, k)$ *with* $n_1 < n_2 < \cdots < n_k$ *and* $(a_i)_{i=1}^k \in S_0$, *we have*

$$\begin{cases} (1-\epsilon)\left|\sum_{i=1}^k a_i e_i\right| - (2\log_t k)\sup_{1\leq i\leq k}|a_i|\sup_n \|x'_n\| \\ \leq \left\|\sum_{i=1}^k a_i x'_{n_i}\right\| \\ \leq (1+\epsilon)\left|\sum_{i=1}^k a_i e_i\right| + (3\log_t k)\sup_{1\leq i\leq k}|a_i|\sup_n \|x'_n\| \end{cases}$$

LEMMA 3. [2] *In any Banach space, every bounded sequence with no norm convergent subsequence has a subsequence which is a good sequence.*

LEMMA 4. [8, Lemma 1] *Let* (x_n) *be a good sequence in a Banach space* X *and* $[F,(e_n)]$ *its spreading model. We put*

$$\rho(k) := \left|\frac{1}{k}\sum_{i=1}^k e_i\right| \qquad (k \in \mathbb{N}).$$

Then $\rho := \lim_{k\to\infty}\rho(k)$ *exists and is equal to* $\inf_k \rho(k)$.

2. Banach-Saks property and spreading model

The following is an application of Brunel-Sucheston's spreading model to weak Banach-Saks property in Banach spaces.

THEOREM 1. *Let* X *be a Banach space. Then the following are equivalent:*

(1) X *has the weak Banach-Saks property.*
(2) *For every weakly null and good sequence* (x_n) *in* X *with its spreading model* $[F,(e_n)]$, $\rho = 0$.

Proof. Suppose that X has the weak Banach-Saks property. Let (y_n) be a weakly null and good sequence with its spreading model $[F,(e_n)]$. Then there exists a subsequence (x_n) of (y_n) such that every subsequence of (x_n) has the arithmetic means converging to zero by Lemma 1. By Lemma 2, there exists a subsequence (x'_n) of (x_n) such that for $k \in \mathbb{N}$,

$$\frac{1}{2}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| - (2\log_2 k)\frac{1}{k}\sup_n \|x'_n\| \leq \left\|\frac{1}{k}\sum_{i=1}^{k} x'_i\right\|$$

Since $\frac{1}{k}\sum_{i=1}^{k} x'_i \to 0$ and $(2\log_2 k)\frac{1}{k} \to 0$ as $k \to \infty$, $\frac{1}{k}\sum_{i=1}^{k} e_i \to 0$ as $k \to \infty$ and $\rho = 0$.

Suppose that for every weakly null and good sequence (x_n) in X, $\rho = 0$. Let (y_n) be a weakly null sequence in X. We show that (y_n) has a subsequence with convergent arithmetic means. If (y_n) has a convergent subsequence, its limit is necessarily zero and so its arithmetic means converge to zero. So we may assume that (y_n) has no convergent subsequence. By Lemma 3, (y_n) has a subsequence (x_n) which is a weakly null and good sequence with its spreading model $[F, (e_n)]$. By Lemma 2, there exists a subsequence (x'_n) of (x_n) such that

$$\left\|\frac{1}{k}\sum_{i=1}^{k} x'_i\right\| \leq \frac{3}{2}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| + (3\log_2 k)\frac{1}{k}\sup_n \|x'_n\|.$$

Since $\lim_{k\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| = \rho = 0$ and $\lim_{k\to\infty}(3\log_2 k)\frac{1}{k} = 0$,

$$\lim_{k\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k} x'_i\right\| = 0.$$

This completes our proof. □

Let (x_n) be a good sequence in a Banach space X and $[F, (e_n)]$ its spreading model. Let F_n be the closed subspace of F generated by the vectors $\{e_i : i \geq n\}$ and $F_\infty = \cap_n F_n$. Then $[F, (e_n)]$ has the following properties which are found in [2].

LEMMA 5. [2] *Let (x_n) be a good sequence in X with its spreading model $[F, (e_n)]$. Then*

(1) *There exists a linear isometry T on F such that $Te_n = e_{n+1}$ for all n. T is called the shift on F.*
(2) *$y \in \cap_n F_n = F_\infty$ if and only if $y = Ty$.*

(3) *If* $F_\infty \neq \{0\}$, F_∞ *is 1-dimensional and* $F = F_\infty \oplus \overline{(I-T)F}$, *where* $\overline{(I-T)F}$ *is the closure of the manifold* $(I-T)F$ *in the norm* $|\cdot|$.

(4) *If* $\lim_{n\to\infty} \frac{1}{n}\sum_{j\le n} e_j$ *exists in* F, (x_n) *has a subsequence whose arithmetic means converge in norm* $\|\cdot\|$.

To prove Theorem 2, we need the following lemma and proposition.

LEMMA 6. *Let* (x_n) *be a good sequence in* X *with its spreading model* $[F,(e_n)]$. *Then*

(1) *If* $F_\infty \neq \{0\}$, *then* (e_n) *has a arithmetic means converging to a non-zero element in* F_∞ *with a norm* $|\cdot|$.

(2) *If* (e_n) *has a convergent arithmetic means with a norm* $|\cdot|$, *then the limit is an element of* F_∞.

Proof. (1) By (3) of Lemma 5, we may assume that $e_1 = e_\infty + x - Tx$ for some $e_\infty \in F_\infty$ and $x \in F$. Then

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n} e_j &= \frac{1}{n}\sum_{j=1}^{n} T^{j-1}e_1\\ &= \frac{1}{n}\sum_{j=1}^{n}\left(e_\infty + T^{j-1}x - T^{j}x\right) \quad \text{by (2) of Lemma 5}\\ &= e_\infty + \frac{1}{n}\left(x - T^n x\right)\end{aligned}$$

and

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=1}^{n} e_j - e_\infty\right| &\le \frac{1}{n}\left(|x| + |T^n x|\right)\\ &\le \frac{2}{n}|x|,\end{aligned}$$

since T is an isometry on F. This implies that $\frac{1}{n}\sum_{i=1}^{n} e_i \to e_\infty \in F_\infty$.

Define a map $f : F \to \mathbb{R}$ by

$$f\left(\sum_{i=1}^{k} a_i e_i\right) = \sum_{i=1}^{k} a_i.$$

Then f is continuous (cf. Proposition 2, [2]) and so there exists $M > 0$ such that

$$\left|\sum_{i=1}^{k} a_i\right| \le M \left|\sum_{i=1}^{k} a_i e_i\right|.$$

Considering $a_i = \frac{1}{k}$, $\left|\frac{1}{n}\sum_{i=1}^{n} e_i\right| \ge \frac{1}{M} > 0$. This implies that $e_\infty \neq 0$. We complete our proof of (1).

(2) Suppose that $\frac{1}{n}\sum_{j=1}^{n} e_j$ converges to e in a norm $|\cdot|$. It suffices to show that $Te = e$ where T is the shift on F, by (2) of Lemma 5. Since

$$T\left(\frac{1}{n}\sum_{j=1}^{n} e_j\right) = \frac{1}{n}\sum_{j=1}^{n} e_{j+1} \to Te \text{ as } n \to \infty,$$

$$\begin{aligned} |Te - e| &= \lim_{n\to\infty} \left| T\left(\frac{1}{n}\sum_{j=1}^{n} e_j\right) - \frac{1}{n}\sum_{j=1}^{n} e_j \right| \\ &= \lim_{n\to\infty} \frac{1}{n}|e_{n+1} - e_1| \\ &\le \lim_{n\to\infty} \frac{2}{n}|e_1| \\ &= 0 \end{aligned}$$

Then $Te = e$. This completes our proof. □

PROPOSITION 7. *Let (x_n) be a good sequence in X with its spreading model $[F, (e_n)]$ and $x \in X$. Then*

(1) *$(x_n - x)$ is also a good sequence.*
(2) *If (f_n) and (e_n) are not weakly null where $[G, (f_n)]$ is a spreading model of $(x_n - x)$, then (f_n) and (e_n) are equivalent.*
(3) *If $F_\infty \neq \{0\}$, $G_\infty \neq \{0\}$ where $[G, (f_n)]$ is a spreading model of $(x_n - x)$, then (f_n) and (e_n) are equivalent.*

Proof. (1) is obvious.

(2) Let $[F, (e_n)]$ and $[G, (f_n)]$ be the spreading models of (x_n) and $(x_n - x)$, respectively. We first show that there exists a $M > 0$ such that

$$\left|\sum_{i=1}^{k} a_i e_i\right| \ge M \left|\sum_{i=1}^{k} a_i\right|$$

for all $k \in \mathbb{N}$ and $(a_i)_{i=1}^k \in S_0$. The following result is followed by Rosenthal l_1 theorem and the spreading model property.

Since (e_n) does not weakly converge to zero, it satisfies one of the following properties :

(1) (e_n) weakly converges to e, $e \neq 0$.

(2) (e_n) is a nontrivial weak cauchy sequence.

(3) (e_n) is equivalent to the unit vector basis of l_1.

First, suppose that (e_n) weakly converges to e, $e \neq 0$. Then for all $k \in \mathbb{N}$ and $n_1 < \cdots < n_k$,

$$\begin{aligned}\left|\sum_{i=1}^{k} a_i e_i\right| &= \left|\sum_{i=1}^{k} a_i e_{n_i}\right| \\ &= \lim_{n_k \to \infty} \left|\sum_{i=1}^{k} a_i e_{n_i}\right| \\ &\geq \left|\sum_{i=1}^{k-1} a_i e_{n_i} + a_k e\right|, \\ &\vdots \\ \text{(i)} \qquad &\geq \left|\sum_{i=1}^{k} a_i\right| |e|, \text{ because } (e_n) \text{ weakly converges to } e.\end{aligned}$$

Now, suppose that (e_n) is non-trivial weak cauchy. Let e'' be the w^* -limit of (e_n). Then we get the following inequality and the proof is similar to the first with e'' instead of e.

$$\text{(ii)} \qquad \left|\sum_{i=1}^{k} a_i e_i\right| \geq \left|\sum_{i=1}^{k} a_i\right| |e''|.$$

Finally, suppose that (e_n) is equivalent to the unit vector basis of l_1. Then there exists a $N > 0$ such that

$$\text{(iii)} \qquad \left|\sum_{i=1}^{k} a_i e_i\right| \geq N \sum_{i=1}^{k} |a_i| \geq N \left|\sum_{i=1}^{k} a_i\right|.$$

By (i), (ii) and (iii), there exists $M > 0$ such that

$$\left|\sum_{i=1}^{k} a_i e_i\right| \geq M \left|\sum_{i=1}^{k} a_i\right|$$

and

$$\begin{aligned}\left|\sum_{i=1}^{k} a_i f_i\right| &= \lim_{\substack{n_1<\cdots<n_k \\ n_1\to\infty}} \left\|\sum_{i=1}^{k} a_i(x_{n_i}-x)\right\| \\ &\le \lim_{\substack{n_1<\cdots<n_k \\ n_1\to\infty}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| + \left|\sum_{i=1}^{k} a_i\right| \|x\| \\ &\le \left|\sum_{i=1}^{k} a_i e_i\right| + \frac{\|x\|}{M}\left|\sum_{i=1}^{k} a_i e_i\right| \\ &= \left(1+\frac{\|x\|}{M}\right)\left|\sum_{i=1}^{k} a_i e_i\right|.\end{aligned}$$

In a similar way, we can get a constant $M' > 0$ such that

$$\left|\sum_{i=1}^{k} a_i f_i\right| \ge M' \left|\sum_{i=1}^{k} a_i\right|$$

and

$$\left|\sum_{i=1}^{k} a_i e_i\right| \le \left(1+\frac{\|x\|}{M'}\right)\left|\sum_{i=1}^{k} a_i f_i\right|.$$

Then

$$\left(1+\frac{\|x\|}{M'}\right)^{-1}\left|\sum_{i=1}^{k} a_i e_i\right| \le \left|\sum_{i=1}^{k} a_i f_i\right| \le \left(1+\frac{\|x\|}{M}\right)\left|\sum_{i=1}^{k} a_i e_i\right|.$$

(3) Suppose that $F_\infty \neq \{0\}$, $G_\infty \neq \{0\}$. We note that if $F_\infty \neq \{0\}$, then

$$f : F \to \mathbb{R} \text{ by } f\left(\sum_{i=1}^{k} a_i e_i\right) = \sum_{i=1}^{k} a_i$$

is continuous (cf. [2, Proposition 2]). Then there exists $M > 0$ such that

$$\left|\sum_{i=1}^{k} a_i e_i\right| \ge M \left|\sum_{i=1}^{k} a_i\right| \quad \text{for} \quad \sum_{i=1}^{k} a_i e_i \in F.$$

$$\begin{aligned}\left|\sum_{i=1}^{k} a_i f_i\right| &= \lim_{\substack{n_1<\cdots<n_k\\ n_1\to\infty}} \left\|\sum_{i=1}^{k} a_i(x_{n_i}-x)\right\| \\ &\leq \lim_{\substack{n_1<\cdots<n_k\\ n_1\to\infty}} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| + \left|\sum_{i=1}^{k} a_i\right| \|x\| \\ &\leq \left|\sum_{i=1}^{k} a_i e_i\right| + \frac{\|x\|}{M}\left|\sum_{i=1}^{k} a_i e_i\right| \\ &= \left(1+\frac{\|x\|}{M}\right)\left|\sum_{i=1}^{k} a_i e_i\right|.\end{aligned}$$

In a similar way, we can get a constant $M' > 0$ such that

$$\left|\sum_{i=1}^{k} a_i f_i\right| \geq M' \left|\sum_{i=1}^{k} a_i\right|$$

and

$$\left|\sum_{i=1}^{k} a_i e_i\right| \leq \left(1+\frac{\|x\|}{M'}\right)\left|\sum_{i=1}^{k} a_i f_i\right|.$$

Then

$$\left(1+\frac{\|x\|}{M'}\right)^{-1}\left|\sum_{i=1}^{k} a_i e_i\right| \leq \left|\sum_{i=1}^{k} a_i f_i\right| \leq \left(1+\frac{\|x\|}{M}\right)\left|\sum_{i=1}^{k} a_i e_i\right|.$$

This completes our proof. □

The following is the main theorem of this paper.

THEOREM 2. *Let X be a Banach space. Then the following are equivalent:*

(1) *X has the Banach-Saks property.*
(2) *For a good sequence (x_n) in X with its spreading model $[F, (e_n)]$, $F_\infty \neq \{0\}$ or $\rho = 0$.*

REMARK. The two conditions of (2) of Theorem 2 are mutually exclusive. Let $F_\infty \neq \{0\}$. We show that $\rho \neq 0$. We note that if $F_\infty \neq \{0\}$, then $f : F \to \mathbb{R}$ by $f\left(\sum_{i=1}^{k} a_i e_i\right) = \sum_{i=1}^{k} a_i$ is continuous (cf. Proposition 2, [2]). Then there exists $M > 0$ such that

$$\left|\sum_{i=1}^{k} a_i\right| \leq M \left|\sum_{i=1}^{k} a_i e_i\right|$$

for $\sum_{i=1}^{k} a_i e_i \in F$. Considering $a_k = \frac{1}{k}$, $\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| \geq \frac{1}{M} > 0$. This implies that $\rho = \lim_{k\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| \geq \frac{1}{M} > 0$.

Proof of Theorem 2. Assume that X have the Banach-Saks property. Let (x_n) be a good sequence in X with its spreading model $[F, (e_n)]$. Since X has the Banach-Saks property, there exists a subsequence (x'_n) of (x_n) such that for all subsequence (x'_{n_i}) of (x'_n),

$$\frac{1}{k}\sum_{i=1}^{k} x'_{n_i} \to x \in X$$

by Lemma 1.

Suppose that $x = 0$. Then by Lemma 2, there exists a subsequence (x'_{n_i}) of (x'_n) such that for all $k \in \mathbb{N}$,

$$\frac{1}{2}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| - (2\log_2 k)\frac{1}{k}\sup_n \|x'_n\| \leq \left\|\frac{1}{k}\sum_{i=1}^{k} x'_{n_i}\right\|.$$

Since $\frac{1}{k}\sum_{i=1}^{k} x_{n_i} \to 0$ and $\frac{1}{k}(2\log_2 k) \to 0$ as $k \to \infty$, $\rho = 0$.

Suppose that $x \neq 0$. By Proposition 7, $(x'_n - x)$ is also a good sequence. Let $[G, (f_i)]$ be a spreading model of $(x'_n - x)$. By Lemma 2, there exists a subsequence $(x'_{n_i} - x)$ such that for all $k \in \mathbb{N}$,

$$\frac{1}{2}\left|\frac{1}{k}\sum_{i=1}^{k} f_i\right| - (2\log_2 k)\frac{1}{k}\sup_n \|x'_n - x\| \leq \left\|\frac{1}{k}\sum_{i=1}^{k}(x'_{n_i} - x)\right\|$$

Since $\frac{1}{k}\sum_{i=1}^{k}(x'_{n_i} - x) \to 0$ and $(2\log_2 k)\frac{1}{k} \to 0$ as $k \to \infty$, $\frac{1}{k}\sum_{i=1}^{k} f_i \to 0$ as $k \to \infty$. We first show that $\frac{1}{k}\sum_{i=1}^{k} e_i$ converges to a non-zero element in F. Since

$$\begin{aligned}\left|\frac{1}{k_1}\sum_{i=1}^{k_1} e_i - \frac{1}{k_2}\sum_{i=1}^{k_2} e_i\right| &= \lim_{\substack{n_1\to\infty \\ n_1<\cdots<n_{k_2}}} \left\|\frac{1}{k_1}\sum_{i=1}^{k_1} x_{n_i} - \frac{1}{k_2}\sum_{i=1}^{k_2} x_{n_i}\right\| \\ &= \lim_{\substack{n_1\to\infty \\ n_1<\cdots<n_{k_2}}} \left\|\frac{1}{k_1}\sum_{i=1}^{k_1}(x_{n_i} - x) - \frac{1}{k_2}\sum_{i=1}^{k_2}(x_{n_i} - x)\right\| \\ &= \left|\frac{1}{k_1}\sum_{i=1}^{k_1} f_i - \frac{1}{k_2}\sum_{i=1}^{k_2} f_i\right| \quad \text{for } k_1 \le k_2\end{aligned}$$

and $\frac{1}{k}\sum_{i=1}^{k} f_i$ converges to zero, $\frac{1}{k}\sum_{i=1}^{k} e_i$ converges in F. Let $\frac{1}{k}\sum_{i=1}^{k} e_i \to \widetilde{e}$ in F. Suppose that $\widetilde{e} = 0$. Then by Lemma 2, there exists a subsequence (x'_{n_i}) of (x'_n) such that for all $k \in \mathbb{N}$

$$\left\|\frac{1}{k}\sum_{i=1}^{k} x'_{n_i}\right\| \le \frac{3}{2}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| + (3\log_2 k)\frac{1}{k}\sup_n \|x'_n\|.$$

Since $\frac{1}{k}\sum_{i=1}^{k} e_i \to 0$ and $(3\log_2 k)\frac{1}{k} \to 0$ as $k \to \infty$, $\frac{1}{k}\sum_{i=1}^{k} x'_{n_i} \to 0$. We get the contradiction, since $\frac{1}{k}\sum_{i=1}^{k} x'_{n_i} \to x \neq 0$. Thus $\frac{1}{k}\sum_{i=1}^{k} e_i$ converges to a non-zero element of $\widetilde{e} \in F$. By (2) of Lemma 6, $\widetilde{e} \in F_\infty$ and so $F_\infty \neq \{0\}$.

Assume that (2) holds. Let (y_n) be a bounded sequence in X. We show that (y_n) has a subsequence whose arithmetic means converge in X. If (y_n) has a convergent subsequence, the subsequence has a convergent arithmetic means. So we may assume that (y_n) has no convergent subsequence. Then by Lemma 3, there exists a subsequence (x_n) of (y_n) such that (x_n) is a good sequence in X with its spreading model $[F, (e_n)]$. Suppose that $F_\infty \neq \{0\}$. Then by (1) of Lemma 6, $\frac{1}{n}\sum_{i=1}^{n} e_i$ converges in F. By (4) of Lemma 5, (x_n) has a subsequence whose arithmetic means converges in norm $\|\cdot\|$.

Suppose that $\rho = 0$. Then there exists a subsequence (x'_n) of (x_n) such that

$$\left\|\frac{1}{k}\sum_{i=1}^{k} x'_i\right\| \le 2\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| + (3\log_2 k)\frac{1}{k}\sup_n \|x'_n\|.$$

Since $\rho = \lim_{k\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| = 0$ and $\lim_{k\to\infty}(3\log_2 k)\frac{1}{k} = 0$, $\frac{1}{k}\sum_{i=1}^{k} x'_i \to 0$ as $k \to \infty$. This completes our proof. □

3. The Schlumprecht space and spreading model

In [9], Th. Schlumprecht introduced the Schlumprecht space S and showed that S is arbitrarily distortable. In [3], the author showed that the Schlumprecht space S is reflexive and not uniformly convex. It is natural to ask the following question.

QUESTION. *Does the Schlumprecht space S or its dual space S^* have the Banach-Saks property?*

In [3], the author proved that the dual space S^* has the Banach-Saks property by the direct computation. In this paper, we show that the Schlumprecht space S does not have the Banach-Saks property using the spreading model properties.

We need the following lemma and corollary.

LEMMA 8. *Let (x_n) be a good sequence in X with its spreading model $[F, (e_n)]$. If (e_n) is a basic sequence in F, then $F_\infty = \{0\}$.*

Proof. Let $x = \sum_{i=1}^\infty a_i e_i \in F_\infty$. Then by (2) of Lemma 5,

$$\sum_{i=1}^{\infty} a_i e_i = x = Tx = \sum_{i=1}^{\infty} a_i e_{i+1}.$$

This implies that $a_2 = a_1 = 0$. Continuing this process, we get that $a_n = 0$ for all n and so $x = 0$. This completes our proof. □

COROLLARY 9. *Let X be a Banach space. If X has a spreading model isomorphic to l_1, then X does not have the Banach-Saks property.*

Proof. Suppose that X has a spreading model $[F, (e_n)]$ isomorphic to l_1. Since X has a spreading model isomorphic to l_1 if and only if it has a spreading model whose fundamental sequence is equivalent to the unit vector basis (x_n) of l_1, we may assume that (e_n) is a basic sequence and there exist $M,\ m > 0$ such that

$$m \left\| \sum_{n=1}^{\infty} a_n x_n \right\|_{l_1} \leq \left| \sum_{n=1}^{\infty} a_n e_n \right| \leq M \left\| \sum_{n=1}^{\infty} a_n x_n \right\|_{l_1}.$$

It suffices to show that $F_\infty = \{0\}$ and $\rho = \lim_{k\to\infty} \left| \frac{1}{k} \sum_{i=1}^k e_i \right| > 0$ by Theorem 2. Since (e_n) is a basic sequence in F, $F_\infty = \{0\}$ by Lemma 8.

Since

$$\rho = \lim_{k\to\infty}\left|\frac{1}{k}\sum_{i=1}^{k} e_i\right| \geq m \lim_{k\to\infty}\left\|\frac{1}{k}\sum_{i=1}^{k} x_n\right\|_{l_1} = m > 0,$$

$\rho > 0$. We complete the proof. □

In [6], D. Kutzarova and P. K. Lin proved that the Schlumprecht space S has an l_1-spreading model. By Corollary 9, [3] and [6], we get the following.

COROLLARY 10. *The Schlumprecht space S does not have the Banach-Saks property and its dual space S^* has the Banach-Saks property.*

In [8], N. Okada and T. Ito proved that if X is a Banach space of type p for some $1 < p \leq 2$, then X has the weak Banach-Saks property. We note that the Banach-Saks property and the weak Banach-Saks property are equivalent in a reflexive space. Then by Corollary 10 and [8], the Schlumprecht space X is not of type p, $1 < p \leq 2$.

If Y is a Banach space with basis (y_i) and if $1 \leq p \leq \infty$ we say that l_p is finitely block represented in Y if for any $\epsilon > 0$ and $n \in \mathbb{N}$ there is a normalized block $(z_i)_{i=1}^n$ of length n of (y_i) which is $(1+\epsilon)$-equivalent to the unit basis of l_p^n and we call (z_i) a block of (y_i) if $z_i = \sum_{j=k_{i-1}+1} \alpha_j y_j$ for $i = 1, 2, ...$ and some $0 = k_0 < k_1 < \cdots$ in $\mathbb{N}_0$ and $(\alpha_j) \subset \mathbb{R}$. In [9], Th. Schlumprecht proved that l_1 is finitely block represented in each infinite block of (e_i), where (e_i) is the unit vector basis of S.

In this paper, our concern is the finitely representability of the Schlumprecht space S. We say that a Banach space X contains $l_1^{(n)}$ uniformly if there exists $\delta > 0$ such that for every $n \geq 1$, we can find $x_1, \cdots, x_n$ in X with $\|x_i\| = 1$ and $\|\sum_{i=1}^n a_i x_i\| \geq \delta \sum_{i=1}^n |a_i|$, for all scalars $a_1, \cdots, a_n$. G. Pisier showed that A Banach space X does not contain $l_1^{(n)}$ uniformly if and only if X has type p, for some $p > 1$ (Theorem 1, p.313 of [1]).

Since the Schlumprecht space is not of type p, $p > 1$, we get the following.

COROLLARY 11. *The Schlumprecht space S contains $l_1^{(n)}$ uniformly.*

Since if a Banach space contains $l_1^{(n)}$ uniformly, then l_1 is finitely representable in it, we get the following.

COROLLARY 12. *l_1 is finitely representable in the Schlumprecht space S.*

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