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NILRADICALS OF POWER SERIES RINGS AND NIL POWER SERIES RINGS

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Abstract. Klein proved that polynomial rings over nil rings of bounded index are also nil of bounded index; while Puczyłowski and Smoktunowicz described the nilradical of a power series ring with an indeterminate. We extend these results to those with any set of commuting indeterminates. We also study prime radicals of power series rings over some class of rings containing the case of bounded index, finding some examples which elaborate our arguments; and we prove that R is a PI ring of bounded index then the power series ring $R[[X]]$, with X any set of indeterminates over R, is also a PI ring of bounded index, obtaining the Klein's result for polynomial rings as a corollary.

1. Introduction

Throughout this note every ring is associative but not necessarily with identity. Given a ring R we use the following notations: $N_*(R)$ is the prime radical of R and $N^*(R)$ is the nilradical (i.e., the sum of nil ideals) of R; $N_1(R)$ is the Wedderburn radical (i.e., the sum of nilpotent ideals) of R and $N_2(R)$ is the ideal of R with $\frac{N_2(R)}{N_1(R)} = N_1(\frac{R}{N_1(R)})$.

Given a ring R the following statements are obtained from definitions and [5, Lemma 5]: (i) $N_1(R) \subseteq N_2(R) \subseteq N_*(R) \subseteq N^*(R)$. (ii) For $r \in R$, we have that $r \in N_1(R)$ if and only if $(rR)^m = 0$, and that

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$r \in N_2(R)$ if and only if $(rR)^n \subseteq N_1(R)$, where m, n are some positive integers.

A ring R is said to be *of bounded index* (of nilpotency) if there is a positive integer n such that $a^n = 0$ whenever $a \in R$ is nilpotent. The least such integer is called the *index* of R.

LEMMA 1.1. [6, Theorem 2] *Let R be a ring and $a \in R$. If aR is nil of bounded index then $a \in N_2(R)$.*

Given a ring R we use $N_b(R)$ to denote the sum of nil one-sided ideals of bounded index in R.

COROLLARY 1.2. *If a ring R is of bounded index then*

$$N_b(R) = N_2(R) = N_*(R) = N^*(R).$$

Proof. Since R is of bounded index, Lemma 1.1 implies $N_b(R) \subseteq N_2(R)$. Thus we get the corollary. □

Throughout this note X denotes a nonempty set of indeterminates over rings. Let R be a ring. The power series ring over R with X is denoted by $R[[X]]$ when X is commuting, and is denoted by $R\{X\}$ when X is noncommuting; if X is singleton, say $X = \{x\}$, then we write $R[[x]]$ in place of $R[[\{x\}]]$. Puczyłowski and Smoktunowicz proved that $N^*(R\{X\}) = N_1(R\{X\})$ for the case of $|X| \geq 2$ and $N^*(R[[x]]) = N_2(R[[x]])$ [8, Corollaries 7 and 10]. Therefore it is natural to conjecture that $N^*(R[[X]]) = N_2(R[[X]])$ for any set X of commuting indeterminates with $|X| \geq 2$. We prove that this is also true, using similar methods to those in [8]; consequently we may extend [8, Corollary 10] to the general case of $|X| \geq 1$. We also study prime radicals of power series rings over some class of rings containing the case of bounded index.

Given a ring R and a nonempty set X, denote the ring R joined with identity and the free abelian monoid on X by R^1 and $F(X)$, respectively. Note that every element in $R[[X]]$ is of the form $\sum_{w \in F(X)} r_w w$ with $r_w \in R$. We use $R[X]$ to denote the polynomial ring over R with X a set of indeterminates; if X is singleton, say $X = \{x\}$, then we write $R[x]$ in place of $R[\{x\}]$. Then each polynomial in $R[X]$ is of the form $r_1w_1 + r_2w_2 + \cdots + r_nw_n$ with $r_i \in R$ and $w_i \in F(X)$. As usual we use $\deg w_i$ to denote the degree of w_i, and define the *degree* of $f(X) \in R[X]$ (write $\deg f(X)$) by the maximal number in $\{\deg w_1, \deg w_2, \ldots, \deg w_n\}$ and write $\min f(X)$ for the minimal one in $\{\deg w_1, \deg w_2, \ldots, \deg w_n\}$. We start with the following.

LEMMA 1.3. *Let R be a ring, m be a positive integer, and $a_0(X)$, $a_1(X), \dots, a_n(X)$ be power series in $R[[X]]$. If x is an elememt in X satisfying $a_0(X) + a_1(X)x^k + a_2(X)x^{2k} + \cdots + a_n(X)x^{nk} = 0$ for all $k \geq m$ then $a_0(X) = a_1(X) = \cdots = a_n(X) = 0$.*

Proof. We proceed by induction on $n \geq 0$. If $n = 0$ then $a_0(X) = 0$ clearly, so we assume $n \geq 1$. By the condition we also have $a_0(X) + a_1(X)x^{2k} + a_2(X)x^{4k} + \cdots + a_n(X)x^{2nk} = 0$ for all $k \geq m$. Hence we get $b_1(X) + b_2(X)x^k + \cdots + b_n(X)x^{(n-1)k} = 0$ for all $k \geq m$ with $b_i(X) = a_i(X)(1 - x^{ik})x^k$ for $i = 1, 2, \dots, n$, after substracting this from the original one. By induction hypothesis $b_1(X) = b_2(X) = \cdots = b_n(X) = 0$. Note that $1 - x^{ik}$ is invertible and x^k is not a zero-divisor in $R^1[[X]]$; so we have $a_1(X) = a_2(X) = \cdots = a_n(X) = 0$, proving the lemma. $\square$

The following two results (i.e., Lemma 1.4 and Theorem 1.5) are extensions of Lemma 8 and Theorem 9 in [8] to the case with any set of commuting indeterminates, respectively. Let x be a fixed indeterminate in X.

LEMMA 1.4. *Let R be a ring and $a(X)$ be a power series in $R[[X]]$ such that $a(X)R[[X]]$ is nil. Then there exist an integer $n \geq 1$ and a polynomial $f(X) \in R[X]$ such that for each $g(X) \in R[X]$ there is $b(X) \in R[[X]]$ with $\{a(X)(f(X) + g(X)x^{\deg f(X)+1} + b(X)x^{\deg f(X)+\deg g(X)})\}^n = 0$.*

Proof. We apply the proof of [8, Lemma 8]. Assume that the result does not hold. Then for any integer $n \geq 1$ there is a polynomial $g_n(X) \in R[X]$ such that

$$\{a(X)(f(X) + g_n(X)x^{\deg f(X)+1} + b(X)x^{\deg f(X)+\deg g_n(X)})\}^n \neq 0$$

for all $f(X) \in R[X]$ and $b(X) \in R[[X]]$. Take $f_1(X) = \alpha x$ with $0 \neq \alpha \in R$, $f_{n+1}(X) = f_n(X) + g_n(X)x^{\deg f_n(X)+1}$ inductively, and $b(X) = 0$. Then we have $\{a(X)(f_n(X) + g_n(X)x^{\deg f_n(X)+1})\}^n \neq 0$ for all $n \geq 1$.

If $g_{n+1}(X) \neq 0$ then we have

$$\begin{aligned} \min\,(g_{n+1}(X)x^{\deg f_{n+1}(X)+1}) &> \deg f_{n+1}(X) \\ &\geq \deg\,(g_n(X)x^{\deg f_n(X)+1}); \end{aligned}$$

hence it is obvious that

$$\begin{aligned} \alpha x + g_1(X)x^{\deg f_1(X)+1} &+ g_2(X)x^{\deg f_2(X)+1} + \cdots \\ &= \alpha x + \sum_{k=1}^{\infty} g_k(X)x^{\deg f_k(X)+1} \end{aligned}$$

is a power series in $R[[X]]$ with ascending degrees, say $c(X)$. Consequently we have $(a(X)c(X))^n \neq 0$ for all $n \geq 1$ since $(a(X)f_{n+1}(X))^n \neq 0$, a contradiction to the hypothesis. $\square$

THEOREM 1.5. Let R be a ring and $a(X) \in R[[X]]$. If $a(X)R[[X]]$ is nil then $a(X)R[[X]]$ is nil of bounded index.

Proof. Let x be an indeterminate in X and suppose that the condition holds. By Lemma 1.4 there exist an integer $n \geq 1$ and a polynomial $f(X) \in R[X]$ such that for each $g(X) \in R[X]$ there is $b(X) \in R[[X]]$ with $\{a(X)(f(X) + g(X)x^{\deg f(X)+1} + b(X)x^{\deg f(X)+\deg g(X)})\}^n = 0$. We will show that $(a(X)g(X))^n = 0$ for every $g(X) \in R[[X]]$. To do this, we apply the proof of [8, Theorem 9]. First we take arbitrarily $0 \neq r \in R$, a positive integer m and a polynomial $h(X) \in R[X]$. Then we also have $\{a(X)(f(X) + (h(X) + rx^m)x^{p+1} + b(X)x^{p+q})\}^n = 0$ with $p = \deg f(X)$ and $q = \deg\,(h(X) + rx^m) \geq m$, substituting $h(X) + rx^m$ in place of $g(X)$. Note $\{a(X)(f(X) + h(X)x^{p+1})\}^n \in x^m R[[X]]$. Here since m is arbitrary, we must have that $\{a(X)(f(X) + h(X)x^{p+1})\}^n = 0$ for all $h(X) \in R[[X]]$. Substituting $h(X)$ by $g(X)x^k$ in the preceding equation, we get $\{a(X)(f(X) + g(X)x^{k+p+1})\}^n = 0$, where k is any positive integer; hence $(a(X)f(X))^n + a_1(X)x^{k+p+1} + a_2(X)x^{2(k+p+1)} + \cdots + a_{n-1}(X)x^{(n-1)(k+p+1)} + (a(X)g(X))^n x^{n(k+p+1)} = 0$ for some $a_i(X)$'s in $R[[X]]$. Since k is arbitrary, it follows that $(a(X)g(X))^n = 0$ by Lemma 1.3, proving that $(a(X)s(X))^n = 0$ for all $s(X) \in R[[X]]$. $\square$

By Lemma 1.1 and Theorem 1.5 we obtain the following.

COROLLARY 1.6. *Given a ring* R,

$$N_2(R[[X]]) = N_*(R[[X]]) = N^*(R[[X]]).$$

2. Prime radicals of power series rings

Klein proved that if R is a nil ring of index n then the polynomial ring over R in one indeterminate is nil of index $\leq n!$ [5, Theorem 9]. This theorem can be extended easily to the polynomial ring with any set of commuting indeterminates. In this chapter we improve this result to the power series ring case. We use $\otimes$ to denote the tensor product. First we obtain the following by [5, Proposition 4 and Lemma 8].

LEMMA 2.1. *Let R be a nil ring of index n and I be the ideal of R generated by the set $\{a^{n-1} \mid a \in R\}$. Then $I \otimes C$ is a nil ring of index n for any commutative ring C.*

We use $\mathbb{Z}$ to denote the ring of integers.

COROLLARY 2.2.*Let R be a nil ring of index n and I be the ideal of R generated by the set $\{a^{n-1} \mid a \in R\}$. Then $I[X]$ is a nil ring of index n.*

Proof. Since $I[X] \cong I \otimes \mathbb{Z}[X]$, $I[X]$ is nil of index n by Lemma 2.1.□

We obtain the following by the same way.

LEMMA 2.3. *Let R be a nil ring of index n and I be the ideal of R generated by the set $\{a^{n-1} \mid a \in R\}$. Then $I[[X]]$ is a nil ring of index n.*

Given a nonempty set X recall that we denote the free abelian monoid on X by $F(X)$. In the following we can extend [5, Theorem 9] for polynomials to the power series ring case.

THEOREM 2.4.*Let R be a nil ring of index $n \geq 2$. Then $R[[X]]$ is a nil ring of index $\leq n!$.*

Proof. We proceed by induction on n. If $n = 2$ then $ab + ba = 0$ for all $a, b \in R$; hence $f(X)^2 = 0$ for all $f(X) = \sum_{w \in F(X)} a_w w$ with $a_w \in R$ in $R[[X]]$. In fact, for any word $w \in F(X)$ the coefficient of w, say b_w, in the power series $f(X)^2$ is the sum of the forms $a_{w_i}a_{w_j} + a_{w_j}a_{w_i}$ or $a_{w_k}^2$ (if any) with $w_iw_j = w = w_k^2$. Since R is nil of index 2, $b_w = 0$ for all $w \in F(X)$ showing $f(X)^2 = 0$. This gives that $R[[X]]$ is of index $2 = 2!$. Now assume $n > 2$ and apply the proof of [5, Theorem 9]. Let I be the ideal of R generated by the set $\{a^{n-1} \mid a \in R\}$ and $\bar{R} = R/I$. Then $\bar{R}$ is nil of index $n - 1$, so $\bar{R}[[X]]$ is nil of index $\leq (n-1)!$ by the induction hypothesis. Since $\bar{R}[[X]] = \dfrac{R}{I}[[X]] \cong \dfrac{R[[X]]}{I[[X]]}$, we have $f(X)^{(n-1)!} \in I[[X]]$ for all $f(X) \in R[[X]]$. Now Lemma 2.3 implies $0 = (f(X)^{(n-1)!})^n = f(X)^{n!}$. □

Puczyłowski[7] proved that if $R[[x]]$ with $x \in X$ is a nil ring then R is nil of bounded index. Immediately we have that if $R[[X]]$ is a nil ring then R is nil of bounded index. So we may obtain the following with help of Theorem 2.4.

THEOREM 2.5. *For a ring R and $x \in X$ the following conditions are equivalent:*

(1) *R is nil of bounded index;*
(2) *$R[[x]]$ is nil of bounded index;*
(3) *$R[[x]]$ is nil;*
(4) *$R[[X]]$ is nil of bounded index;*
(5) *$R[[X]]$ is nil.*

It is well-known that $N_*(R[[X]]) \subseteq N_*(R)[[X]]$ [2, Corollary 1.2], so we also obtain the following with the help of [5, Lemma 5].

COROLLARY 2.6. *Given a ring R the following conditions are equivalent:*

(1) *$N_*(R)$ is of bounded index;*
(2) *$N_*(R)[[X]]$ is nil of bounded index;*
(3) *$N_*(R)[[X]]$ is nil;*
(4) *$N_*(R)[[X]] = N_*(R[[X]])$.*

REMARK. Corollary 2.6 need not hold for rings whose indices are not bounded as can be seen by the following example. Let F be a field and let V be an infinite dimensional left vector space over F with $\{v_1, v_2, \dots\}$ a basis. According to [3, Example 1.1], define $A_1 = \{f \in A \mid \operatorname{rank}(f) < \infty \text{ and } f(v_i) = a_1v_1 + \cdots + a_iv_i \text{ for } i = 1, 2, \dots \text{ with } a_j \in F\}$ and let R be the F-subalgebra of A generated by A_1 and 1_A, where $A = End_F(V)$ is the endomorphism ring of V over F. Then $\dfrac{R}{N_*(R)} \cong \{(a_1, a_2, \dots, a_n, b, b, \dots) \mid a_i, b \in F \text{ and } n = 1, 2, \dots\} \subset \prod_{i=1}^{\infty} F_i$ by [3, Example 1.1], where $F_i = F$ for all i. Let e_{ij} be the infinite matrix over F with (i, j)-entry 1 and elsewhere 0. The following argument is due to [3, Example 1.1]. Take power series

$$f(x) = e_{12} + e_{34}x + \cdots + e_{(2n+1)(2n+2)}x^n + \cdots$$

and

$$g(x) = e_{23} + e_{45}x + \cdots + e_{(2n+2)(2n+3)}x^n + \cdots$$

in $N_*(R)[[x]]$. Then $f(x)^2 = 0$ and $g(x)^2 = 0$; however the coefficients of $(f(x) + g(x))^k$ are

$$e_{1(k+1)}, e_{2(k+2)}, \dots, e_{n(k+n)}, \dots \text{ for } k = 2, 3, \dots,$$

and so $f(x)+g(x)$ is not nilpotent and $f(x)+g(x) \notin N_*(R[[X]])$. Consequently $f(x) \notin N_*(R[[x]])$ or $g(x) \notin N_*(R[[x]])$, and thus we have $N_*(R[[x]]) \subsetneqq N_*(R)[[x]]$. $\square$

By the preceding remark the condition "of bounded index" in Corollary 2.6 is not superfluous.

COROLLARY 2.7. *Let R be a ring and I be a one-sided ideal of R. Then I is nil of bounded index if and only if $I[[X]] \subseteq N_*(R[[X]])$.*

Proof. (Necessity) If I is nil of bounded index, then so is $I[[X]]$ by Theorem 2.4; hence $I[[X]]$ is contained in $N_*(R[[X]])$ by Lemma 1.1 and Corollary 1.6. (Sufficiency) If $I[[X]] \subseteq N_*(R[[X]])$ then $I[[X]]$ is clearly nil and so I is nil of bounded index by Theorem 2.5. $\square$

We next consider other useful conditions under which $N_*(R)[[X]] = N_*(R[[X]])$ holds.

LEMMA 2.8 [4, A Theorem of Nagata-Higman, Appendix C].*Let R be a nil algebra of index n over a field of characteristic zero or a prime $p > n$. Then R is nilpotent with $R^{2^n-1} = 0$.*

By Corollary 2.6 and Lemma 2.8 we have the following.

COROLLARY 2.9. *Let R be an algebra over a field K of characteristic zero. Then the following conditions are equivalent:*
(1) *$N_*(R)$ is of bounded index;*
(2) *$N_*(R)$ is nilpotent;*
(3) *$N_*(R)[[X]]$ is nilpotent;*
(4) *$N_*(R)[[X]] = N_*(R[[X]])$;*
(5) *$N_*(R)[[X]]$ is nil.*

Though easy to prove, the following result contains some useful relations between prime radicals of rings and those of their power series rings, comparing with Corollary 1.6 (i.e., $N_*(R)[[X]]) = N^*(R[[X]]) = N_2(R[[X]])$ for any ring R).

PROPOSITION 2.10. *Let R be a ring.*
(1) *If $N_*(R)[[X]] = N_1(R[[X]])$ then $N_*(R) = N_1(R)$.*
(2) *If $N_*(R) = N_1(R)$ then $N_*(R) \subseteq N_*(R[[X]])$.*
(3) *If $N_*(R) \subseteq N_*(R[[X]])$ then $N_*(R) = N_2(R)$.*

Proof. (1) If $N_*(R)[[X]] = N_1(R[[X]])$ and $a \in N_*(R)$, then $aR[[X]]$ is nilpotent and so is aR, implying $a \in N_1(R)$.
(2) The proof follows from the fact $N_1(R) \subseteq N_*(R[[X]])$.

(3) Assume that $N_*(R) \subseteq N_*(R[[X]])$, and let $a \in N_*(R)$. Since $N_*(R[[X]]) = N_2(R[[X]])$ by Corollary 1.6, $(aR[[X]])^n \subseteq N_1(R[[X]])$ for some positive integer n. Thus for each $b \in (aR)^n$, $bR[[X]]$ is nilpotent and so is bR; hence $b \in N_1(R)$. Consequently we have $a \in N_2(R)$, obtaining $N_*(R) = N_2(R)$. □

Every converse of Proposition 2.10 is not true in general by the examples in the next section. But the converses of (2) and (3) may hold under some conditions as follows.

Proposition 2.11. *Let R be an algebra over a field K of characteristic zero. Then $N_*(R) = N_1(R)$ if and only if $N_*(R) \subseteq N_*(R[[X]])$.*

Proof. It suffices to show the Sufficiency by Proposition 2.10(2). If $a \in N_*(R)$ and $N_*(R) \subseteq N_*(R[[X]])$, then $aR[[X]]$ is nil subring of bounded index in $R[[X]]$ by Theorem 1.5; hence $aR[[X]]$ is nilpotent by Lemma 2.8 and so is aR, implying $a \in N_1(R)$. □

Proposition 2.12.*Given a ring R suppose that $N_2(R)$ is of bounded index. Then the following conditions are equivalent:*

(1) $N_*(R) = N_2(R)$;

(2) $N_*(R)[[X]] = N_*(R[[X]])$;

(3) $N_*(R) \subseteq N_*(R[[X]])$.

Proof. (1)⇒(2) is shown by Corollary 2.7 and [2, Corollary 1.2]; while, (2)⇒(3) is obvious and (3)⇒(1) is obtained by Proposition 2.10(3). □

A ring R is called *PI* (or a ring with a polynomial identity) if there is a polynomial $f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$ with noncommuting indeterminates $x_1, x_2, \ldots, x_n$ such that at least one coefficient of $f(x_1, x_2, \ldots, x_n)$ is 1 or -1 and $f(a_1, a_2, \ldots, a_n) = 0$ for every $a_1, a_2, \ldots, a_n$ in R. The class of PI rings include commutative rings obviously. Klein proved that if R is a PI ring of bounded index then so is the polynomial ring over R [5, Theorem 12]. In the following we extend this result to power series rings with any set of indeterminates.

Theorem 2.13. *If R is a PI ring of bounded index then the power series ring $R[[X]]$ is also a PI ring of bounded index, where X is any set of indeterminates over R.*

Proof. It is well known that $R[[X]]$ is PI if so is R. Suppose that R satisfies a polynomial identity of degree d. Then, by [9, Theorem 6.1.26], there is a commutative ring C which is a direct product of fields such that R/B is isomorphic to a subring of $Mat_n(C)$, where $B = N_b(R)$,

$n = [\frac{d}{2}]$ (i.e., the largest integer $\leq \frac{d}{2}$) and $Mat_n(C)$ is the n by n matrix ring over C. Thus we have

$$\frac{R[[X]]}{B[[X]]} \cong \frac{R}{B}[[X]] \hookrightarrow Mat_n(C)[[X]] \cong Mat_n(C[[X]]).$$

By the Cayley-Hamilton Theorem, $Mat_n(C[[X]])$ is of bounded index; hence so is $\frac{R[[X]]}{B[[X]]}$ by the preceding result. Now since B is a nil ideal of bounded index, it follows from Theorem 2.5 that $B[[X]]$ is nil of bounded index. Therefore $R[[X]]$ is of bounded index. ☐

COROLLARY 2.14. (1) *If R is a PI ring of bounded index then the polynomial ring $R[X]$ is also a PI ring of bounded index, where X is any set of indeterminates over R.*

(2) [5, *Theorem* 12] *If R is a PI ring of bounded index then the polynomial ring $R[x]$ is also a PI ring of bounded index, where x is an indeterminate over R.*

3. Related examples

In this section we find counterexamples which are concerned with the converses of Proposition 2.10.

EXAMPLE 3.1. Let K be any field and $B = \{t_n \mid n = 1, 2, \dots\}$ be a set of noncommuting indeterminates over K. Next set R be the exterior algebra on B over K, that is, R is an algebra over K generated by the elements in B subject to the following relations: $t_i t_j = -t_j t_i$ for all i, j with $i \neq j$, and $t_n^2 = 0$ for all n. Then $N_*(R) = \oplus_{n=1}^{\infty} t_n R$, but this is not nilpotent. We have the following properties for the ring R:

(1) $N_*(R) = N_1(R)$ is a maximal ideal of R with $\frac{R}{N_*(R)} \cong K$.

(2) If the characteristic of K is $p \neq 0$ then $N_*(R)$ is of index p and so Corollary 2.6 implies $N_*(R[[X]]) = N_*(R)[[X]]$.

(3) R is commutative if and only if K is of characteristic 2.

(4) If K is of characteristic zero then $N_*(R)$ is not of bounded index by Lemma 2.8. Thus we have $N_*(R[[X]]) = N_2(R[[X]]) \subsetneqq N_*(R)[[X]]$ by Corollary 2.6 and $N_1(R[[X]]) \subsetneqq N_*(R)[[X]]$.

Therefore for any field of characteristic zero, constructing an exterior algebra over it as above gives a counterexample to the converse of Proposition 2.10(1), by properties (1) and (4).

Next to construct counterexamples for the converses of Proposition 2.10(2, 3), we refer to the example of Amitsur [1]. Let K be a field and R be the exterior algebra on the set B over K as in Example 3.1. Let T be the ring of $\aleph_0$ by $\aleph_0$ matrices of the form

$$\left(\begin{array}{c|cccccc} A & & O & & & & \\ \hline & r & 0 & 0 & & & \\ & 0 & r & 0 & & O & \\ & 0 & 0 & r & & & \\ O & & & & r & & \\ & & O & & & \ddots & \\ & & & & & & \ddots \end{array}\right),$$

where $r \in R$, A is an n by n matrix over R for some positive integer n, and each O is a zero matrix. Denote the identity of T by 1_T and let e_{ij} be the matrix in T such that (i, j)-entry is 1_K and zero elsewhere.

Next let S be the subalgebra of T consisting of all matrices of the form

$$r1_T + \sum_{i>j} r_{ij}e_{ij} + \sum_{i \le j} a_{ij}e_{ij}$$

with $r, r_{ij} \in R$ and $a_{ij} \in N_*(R)$, where each sum is taken finitely. Define an ideal Q of S by

$$\{a1_T + \sum_{i>j} r_{ij}e_{ij} + \sum_{i \le j} a_{ij}e_{ij} \mid a, a_{ij} \in N_*(R) \text{ and } r_{ij} \in R\}$$

and for a given nil ideal I of R define

$$\{b1_T + \sum b_{ij}e_{ij} \mid b, b_{ij} \in I\} \stackrel{\text{let}}{=} I'.$$

LEMMA 3.2. *Let R, S, Q and I' be as above. Then we have the following properties:*

(1) *R can be imbedded in S as scalar matrices, i.e., $R \hookrightarrow S$ with $r \mapsto r1_T$.*

(2) *Q is a maximal ideal of S with $\dfrac{S}{Q} \cong \dfrac{R}{N_*(R)} \cong K$.*

(3) *For any nil ideal I of R, I' is an ideal of S with $I' \subseteq Q$; in particular $N_*(R)' \subseteq Q$. Moreover I is nilpotent if and only if so is I'.*

(4) $N_*(R)' \subseteq N_1(S)$.

(5) $Q = N_2(S) = N_*(S)$.

(6) $N_1(S) \subsetneqq N_2(S)$.

Proof. The proofs of (1), (2) and (3) are obvious from the definitions.

(4) Let $u = a + \sum a_{ij}e_{ij} \in N_*(R)'$. Since $N_*(R) = N_1(R)$, there exists a nilpotent ideal J of R such that $a, a_{ij} \in J$; hence $u \in J'$. But J' is nilpotent since J is nilpotent, and so $u \in N_1(S)$ showing the result.

(5) Let $p > q$ be any positive integers. Note that $e_{pq}Se_{pq} \subseteq \{ae_{pq} \mid a \in N_*(R)\} \subseteq N_*(R)'$. It then follows that $(e_{pq}S)^2 \subseteq N_*(R)' \subseteq N_1(S)$ by (4); hence $e_{pq} \in N_2(S)$ for all such p, q. This result, together with (4), implies $Q \subseteq N_2(S)$. But Q is maximal, so we obtain $Q = N_2(S) = N_*(S)$.

(6) We will show $e_{21} \notin N_1(S)$, then we get the proof since $e_{21} \in N_2(S)$ by (5). Assume on the contrary that $e_{21} \in N_1(S)$, then $(e_{21}S)^n = 0$ for some positive integer n. Since $N_*(R)$ is not nilpotent, there exist elements $a_1, a_2, \ldots, a_n$ in $N_*(R)$ such that $a_1a_2\cdots a_n \neq 0$. Take $s_k = a_ke_{12} \in S$ for each $k \in \{1, 2, \ldots, n\}$, then $a_ke_{22} = e_{21}s_k \in e_{21}S$ and $(e_{21}s_1)(e_{21}s_2)\cdots(e_{21}s_n) = (a_1a_2\cdots a_n)e_{22} \neq 0$ in $(e_{21}S)^n$, a contradiction. □

The following is a counterexample to the converse of Proposition 2.10(2).

EXAMPLE 3.3. Let K be the field of integers modulo 2. Set R be the exterior algebra on the set B over K as in Example 3.1 and $S = \{r1_T + \sum_{i>j} r_{ij}e_{ij} + \sum_{i\leq j} a_{ij}e_{ij} \mid r, r_{ij} \in R \text{ and } a_{ij} \in N_*(R)\}$ as in Lemma 3.2. First we have $N_1(S) \subsetneqq N_2(S) = N_*(S)$ by Lemma 3.2(5, 6). Since $N_*(R)' \subseteq N_1(S)$ by Lemma 3.2(4) and $N_1(S) \subseteq N_*(S[[X]])$, it suffices to prove that $e_{pq} \in N_*(S[[X]])$ for all positive integers $p > q$. Note $(e_{pq}S)^2 \subseteq e_{pq}N_*(R)'$.

Next we claim that $(uv + vu)e_{pq} = u^2e_{pq} = 0$ for all $u, v \in e_{pq}N_*(R)'$. Let $u = a_1e_{p1} + a_2e_{p2} + \cdots + a_pe_{pp} + a_{p+1}e_{p(p+1)} + \cdots + a_ne_{pn}$ and $v = b_1e_{p1} + b_2e_{p2} + \cdots + b_pe_{pp} + b_{p+1}e_{p(p+1)} + \cdots + b_ne_{pn}$ with $a_i, b_i \in N_*(R)$ and n some positive integer. Then since R is commutative and $N_*(R)$ is nil of index 2, it follows that $(uv + vu)e_{pq} = 0 = u^2e_{pq}$; hence $(uv + vu)e_{pq}N_*(R)' = u^2e_{pq}N_*(R)' = 0$ for all $u, v \in e_{pq}N_*(R)'$.

If $f(X) = \sum_{w\in F(X)} u_ww \in (e_{pq}N_*(R)')[[X]]$, then the coefficient at

w in $f(X)^2$ is

$$\left(\sum_{w_1 w_2 = w \text{ and } w_1, w_2 \in F(X)} (u_{w_1} u_{w_2} + u_{w_2} u_{w_1}) \right) + u_{w_3}^2$$

for any w, where $w_3^2 = w$ (if any). Thus $f(X)^3 = 0$ for all $f(X) \in (e_{pq}N_*(R)')[[X]]$ by the previous argument.

Finally if $g(X) \in (e_{pq}S)[[X]] = e_{pq}S[[X]]$, then

$$g(X)^2 \in (e_{pq}S[[X]])^2 \subseteq (e_{pq}N_*(R)')[[X]]$$

and so $g(X)^6 = (g(X)^2)^3 = 0$; hence $(e_{pq}S)[[X]]$ is nil of index ≤ 6. This implies that $e_{pq} \in N_*(S[[X]])$. Therefore we obtain that $N_*(S) \subset N_*(S[[X]])$ but $N_1(S) \subsetneqq N_*(S)$. □

The following is a counterexample to the converse of Proposition 2.10(3).

EXAMPLE 3.4. Let K be a field of characteristic zero. Let R be the exterior algebra on the set B over K as in Example 3.1 and $S, Q, N_*(R)'$ be the same ones as in Lemma 3.2. Then $N_2(S) = N_*(S)$ by Lemma 3.2(5), but $N_1(S) \subsetneqq N_2(S)$ by Lemma 3.2(6). So $N_*(S) \not\subseteq N_*(S[[X]])$ by Proposition 2.11. □

References

[1] S. A. Amitsur, *Nil radicals. Historical notes and some new results*, Associative Rings, Modules, and Radicals (Proc. Colloquium at Keszthely 1971, edited by A. Kertész) (1973), János Bolyai Mathematical Society and North-Holland Publishing Company, Amsterdam, London, and Budapest, 47–65.

[2] C. Huh, H. K. Kim, D. S. Lee, Y. Lee, *Prime radicals of formal power series rings*, Bull. Korean Math. Soc. **38** (2001), 623–633.

[3] C. Huh, H. K. Kim, Y. Lee, *Questions on 2-primal rings*, Comm. Algebra **26** (1998), 595–600.

[4] N. Jacobson, *Structure of Rings*, vol. 37, Amer. Math. Soc. Colloq. Publ., 1964.

[5] A. A. Klein, *Rings of bounded index*, Comm. Algebra **12** (1984), 9–21.

[6] ———, *The sum of nil one-sided ideals of bounded index of a ring*, Israel J. Math. **88** (1994), 25–30.

[7] E. R. Puczyłowski, *On radicals of polynomial rings, power series rings and tensor products*, Comm. Algebra **8** (1980), 1699–1709.

[8] E. R. Puczyłowski and A. Smoktunowicz, *The nil radical of power series rings*, Israel J. Math. **110** (1999), 317–324.

[9] L. H. Rowen, *Ring Theory*, Academic Press, Inc., 1991.

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