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WEIGHTED HARMONIC BERGMAN FUNCTIONS ON HALF-SPACES

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ABSTRACT. On the setting of the upper half-space $\mathbf{H}$ of the Euclidean n-space, we show the boundedness of weighted Bergman projection for $1 < p < \infty$ and nonorthogonal projections for $1 \leq p < \infty$. Using these results, we show that Bergman norm is equivalent to the normal derivative norms on weighted harmonic Bergman spaces. Finally, we find the dual of b^1_α.

1. Introduction

For a fixed positive integer n, let $\mathbf{H} = \mathbf{R}^{n-1} \times \mathbf{R}_+$ be the upper half-space where $\mathbf{R}_+$ denotes the set of all positive real numbers. We write point $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$, $1 \leq p < \infty$, and $\Omega \subset \mathbf{R}^n$, let $b^p_\alpha(\Omega)$ denote *the weighted harmonic Bergman space* consisting of all real-valued harmonic functions u on Ω such that

$$||u||_{L^p_\alpha(\Omega)} := \left(\int_\Omega |u(z)|^p \, dV_\alpha(z) \right)^{1/p} < \infty,$$

where $dV_\alpha(z) = \operatorname{dist}(z, \partial\Omega)^\alpha \, dz$ and dz is the Lebesgue measure on $\mathbf{R}^n$. Here $\operatorname{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to the boundary of Ω. We let $b^p_\alpha = b^p_\alpha(\mathbf{H})$ and $b^p = b^p_0$.

Harmonic Bergman spaces are not studied as extensively as their holomorphic counterparts and most work on Bergman spaces (even in the holomorphic case) has been done for bounded domains.

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Because **H** is a unbounded domain, it causes some problems. For example, the weighted harmonic Bergman kernel is not even integrable unlike the case of bounded domains. However we overcome this difficulty by noticing the b^1_α-*cancellation property* which we mention in section 5. Note that **H** is a product space. This fact allows us to use the integration by parts (especially) with respect to the last component and this gives us reproducing properties of weighted harmonic Bergman functions. Furthermore, unlike the case of the unit disc, **H** is invariant under dilations, i.e., for every $r > 0$,

$$\{rz \mid z \in \mathbf{H}\} = \mathbf{H}.$$

Therefore we can use the change of variable freely with respect to the last coordinate which helps us to estimate the size of some integrals that appear in this paper.

$b^p(\Omega)$ is studied in [10] and [7] on the setting of upper half-space and bounded smooth domain in $\mathbf{R}^n$ respectively.

Recently, for any range $\alpha > -1$, the explicit formula of weighted harmonic Bergman kernel for b^p_α was found in [8]. In this paper, we show that some of the known results for b^p as well as weighted holomorphic Bergman spaces continue to hold on b^p_α for any range $\alpha > -1$ with this weighted harmonic Bergman kernel for b^p_α.

This paper is organized as follows. In section 2, we review the weighted harmonic Bergman kernel of $b^2_\alpha(\mathbf{H})$ and some useful results that are proved in [8].

Section 3 is devoted to proving that the weighted harmonic Bergman projection, initially defined as the orthogonal projection of $L^2_\alpha(\mathbf{H})$ onto b^2_α, extends to a bounded projection of $L^p_\alpha(\mathbf{H})$ onto b^p_α for the range $1 < p < \infty$ (Theorem 3.1). From this result, we easily get $(b^p_\alpha)^* \cong b^q_\alpha$ (Theorem 3.2).

In section 4, we define nonorthogonal projections of $L^p_\alpha(\mathbf{H})$ onto b^p_α and then we find a necessary and sufficient condition for these projections to be bounded including the case $p = 1$ (Theorem 4.3). We use these projections to find dense subspaces of b^p_α which have "*nice*" vanishing properties near ∞ (Theorem 4.4) and then we show that every weighted harmonic Bergman function can be reformulated in terms of its fractional normal derivatives (Theorem 4.7). We also give the norm equivalence result for weighted harmonic Bergman functions through these projections (Theorem 4.8).

In final section, we show that any Bloch function can be reproduced from its fractional normal derivative of any positive order (Theorem 5.9). Also, we show that the dual space of b^1_α can be identified with the harmonic Bloch space (Theorem 5.12) and then we give the norm equivalence result for harmonic Bloch functions (Theorem 5.13).

CONSTANTS. Throughout the paper we use the same letter C to denote various constants which may change at each occurrence. The constant C may often depend on the dimension n and some other parameters, but it is always independent of particular functions, points or parameters under consideration. For nonnegative quantities A and B, we often write $A \lesssim B$ or $B \gtrsim A$ if A is dominated by B times some *inessential* positive constant. Also, we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Bergman kernel for b^2_α and some results

In this section, we review the weighted harmonic Bergman kernel for b^2_α and recent results which are proved in [8].

Let $P(z, w)$ be the extended Poisson kernel on $\mathbf{H}$, i.e.,

$$P_z(w) := P(z, w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \overline{w}|^n} \tag{2.1}$$

where $z, w \in \mathbf{H}$ and $\overline{w} = (w', -w_n)$. Note that for each $j = 1, \ldots, n-1$, $D_{z_j} P(z, w) = -D_{w_j} P(z, w)$ and $D_{z_n} P(z, w) = D_{w_n} P(z, w)$. Therefore we can show from (2.1) that for multi-indices $\beta = (\beta_1, \ldots, \beta_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$,

$$\begin{aligned} D^\beta_z D^\gamma_w P(z, w) &= D^{\beta_1}_{z_1} \cdots D^{\beta_n}_{z_n} D^{\gamma_1}_{w_1} \cdots D^{\gamma_n}_{w_n} P(z, w) \\ &= (-1)^{\gamma_1 + \cdots + \gamma_{n-1}} D^{\beta_1 + \gamma_1}_{z_1} \cdots D^{\beta_n + \gamma_n}_{z_n} P(z, w) \\ &= (-1)^{\gamma_1 + \cdots + \gamma_{n-1}} \frac{f_{\beta,\gamma}(z - \overline{w})}{|z - \overline{w}|^{n + 2|\beta| + 2|\gamma|}}, \end{aligned} \tag{2.2}$$

where $f_{\beta,\gamma}$ is a homogeneous polynomial of degree $1 + |\beta| + |\gamma|$. (In fact, $f_{\beta,\gamma}$ is harmonic but we do not need this fact here.)

Let k be a nonnegative integer and let D denote the differentiation with respect to the last component. If $u \in b^p_\alpha(\Omega)$, then we know from the mean value property, Jensen's inequality and then Cauchy's estimate that

$$|D^k u(z)| \lesssim \operatorname{dist}(z, \partial\Omega)^{-(n+\alpha)/p - k} \tag{2.3}$$

for each $z \in \Omega$.

Let $\mathcal{F}_\beta$ ($\beta > 0$) be the collection of all functions v on $\mathbf{H}$ satisfying $|v(z)| \lesssim z_n^{-\beta}$ and let $\mathcal{F} = \cup_{\beta > 0} \mathcal{F}_\beta$. If $v \in \mathcal{F}$, then $v \in \mathcal{F}_\beta$ for some

$\beta > 0$. In this case, we define the fractional derivative of v of order $-s$ by

$$\mathcal{D}^{-s}v(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} v(z', z_n + t)\, dt$$

for the range $0 < s < \beta$. Here, Γ is the Gamma function.

If $u \in b^p_\alpha$, then for every nonnegative integer k, $D^k u \in \mathcal{F}$ by (2.3). Thus for $s \geq 0$, we define the fractional derivative of u of order s by

$$\mathcal{D}^s u = \mathcal{D}^{-([s]-s)} D^{[s]} u,$$

where $[s]$ is the smallest integer greater than or equal to s and $\mathcal{D}^0 = D^0$ is the identity operator. If $s > 0$ is not an integer, then $-1 < [s] - s - 1 < 0$ and $[s] \geq 1$. Thus we know from (2.3) that for each $z \in \mathbf{H}$ and for every $u \in b^p_\alpha$,

$$\mathcal{D}^s u(z) = \frac{1}{\Gamma([s]-s)} \int_0^\infty t^{[s]-s-1} D^{[s]} u(z', z_n + t)\, dt$$

always makes sense.

For $\alpha > -1$, the Bergman kernel for b^2_α is given by

$$R_\alpha(z, w) = C_\alpha \mathcal{D}^{\alpha+1} P_z(w),$$

where

$$C_\alpha = \frac{(-1)^{[\alpha]+1} 2^{\alpha+1}}{\Gamma(\alpha+1)}. \tag{2.4}$$

It is shown in [1] that the Bergman kernel for b^2 is $-2DP_z(w)$.

Let $s > -n - \alpha$ and let β be a multi-index. Then

$$\left| D^\beta_z \mathcal{D}^s_{z_n} R_\alpha(z, w) \right| \lesssim \frac{1}{|z - \overline{w}|^{n+\alpha+|\beta|+s}} \tag{2.5}$$

for $z, w \in \mathbf{H}$. Thus, we have

$$\|R_\alpha(z, \cdot)\|_{L^q_\alpha(\mathbf{H})} \lesssim z_n^{(n+\alpha)(1/q-1)} \tag{2.6}$$

for $1 < q < \infty$.

For a function u on $\mathbf{H}$, define $u_\delta(z) = u(z', z_n + \delta)$ for $\delta > 0$. The following propositions are used many times in this paper.

Proposition 2.1. *Let $\alpha > -1$, $1 \leq p < \infty$ and let $u \in b^p_\alpha$. Then*

$$\lim_{\delta \to 0^+} \|u_\delta - u\|_{L^p_\alpha(\mathbf{H})} = 0.$$

PROPOSITION 2.2. *Let $b < 0$ and let $a + b > -1$. Then,*

$$\int_{\mathbf{H}} \frac{w_n^{a+b}}{|z - \overline{w}|^{n+a}}\, dw \approx z_n^b$$

as z ranges over all points in $\mathbf{H}$.

The following lemma comes from integration by parts with respect to the w_n-variable and this plays an important role in this paper. Let $\mathbf{H}_\delta = \{z \in \mathbf{R}^n \mid z_n > -\delta\}$ for $\delta > 0$. Thus for each $\delta > 0$, $\mathbf{H}_\delta$ is a half-space that contains $\mathbf{H}$ properly.

LEMMA 2.3. *Let $\delta > 0$, $1 \le p < \infty$ and let $u \in b^p_\alpha(\mathbf{H}_\delta)$. Suppose that k and m are nonnegative integers. Then for every $z \in \mathbf{H}$ and for each $a, b > 0$,*

$$\int_{\mathbf{H}} \left[D^{k+1} P_z\left(w', a w_n\right)\right]\left[D^m u\left(w', b w_n\right)\right] w_n^{m+k} dw$$
$$= \frac{(-1)^{m+k+1}(m+k)!}{(a+b)^{m+k+1}} u(z).$$

We have the following reproducing properties of integral operators with the weighted harmonic Bergman kernel.

PROPOSITION 2.4. *Let $\delta > 0$, $\alpha > -1$, $1 \le p < \infty$ and let $s > -1$. Then for every $u \in b^p_\alpha(\mathbf{H}_\delta)$ and for each $z \in \mathbf{H}$,*

$$u(z) = \int_{\mathbf{H}} R_s(z, w) u(w)\, dV_s(w).$$

THEOREM 2.5. *Let $\alpha > -1$ and let $1 \le p < \infty$. If $u \in b^p_\alpha$, then for every $z \in \mathbf{H}$,*

$$u(z) = \int_{\mathbf{H}} u(w) R_\alpha(z, w)\, dV_\alpha(w).$$

3. Bergman projection

In this section, we study the weighted harmonic Bergman projection. Because b^2_α is a closed subspace of $L^2_\alpha(\mathbf{H})$, there is a unique orthogonal projection Π_α of $L^2_\alpha(\mathbf{H})$ onto b^2_α:

$$\Pi_\alpha f(z) = \int_{\mathbf{H}} f(w) R_\alpha(z, w)\, dV_\alpha(w)$$

for every $f \in L^2_\alpha(\mathbf{H})$ and for each $z \in \mathbf{H}$. We know from (2.6) that $R_\alpha(z, \cdot) \in b^q_\alpha$ for all $1 < q \le \infty$. Thus, Π_α is well defined whenever

$f \in L^p_\alpha(\mathbf{H})$ for $1 \le p < \infty$. Moreover Theorem 2.5 implies that Π_α is the identity on b^p_α for $1 \le p < \infty$. Therefore we only need to show the boundedness of Π_α to complete the proof of the following theorem.

THEOREM 3.1. *Let $\alpha > -1$. If $1 < p < \infty$, then Π_α is a bounded projection of $L^p_\alpha(\mathbf{H})$ onto b^p_α.*

Proof. Let $f \in L^p_\alpha$. Then (2.5) implies that for $z \in \mathbf{H}$,

$$|\Pi_\alpha f(z)| \lesssim \int_{\mathbf{H}} |f(w)| \frac{1}{|z-\overline{w}|^{n+\alpha}}\, dV_\alpha(w).$$

Let q denote the index conjugate of p. After applying Hölder's inequality to the following two functions

$$|f(w)| \left(\frac{w_n^{(1+\alpha)/q}}{|z-\overline{w}|^{n+\alpha}} \right)^{1/p}, \quad \left(\frac{w_n^{-(1+\alpha)/p}}{|z-\overline{w}|^{n+\alpha}} \right)^{1/q},$$

we see that

$$|\Pi_\alpha f(z)|^p \lesssim \left(\int_{\mathbf{H}} |f(w)|^p \frac{w_n^{(1+\alpha)/q}}{|z-\overline{w}|^{n+\alpha}} w_n^\alpha\, dw \right) \left(\int_{\mathbf{H}} \frac{w_n^{-(1+\alpha)/p}}{|z-\overline{w}|^{n+\alpha}} w_n^\alpha\, dw \right)^{p/q}$$

Hence we get from Proposition 2.2 that

$$\int_{\mathbf{H}} |\Pi_\alpha f(z)|^p\, dV_\alpha(z) \lesssim \int_{\mathbf{H}} \int_{\mathbf{H}} |f(w)|^p \frac{w_n^{(1+\alpha)/q}}{|z-\overline{w}|^{n+\alpha}}\, w_n^\alpha\, dw\, z_n^{-(1+\alpha)/q}\, z_n^\alpha\, dz.$$

After applying Fubini's Theorem and then Proposition 2.2 once again, we get

$$\|\Pi_\alpha f\|^p_{L^p_\alpha(\mathbf{H})} \lesssim \|f\|^p_{L^p_\alpha(\mathbf{H})}.$$

Therefore the proof is complete. □

After we define nonorthogonal projections Π_β in section 4, we generalize Theorem 3.1. (See Theorem 4.3.) We also show that $\Pi_\alpha f$ does not belong to $L^1_\alpha(\mathbf{H})$ for some $f \in L^1_\alpha(\mathbf{H})$ in Theorem 4.3. This shows the failure of Theorem 3.1 for $p = 1$.

The following theorem with $\alpha = 0$ case is proved in [10]. Because we have L^p_α-boundedness of Π_α, the proof of the following theorem is very similar to that of [10]. Therefore we omit the proof.

THEOREM 3.2. *If $\alpha > -1$ and $1 < p < \infty$, then $(b^p_\alpha)^* \cong b^q_\alpha$, where q is the index conjugate of p.*

4. Nonorthogonal projections

In this section, we study nonorthogonal projections of $L^p_\alpha(\mathbf{H})$ onto b^p_α for all range $1 \le p < \infty$. We call them nonorthogonal because they are not orthogonal when acting on the Hilbert-space $L^2_\alpha(\mathbf{H})$. Let $\alpha, \beta > -1$ and let $1 \le p < \infty$. Define Π_β on $L^p_\alpha(\mathbf{H})$ by

$$\Pi_\beta f(z) = \int_{\mathbf{H}} f(w) R_\beta(z,w)\, dV_\beta(w)$$

for $z \in \mathbf{H}$. Here $R_\beta(z,w) = C_\beta \mathcal{D}^{\beta+1} P_z(w)$ and C_β is the constant given in (2.4).

One of the advantage of Π_β is that Π_β is bounded on $L^1_\alpha(\mathbf{H})$ whenever $\alpha < \beta$ unlike the Bergman projection Π_α. From this L^1_α-boundedness of Π_β, we obtain dense subspaces of b^1_α which have nice vanishing properties near ∞ ; we use these subspaces to find the dual space of b^1_α. (See section 5.) Also, Π_β leads to fractional normal derivative norm equivalence on weighted harmonic Bergman spaces.

We first estimate the size of $R_\beta(z,\cdot)$ on a thin cone with vertex $\overline{z}$ and axis of symmetry parallel to the z_n-axis. Fix $z_0 = (0,1)$ for the rest of this paper. Then we can check easily from (2.2) that for each nonnegative integer k

$$f_{k+1}(z_0) = (-1)^{k+1} 2(n-1)n\cdots(n+k-1)/nV(B) \neq 0,$$

where $f_{k+1}(z_0) = f_{(0,\dots,0),(0,\dots,0,k+1)}(z_0)$ in (2.2). Because f_{k+1} is a homogeneous polynomial of degree $k+2$, there exists $\varepsilon_0 > 0$ such that

$$0 < f_{k+1}(z_0) f_{k+1}(z) \approx |z|^{k+2}, \quad |z| \approx z_n, \tag{4.1}$$

as z ranges over all points in $\Gamma_{\varepsilon_0}(0) := \{z \in \mathbf{H} \mid z_n > \varepsilon_0 |z'|\ \}$. Thus we have from (4.1) that

$$\begin{aligned} f_{k+1}(z_0) D^{k+1} P_z(w) &= \frac{f_{k+1}(z_0) f_{k+1}(z - \overline{w})}{|z - \overline{w}|^{n+2+2k}} \\ &\approx \frac{(z_n + w_n)^{k+2}}{|z - \overline{w}|^{n+2+2k}} \\ &\approx \frac{1}{(z_n + w_n)^{n+k}}, \end{aligned} \tag{4.2}$$

for $z \in \mathbf{H},\ w \in \Gamma_{\varepsilon_0}(\overline{z}) := \{w \in \mathbf{H} \mid (z_n + w_n) > \varepsilon_0 |z' - w'|\ \}$.

This estimate guarantees the following lemma, which is a key in showing the failure of boundedness of Π_β on $L^p_\alpha(\mathbf{H})$ for $\alpha + 1 \ge (\beta+1)p$ in Theorem 4.3.

LEMMA 4.1. *Let $\beta > -1$ and let $z \in \mathbf{H}$. Then we have*

$$|R_\beta(z,w)| \approx \frac{1}{(z_n + w_n)^{n+\beta}}$$

as w ranges over all points in $\Gamma_{\varepsilon_0}(\overline{z})$.

Proof. If β is an integer, then the proof follows easily from (4.2). Assume that β is not an integer. Let $k = [\beta]$. Note that $k - \beta - 1 > -1$ and note also that $w \in \Gamma_{\varepsilon_0}(\overline{z})$ implies $w + (0,t) \in \Gamma_{\varepsilon_0}(\overline{z})$ for every $t > 0$. Hence we get from the definition of R_β and (4.2) that

$$\begin{aligned} |R_\beta(z,w)| &\approx \left| \int_0^\infty t^{k-\beta-1} D^{k+1} P_z(w', w_n + t)\, dt \right| \\ &\approx \int_0^\infty t^{k-\beta-1} f_{k+1}(z_0) D^{k+1} P_z(w', w_n + t)\, dt \\ &\approx \int_0^\infty \frac{t^{k-\beta-1}}{(z_n + w_n + t)^{n+k}}\, dt \\ &\approx \frac{1}{(z_n + w_n)^{n+\beta}}, \end{aligned}$$

as w ranges over all points in $\Gamma_{\varepsilon_0}(\overline{z})$. Therefore the proof is complete. □

Before we prove Theorem 4.3, we need one more lemma.

LEMMA 4.2. *Let $\alpha > -1$, $1 \le p < \infty$ and let $z \in \mathbf{H}$. If $\alpha + 1 < (\beta + 1)p$ and q is the index conjugate of p, then $R_\beta(z,w) w_n^{\beta-\alpha} \in L^q_\alpha(\mathbf{H})$ as a function of w.*

Proof. If $p = 1$, then $\alpha < \beta$. Thus we get from (2.5) that

$$\|R_\beta(z,w) w_n^{\beta-\alpha}\|_\infty \lesssim \sup_{w \in \mathbf{H}} \frac{w_n^{\beta-\alpha}}{|z - \overline{w}|^{n+\beta}} \le z_n^{-(n+\alpha)} < \infty.$$

Therefore $R_\beta(z,w) w_n^{\beta-\alpha}$ is uniformly bounded on $\mathbf{H}$.

Assume that $p > 1$. Then we have $-1 < (\beta - \alpha)q + \alpha$, because $(\alpha + 1) < (\beta + 1)p$. Therefore we see from (2.5) and Proposition 2.2 with $a = (n + \beta)q - n$ and $b = (n + \alpha)(1 - q)$ that

$$\|R_\beta(z,w) w_n^{\beta-\alpha}\|_{L^q_\alpha(\mathbf{H})} \lesssim z_n^{-(n+\alpha)/p}. \tag{4.3}$$

This completes the proof. □

In the following theorem, we find a necessary and sufficient condition for Π_β to be a bounded projection on $L^p_\alpha(\mathbf{H})$. We use this theorem to find some useful dense subspaces of b^p_α and to show the norm equivalence result on weighted harmonic Bergman spaces.

THEOREM 4.3. *Let $\alpha, \beta > -1$ and let $1 \le p < \infty$. Then Π_β is a bounded projection of $L^p_\alpha(\mathbf{H})$ onto b^p_α if and only if $\alpha + 1 < (\beta + 1)p$.*

Proof. Suppose that Π_β is bounded on $L^p_\alpha(\mathbf{H})$ and $\alpha + 1 \ge (\beta + 1)p$. If $p = 1$, then $\alpha \ge \beta$. Consider $f(z) = z_n^{-\beta} \chi_{B(z_0,1)}/V(B(z_0, 1))$ where $B(z_0, 1)$ is an open ball in $\mathbf{R}^n$ centered at z_0 with radius 1. Then clearly $f \in L^1_\alpha(\mathbf{H})$. We know from the mean value property that

$$\begin{aligned}\Pi_\beta f(z) &= \int_{\mathbf{H}} f(w) R_\beta(z, w)\, dV_\beta(w) \\ &= \frac{1}{V(B(z_0,1))} \int_{B(z_0,1)} R_\beta(z, w)\, dw \\ &= R_\beta(z, z_0).\end{aligned}$$

Note that $\Gamma_{\varepsilon_0}(0) \subset \Gamma_{\varepsilon_0}(\overline{z}_0)$. Then Lemma 4.1 and (4.1) imply that

$$\begin{aligned}\|\Pi_\beta f\|_{L^1_\alpha(\mathbf{H})} &= \int_{\mathbf{H}} |R_\beta(z, z_0)|\, dV_\alpha(z) \\ &\gtrsim \int_{\Gamma_{\varepsilon_0}(0)} \frac{z_n^\alpha}{(1 + z_n)^{n+\beta}}\, dz \\ &\approx \int_{\Gamma_{\varepsilon_0}(0)} \frac{z_n^\alpha}{1 + |z|^{n+\beta}}\, dz.\end{aligned}$$

Let $S_{\epsilon_0} = \{z \in \mathbf{H} \mid z \in \Gamma_{\varepsilon_0}(0),\ |z| = 1\ \}$. Then we get from polar coordinates that

$$\begin{aligned}\|\Pi_\beta f\|_{L^1_\alpha(\mathbf{H})} &\gtrsim \int_0^\infty \frac{r^{n-1}}{1 + r^{n+\beta}} \int_{S_{\epsilon_0}} r^\alpha\, \zeta_n^\alpha\, d\sigma(\zeta)\, dr \\ &\approx \int_0^\infty \frac{r^{n-1+\alpha}}{1 + r^{n+\beta}} dr \\ &= \infty.\end{aligned}$$

(Here $d\sigma$ denotes the normalized surface area measure on the unit sphere in $\mathbf{R}^n$.) This shows that $\Pi_\beta f$ is not in $L^1_\alpha(\mathbf{H})$. Therefore $\alpha < \beta$.

If $1 < p < \infty$, then $(\beta - \alpha)q + \alpha \le -1$, because $\alpha + 1 \ge (\beta + 1)p$. (Here q is the index conjugate of p.) Fix $z \in \mathbf{H}$. Notice that

$$\{w' \in \partial\mathbf{H} \mid z_n > \varepsilon_0 |z' - w'|\ \} \times (0, 1) \subset \Gamma_{\varepsilon_0}(\overline{z}).$$

Thus we get from Lemma 4.1 that

$$\begin{aligned}
&\int_{\mathbf{H}} \left| R_\beta(z,w) w_n^{\beta-\alpha} \right|^q dV_\alpha(w) \\
&\gtrsim \int_{\Gamma_{\varepsilon_0}(\overline{z})} \frac{w_n^{(\beta-\alpha)q+\alpha}}{(z_n+w_n)^{(n+\beta)q}}\, dw \\
&\gtrsim \int_{\{w'\in\partial\mathbf{H} | z_n > \varepsilon_0 |z'-w'|\}} \int_0^1 \frac{w_n^{(\beta-\alpha)q+\alpha}}{(z_n+w_n)^{(n+\beta)q}}\, dw_n\, dw' \\
&\gtrsim \left(\frac{z_n}{\varepsilon_0}\right)^{n-1} \frac{1}{(z_n+1)^{(n+\beta)q}} \int_0^1 w_n^{(\beta-\alpha)q+\alpha}\, dw_n \\
&= \infty.
\end{aligned}$$

This shows that $\Pi_\beta g$ fails to exist at z for some $g \in L^p_\alpha(\mathbf{H})$, because dV_α is a positive σ-finite measure on $\mathbf{H}$.

For the other direction, assume that $\alpha + 1 < (\beta+1)p$. We first show that Π_β is the identity operator on b^p_α. Let $u \in b^p_\alpha$. Then $u_\delta \in b^p_\alpha(\mathbf{H}_\delta)$ for $\delta > 0$. Fix $z \in \mathbf{H}$. Then Proposition 2.4 implies that

$$|\Pi_\beta u(z) - u(z)| \le \left| \int_{\mathbf{H}} \big(u(w) - u_\delta(w)\big) R_\beta(z,w)\, dV_\beta(w) \right| + |u_\delta(z) - u(z)|.$$

Applying Hölder's inequality and Lemma 4.2 to the above integral and then letting $\delta \to 0^+$, we see from Proposition 2.1 that $\Pi_\beta u(z) = u(z)$. Therefore Π_β is the identity on b^p_α. Next, we show that Π_β is bounded on $L^p_\alpha(\mathbf{H})$. If $p = 1$, then we get from (2.5) and Fubini's Theorem that

$$\begin{aligned}
\|\Pi_\beta f\|_{L^1_\alpha(\mathbf{H})} &\le \int_{\mathbf{H}} \int_{\mathbf{H}} |f(w) R_\beta(z,w)|\, dV_\beta(w)\, dV_\alpha(z) \\
&\lesssim \int_{\mathbf{H}} |f(w)| \int_{\mathbf{H}} \frac{z_n^\alpha}{|z-\overline{w}|^{n+\beta}}\, dz\, w_n^\beta\, dw \qquad (4.4)
\end{aligned}$$

for $f \in L^1_\alpha(\mathbf{H})$. Because $\alpha < \beta$, we get $\|\Pi_\beta f\|_{L^1_\alpha(\mathbf{H})} \lesssim \|f\|_{L^1_\alpha(\mathbf{H})}$, after applying Proposition 2.2 to the inner integral in (4.4).

Let $1 < p < \infty$ and let $f \in L^p_\alpha(\mathbf{H})$. Then we obtain from (2.5) that

$$\begin{aligned}
|\Pi_\beta f(z)| &\lesssim \int_{\mathbf{H}} |f(w)| \frac{1}{|z-\overline{w}|^{n+\beta}}\, dV_\beta(w) \\
&= \int_{\mathbf{H}} |f(w)| \frac{w_n^{\beta-\alpha}}{|z-\overline{w}|^{n+\beta}}\, w_n^\alpha\, dw.
\end{aligned}$$

Note that $0 < \alpha + 1 < (\beta+1)p$ implies $-(1+\beta)/q < (\beta-\alpha)/p$ and $-(1+\alpha)p < 0$. Therefore we can choose a real number s which satisfies both $-(1+\beta)/q < s < 0$ and $-(1+\alpha)/p < s < (\beta-\alpha)/p$. After applying Hölder's inequality to the following two functions

$$w_n^{-s}|f(w)|\left(\frac{w_n^{\beta-\alpha}}{|z-\overline{w}|^{n+\beta}}\right)^{1/p}, \quad w_n^{s}\left(\frac{w_n^{\beta-\alpha}}{|z-\overline{w}|^{n+\beta}}\right)^{1/q},$$

we see that

$$|\Pi_\beta f(z)| \lesssim \left(\int_{\mathbf{H}} |f(w)|^p \frac{w_n^{\beta-ps}}{|z-\overline{w}|^{n+\beta}}\,dw\right)^{1/p} \left(\int_{\mathbf{H}} \frac{w_n^{\beta+qs}}{|z-\overline{w}|^{n+\beta}}\,dw\right)^{1/q}.$$

Because $-(1+\beta)/q < s < 0$, we see $-1 < \beta+qs$ and $qs < 0$. Therefore we get from Proposition 2.2 that

$$|\Pi_\beta f(z)| \lesssim z_n^{s}\left(\int_{\mathbf{H}} |f(w)|^p \frac{w_n^{\beta-ps}}{|z-\overline{w}|^{n+\beta}}\,dw\right)^{1/p}.$$

Note that $-(1+\alpha)/p < s < (\beta-\alpha)/p$ implies $-1 < ps+\alpha$ and $ps+\alpha-\beta < 0$. Hence after applying Proposition 2.2 once again, we see that

$$\begin{aligned}\|\Pi_\beta f\|^p_{L^p_\alpha(\mathbf{H})} &\lesssim \int_{\mathbf{H}} |f(w)|^p\, w_n^{\beta-ps} \int_{\mathbf{H}} \frac{z_n^{ps+\alpha}}{|z-\overline{w}|^{n+\beta}}\,dz\,dw \\ &\lesssim \|f\|^p_{L^p_\alpha(\mathbf{H})}.\end{aligned}$$

Therefore the proof is complete. □

In the proof of the above theorem, we showed that $R_\beta(z_0,\cdot) \notin L^1_\alpha(\mathbf{H})$ for $\alpha \geq \beta$. With a simple modification of the proof, we can also show that for each $z \in \mathbf{H}$, $R_\beta(z,\cdot) \notin L^1_\alpha(\mathbf{H})$ whenever $\alpha \geq \beta$. This implies that the weighted harmonic Bergman kernel $R_\alpha(z,\cdot) \notin L^1_\alpha(\mathbf{H})$.

Now we obtain useful dense subspaces of b^p_α for all range $1 \leq p < \infty$ from the boundedness of Π_β. For $\beta > -1$, denote $\mathcal{S}_\beta$ be the vector space of functions u harmonic on $\mathbf{H}$ that satisfy

$$|u(z)| \lesssim \frac{1}{1+|z|^\beta}$$

as z ranges over all points in $\mathbf{H}$.

THEOREM 4.4. *Let $\alpha > -1$ and let $1 \leq p < \infty$. If $\alpha+1 < (\beta+1)p$, then $\mathcal{S}_{n+\beta}$ is dense in b^p_α.*

Proof. Let $u \in \mathcal{S}_{n+\beta}$. Because $1 + |z| \approx |z - \overline{z}_0|$,

$$\int_{\mathbf{H}} |u(z)|^p \, dV_\alpha(z) \lesssim \int_{\mathbf{H}} \frac{z_n^\alpha}{(1+|z|^{n+\beta})^p} \, dz \approx \int_{\mathbf{H}} \frac{z_n^\alpha}{|z-\overline{z}_0|^{(n+\beta)p}} \, dz.$$

Since $\alpha + 1 < (\beta+1)p$, we see $\alpha + n < (\beta+n)p$. Thus after applying Proposition 2.2 to the above integral, we see that $u \in b^p_\alpha$.

To prove density, let $u \in b^p_\alpha$. Choose a sequence of compact sets $\langle K_j \rangle$ such that $K_j \subset K_{j+1}$ and $\mathbf{H} = \cup K_j$. Setting $u_j = u\chi_{K_j}$, we have

$$|\Pi_\beta u_j(z)| \lesssim \int_{K_j} \frac{|u(w)|}{|z-\overline{w}|^{n+\beta}} \, dV_\beta(w) \lesssim \frac{1}{1+|z|^{n+\beta}}.$$

Thus $\Pi_\beta u_j \in \mathcal{S}_{n+\beta}$. Furthermore we see from Theorem 4.3 that

$$\|\Pi_\beta u_j - u\|_{L^p_\alpha(\mathbf{H})} = \|\Pi_\beta(u_j - u)\|_{L^p_\alpha(\mathbf{H})} \leq \|\Pi\|_\beta \, \|u_j - u\|_{L^p_\alpha(\mathbf{H})} \to 0$$

as $j \to \infty$. Hence $\mathcal{S}_{n+\beta}$ is dense in b^p_α and the proof is complete. □

The following lemma is used to show the fractional normal derivative norm equivalence result on weighted harmonic Bergman spaces in Theorem 4.8. If γ is a nonnegative integer, then it is proved in [10]. Therefore to complete the proof of the following lemma, we only need to show the case that γ is not a nonnegative integer.

LEMMA 4.5. *Let $\alpha > -1$ and let $1 \leq p < \infty$. If $(1+\alpha)/p + \gamma > 0$, then*

$$\|z_n^\gamma \mathcal{D}^\gamma u\|_{L^p_\alpha(\mathbf{H})} \lesssim \|u\|_{L^p_\alpha(\mathbf{H})}$$

for $u \in b^p_\alpha$.

Proof. Let $z \in \mathbf{H}$, $u \in b^p_\alpha$ and let $\beta = \alpha + 1$. Because $\alpha + 1 < (\beta+1)p$, we know from Theorem 4.3 that

$$u(z) = \Pi_\beta u(z) = \int_{\mathbf{H}} u(w) R_\beta(z,w) \, dV_\beta(w).$$

Let $k = [\gamma]$ if $\gamma > -1$ and let $k = 0$ if $\gamma \leq -1$. Then we get from (2.5) and Fubini's theorem that

$$\begin{aligned}|\mathcal{D}^\gamma u(z)| &\lesssim \int_0^\infty |D^k u(z', z_n + t)| t^{k-\gamma-1}\, dt \\ &\lesssim \int_0^\infty \int_{\mathbf{H}} |u(w)|\, \left|D^k_{z_n} R_\beta\big((z', z_n + t), w\big)\right| dV_\beta(w)\, t^{k-\gamma-1}\, dt \\ &\lesssim \int_{\mathbf{H}} |u(w)| \int_0^\infty \frac{t^{k-\gamma-1}}{|(z', z_n + t) - \overline{w}|^{n+\beta+k}}\, dt\, dV_\beta(w).\end{aligned}$$

Note that $|(z', z_n + t) - \overline{w}| \approx |z - \overline{w}| + t$ for $w \in \mathbf{H}$, $t > 0$. Thus we have

$$\begin{aligned}|\mathcal{D}^\gamma u(z)| &\lesssim \int_{\mathbf{H}} |u(w)| \int_0^\infty \frac{t^{k-\gamma-1}}{(|z - \overline{w}| + t)^{n+\beta+k}}\, dt\, dV_\beta(w) \\ &\approx \int_{\mathbf{H}} |u(w)| \frac{w_n^\beta}{|z - \overline{w}|^{n+\beta+\gamma}}\, dw, \qquad (4.5)\end{aligned}$$

where we used change of variable $t \mapsto |z - \overline{w}| t$. If $p = 1$, then $\alpha + \gamma > -1$. Therefore we get from Fubini's Theorem that

$$\begin{aligned}\|z_n^\gamma \mathcal{D}^\gamma u\|_{L^1_\alpha(\mathbf{H})} &\lesssim \int_{\mathbf{H}} z_n^\gamma \int_{\mathbf{H}} |u(w)| \frac{w_n^\beta}{|z - \overline{w}|^{n+\beta+\gamma}}\, dw\, dV_\alpha(z) \\ &= \int_{\mathbf{H}} |u(w)| w_n^\beta \int_{\mathbf{H}} \frac{z_n^{\gamma+\alpha}}{|z - \overline{w}|^{n+\beta+\gamma}}\, dz\, dw. \qquad (4.6)\end{aligned}$$

After applying Proposition 2.2 to the inner integral in (4.6), we see that

$$\|z_n^\gamma \mathcal{D}^\gamma u\|_{L^1_\alpha(\mathbf{H})} \lesssim \|u\|_{L^1_\alpha(\mathbf{H})}.$$

Assume that $1 < p < \infty$. Because $(1 + \alpha)/p + \gamma > 0$, we see that $0 < n/p < (n + \beta)/p + \gamma < n + \beta + \gamma$. Choose $n/p < \lambda < (n + \beta)/p + \gamma$. Then after applying Hölder's inequality in (4.5) to the following two functions

$$|u(w)| \frac{w_n^{\beta/p}}{|z - \overline{w}|^\lambda}, \qquad \frac{w_n^{\beta/q}}{|z - \overline{w}|^{n+\beta+\gamma-\lambda}},$$

we see that (4.5) is less than or equal to

$$\left(\int_{\mathbf{H}} |u(w)|^p \frac{w_n^\beta}{|z - \overline{w}|^{\lambda p}}\, dw\right)^{1/p} \left(\int_{\mathbf{H}} \frac{w_n^\beta}{|z - \overline{w}|^{(n+\beta+\gamma-\lambda)q}}\, dw\right)^{1/q}. \qquad (4.7)$$

Here q denotes the index conjugate of p. Because $\lambda < (n + \beta)/p + \gamma$, we see $n + \beta - (n + \beta + \gamma - \lambda)q < 0$. Thus after applying Proposition 2.2

to the second integral in (4.7), we know that

$$\left(\int_{\mathbf{H}} \frac{w_n^{\beta}}{|z-\overline{w}|^{(n+\beta+\gamma-\lambda)q}}\, dw\right)^{1/q} \approx z_n^{\lambda-\gamma-(n+\beta)/p}.$$

Therefore, we get from Fubini's Theorem that

$$\|z_n^{\gamma}\mathcal{D}^{\gamma}u\|^p_{L^p_\alpha(\mathbf{H})} \lesssim \int_{\mathbf{H}}\int_{\mathbf{H}} |u(w)|^p \frac{w_n^{\beta}}{|z-\overline{w}|^{\lambda p}}\, dw\, z_n^{\alpha-\beta+\lambda p-n}\, dz$$

$$= \int_{\mathbf{H}} |u(w)|^p w_n^{\beta} \int_{\mathbf{H}} \frac{z_n^{\alpha-\beta+\lambda p-n}}{|z-\overline{w}|^{\lambda p}}\, dz\, dw. \tag{4.8}$$

Because $n/p < \lambda$, we see $\alpha-\beta+\lambda p-n > -1$. Therefore after applying Proposition 2.2 once again to the inner integral in (4.8), we get

$$\|z_n^{\gamma}\mathcal{D}^{\gamma}u\|^p_{L^p_\alpha(\mathbf{H})} \lesssim \|u\|^p_{L^p_\alpha(\mathbf{H})}.$$

This completes the proof. □

We use the following lemma to switch the order of integrations that appear in Theorem 4.7.

LEMMA 4.6. *Let $\alpha > -1$, $1 \le p < \infty$ and let $(n+\alpha)/p+\gamma > 0$. If $s > -1$ is not an integer and $s+\gamma+1 > 0$, then for $z \in \mathbf{H}$*

$$I := \int_0^{\infty}\int_{\mathbf{H}} \frac{w_n^{[s]+\gamma}}{|z-(w',-(1+t)w_n)|^{n+[s]}(1+w_n)^{(n+\alpha)/p+\gamma}}\, dw\, t^{[s]-s-1}\, dt < \infty.$$

Proof. Let $z \in \mathbf{H}$ and let $k = [s]$. Then k is a nonnegative integer and $k-s > 0$. Note that

$$\frac{1}{|z-(w',-(1+t)w_n)|^{n+k}} \lesssim \frac{P(z,(w',(1+t)w_n))}{(z_n+(1+t)w_n)^{k+1}}.$$

Then we see from change of variable $(1+t)w_n \mapsto r$ that

$$\begin{aligned} I &\lesssim \int_0^{\infty}\int_0^{\infty} \frac{w_n^{k+\gamma}}{(z_n+(1+t)w_n)^{k+1}(1+w_n)^{(n+\alpha)/p+\gamma}}\, dw_n\, t^{k-s-1}\, dt \\ &= \int_0^{\infty}\int_0^{\infty} \frac{r^{k+\gamma}}{(z_n+r)^{k+1}(1+t+r)^{(n+\alpha)/p+\gamma}}\, dr \\ &\quad \times (1+t)^{(n+\alpha)/p-(k+1)} t^{k-s-1}\, dt \\ &:= II. \end{aligned}$$

If $\gamma \geq 0$, choose $0 < \lambda_1 < s+1$ satisfying $\lambda_1 < (n+\alpha)/p$. Then

$$II \lesssim \int_0^\infty \int_0^\infty \frac{1}{(z_n+r)(1+r)^{\lambda_1}}\, dr\, \frac{(1+t)^{(n+\alpha)/p-(k+1)}}{(1+t)^{(n+\alpha)/p-\lambda_1}} t^{k-s-1}\, dt$$
$$< \infty,$$

because $\gamma \geq 0$, $\lambda_1 > 0$, $k-s-1 > -1$ and $s-\lambda_1+2 > 1$.

If $\gamma < 0$, choose $0 < \lambda_2 < s+\gamma+1$ satisfying $\lambda_2 < (n+\alpha)/p+\gamma$. Then

$$II \lesssim \int_0^\infty \int_0^\infty \frac{r^{k+\gamma}}{(z_n+r)^{k+1}(1+r)^{\lambda_2}}\, dr\, \frac{(1+t)^{(n+\alpha)/p-(k+1)}}{(1+t)^{(n+\alpha)/p+\gamma-\lambda_2}} t^{k-s-1}\, dt$$
$$< \infty,$$

because $k+\gamma > -1$, $1+\lambda_2-\gamma > 1$, $k-s-1 > -1$ and $\gamma-\lambda_2+s+2 > 1$. Therefore the proof is complete. □

In the following theorem, we show that every function $u \in b^p_\alpha$ can be reproduced by its fractional normal derivatives.

THEOREM 4.7. *Let $\alpha > -1$, $1 \leq p < \infty$ and let $\alpha+1 < (\beta+1)p$. If $u \in b^p_\alpha$ and $(1+\alpha)/p+\gamma > 0$, then for $z \in \mathbf{H}$*

$$\Pi_\beta(w_n^\gamma \mathcal{D}^\gamma u)(z) = \frac{C_\beta}{C_{\beta+\gamma}} u(z),$$

where

$$l = \begin{cases} [\gamma] & \text{if } \gamma > -1 \\ 0 & \text{if } \gamma \leq -1. \end{cases}$$

Proof. Let $z \in \mathbf{H}$. Then we know from Lemma 4.5 that $\Pi_\beta(w_n^\gamma \mathcal{D}^\gamma u)(z)$ is well defined and clearly, $u_\delta \in b^p_\alpha(\mathbf{H}_\delta)$ for $\delta > 0$. Suppose β and γ are not integers. Set $k = [\beta]$. Then $k-\beta > 0$. Hence, by similar argument to the proof of Proposition 2.4, we see that $\Pi_\beta(w_n^\gamma \mathcal{D}^\gamma u_\delta)(z)$ becomes

$$\int_{\mathbf{H}} w_n^\gamma\, \mathcal{D}^\gamma u_\delta(w) R_\beta(z,w)\, dV_\beta(w)$$
$$= \frac{C_\beta}{\Gamma(k-\beta)} \int_{\mathbf{H}} \int_0^\infty [D^{k+1} P_z(w', (1+t)w_n)][\mathcal{D}^\gamma u_\delta(w) t^{k-\beta-1}]\, w_n^{\gamma+k} dt\, dw,$$

after applying change of variable $t \mapsto tw_n$. Note from (2.3) that

$$
\begin{aligned}
\left|\mathcal{D}^\gamma u_\delta(w)\right| &\lesssim \int_0^\infty \left|D^l u_\delta(w', w_n + t)\right| t^{l-\gamma-1}\, dt \\
&\lesssim \int_0^\infty \frac{t^{l-\gamma-1}}{(w_n+\delta+t)^{(n+\alpha)/p+l}}\, dt \\
&\approx \frac{1}{(w_n+\delta)^{(n+\alpha)/p+\gamma}} \approx \frac{1}{(1+w_n)^{(n+\alpha)/p+\gamma}}.
\end{aligned}
$$

Because $\alpha+1<(\beta+1)p$ and $(1+\alpha)/p+\gamma>0$, we have $\beta+\gamma+1>0$ and $(n+\alpha)/p+\gamma>0$. Therefore we know from (2.2) and Lemma 4.6 that

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbf{H}} \left|\left[\mathcal{D}^\gamma u_\delta(w)\right] D^{k+1} P_z\big(w', (1+t)w_n\big)\right| w_n^{\gamma+k}\, dw\, t^{k-\beta-1}\, dt \\
&\lesssim \int_0^\infty \int_{\mathbf{H}} \frac{w_n^{\gamma+k}}{|z-(w', -(1+t)w_n)|^{n+k}(1+w_n)^{(n+\alpha)/p+\gamma}}\, dw\ t^{k-\beta-1}\, dt \\
&< \infty.
\end{aligned}
$$

This estimate guarantees us switching the order of integrations above. Therefore we know from Fubini's Theorem that $\Pi_\beta(w_n^\gamma \mathcal{D}^\gamma u_\delta)(z)$ equals

$$
\frac{C_\beta}{\Gamma(k-\beta)} \int_0^\infty \int_{\mathbf{H}} \left[D^{k+1} P_z\big(w', (1+t)w_n\big)\right] \left[\mathcal{D}^\gamma u_\delta(w)\right] w_n^{\gamma+k}\, dw\, t^{k-\beta-1}\, dt.
$$

Hence, we see from the definition of $\mathcal{D}^\gamma u(w)$, Fubini's Theorem and then change of variable $s \mapsto sw_n$ that $\Pi_\beta(w_n^\gamma \mathcal{D}^\gamma u_\delta)(z)$ becomes

$$
\begin{aligned}
&\frac{C_\beta}{\Gamma(k-\beta)\Gamma(l-\gamma)} \int_0^\infty \int_{\mathbf{H}} \left[D^{k+1} P_z\big(w', w_n(1+t)\big)\right] \\
&\qquad \int_0^\infty \left[D^l u_\delta(w', w_n+s)\right]\, s^{l-\gamma-1}\, ds\, w_n^{\gamma+k}\, dw\ t^{k-\beta-1}\, dt \\
&= \frac{C_\beta}{\Gamma(k-\beta)\Gamma(l-\gamma)} \int_0^\infty \int_0^\infty \int_{\mathbf{H}} \left[D^{k+1} P_z\big(w', (1+t)w_n\big)\right] \\
&\qquad \left[D^l u_\delta\big(w', (1+s)w_n\big)\right] w_n^{l+k}\, dw\ s^{l-\gamma-1}\, ds\ t^{k-\beta-1}\, dt.
\end{aligned}
$$

(Here we see with a simple estimate much easier than the proof of Lemma 4.6 that switching the order of integrations is permissible.) Therefore we know from Lemma 2.3 that the above becomes

$$
\left(\frac{(-1)^{l+k+1}(l+k)!\,C_\beta}{\Gamma(k-\beta)\Gamma(l-\gamma)} \int_0^\infty \int_0^\infty \frac{s^{l-\gamma-1} t^{k-\beta-1}}{(2+s+t)^{l+k+1}}\, ds\, dt\right) u_\delta(z).
$$

After changing of variables, we see that

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{s^{l-\gamma-1}\, t^{k-\beta-1}}{(2+s+t)^{l+k+1}}\, ds\, dt \\
&= \frac{1}{2^{\beta+\gamma+1}} \int_0^1 s^{\beta+\gamma}(1-s)^{l-\gamma-1}\, ds \int_0^1 t^{l+\beta}(1-t)^{k-\beta-1}\, dt \\
&= \frac{1}{2^{\beta+\gamma+1}} \frac{\Gamma(\beta+\gamma+1)\Gamma(l-\gamma)\Gamma(k-\beta)}{\Gamma(l+k+1)}.
\end{aligned}$$

Consequently, $\Pi_\beta(w_n^\gamma \mathcal{D}^\gamma u_\delta)(z) = C_\beta / C_{\beta+\gamma} u_\delta(z)$.

The case that either β or γ is an integer can be proved similarly.

Now, we can make our usual limiting argument for u_δ with Lemma 4.5 and Theorem 4.3 to obtain the desired result for an arbitrary function $u \in b_\alpha^p$. Therefore the proof is complete. □

In the following theorem, we show that the norm of a weighted Bergman function is equivalent to the norm of the fractional normal derivative of this function.

THEOREM 4.8. *Let $\alpha > -1$ and let $1 \le p < \infty$. If $(1+\alpha)/p + \gamma > 0$, then*

$$\|u\|_{L_\alpha^p(\mathbf{H})} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_{L_\alpha^p(\mathbf{H})}$$

as u ranges over all b_α^p-functions.

Proof. We know from Lemma 4.5 that $w_n^\gamma \mathcal{D}^\gamma u \in L_\alpha^p(\mathbf{H})$ for $u \in b_\alpha^p$. Thus we get from Theorem 4.7, Theorem 4.3 and Lemma 4.5 that

$$\|u\|_{L_\alpha^p(\mathbf{H})} \approx \|\Pi_{\alpha+1}(w_n^\gamma \mathcal{D}^\gamma u)\|_{L_\alpha^p(\mathbf{H})} \lesssim \|w_n^\gamma \mathcal{D}^\gamma u\|_{L_\alpha^p(\mathbf{H})} \lesssim \|u\|_{L_\alpha^p(\mathbf{H})}$$

as u ranges over all b_α^p-functions. This completes the proof. □

See Theorem 5.13 for the corresponding result of the case $p = \infty$.

5. The harmonic Bloch space as the dual space of b_α^1

It is shown in [10] that the dual of b^1 can be identified with the space of harmonic Bloch functions modulo constant because of the following b^1-cancellation property: If $u \in b^1$, then $\int_{\mathbf{H}} u(z)\, dV(z) = 0$. It is shown in [4] and [5] that this kind of vanishing property also holds not only for b_α^1 but for b_α^p with an appropriate range of p. However at the present paper, we only need the following b^1_α-*cancellation property*: If $u \in b^1_\alpha$, then

$$\int_{\mathbf{H}} u(z)\, dV_\alpha(z) = 0. \tag{5.1}$$

(In fact for $u \in b^1_\alpha$, $\int_{\partial\mathbf{H}} u(z', \delta)\, dz' = 0$ for every $\delta > 0$. See [4] and [5] for details and related facts.)

Now we describe the harmonic Bloch space. A harmonic function u on $\mathbf{H}$ is called a *Bloch function* if

$$\|u\|_{\mathcal{B}} = \sup w_n |\nabla u(w)| < \infty,$$

where the supremum is taken over all $w \in \mathbf{H}$ and ∇u denotes the gradient of u. We let $\mathcal{B}$ denote the set of Bloch functions on $\mathbf{H}$ and let $\widetilde{\mathcal{B}}$ denote the subspace of functions in $\mathcal{B}$ that vanish at z_0. The space $\widetilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\| \ \|_{\mathcal{B}}$.

In this section, we prove that the dual space of b^1_α can be also identified with $\widetilde{\mathcal{B}}$ as is the case for b^1. Because $R_\alpha(z, \cdot)$ is not in $L^1_\alpha(\mathbf{H})$, we modify R_α to $\widetilde{R}_\alpha$ so that for each $z \in \mathbf{H}$, $\widetilde{R}_\alpha(z, \cdot) \in L^1_\alpha(\mathbf{H})$. (See Proposition 5.2.) With this modified Bergman kernel $\widetilde{R}_\alpha$, we define $\widetilde{\Pi}_\alpha$ on $L^\infty(\mathbf{H})$ and then we show that $\widetilde{\Pi}_\alpha$ is bounded and linear from $L^\infty(\mathbf{H})$ into $\widetilde{\mathcal{B}}$. (In fact, this map is onto. See Corollary 5.10.) We use this map $\widetilde{\Pi}_\alpha$ to get $(b^1_\alpha)^* \cong \widetilde{\mathcal{B}}$.

For this purpose, we first define

$$\widetilde{R}_\alpha(z, w) = R_\alpha(z, w) - R_\alpha(z_0, w)$$

for $z, w \in \mathbf{H}$ and then we estimate the size of $\widetilde{R}_\alpha(z, w)$. To do so, we need a lemma.

LEMMA 5.1. *Let l and m be nonnegative integers. If $0 < a \le b$ and $0 < c \le d$, then*

$$\left| \frac{a^l}{b^{m+l}} - \frac{c^l}{d^{m+l}} \right| \lesssim (|a - c| + |b - d|) \left(\frac{1}{b^m d} + \frac{1}{b d^m} \right).$$

Proof. If $l = 0$, then the proof is trivial. So assume that $l > 0$. Without loss of generality, we may assume $b \le d$. Let $B = b^{(m+l)/l}$ and

$D = d^{(m+l)/l}$. Then,

$$
\begin{aligned}
\left|\frac{a^l}{B^l} - \frac{c^l}{D^l}\right| &\lesssim \frac{1}{b^{(l-1)m/l}}\left|\frac{a}{B} - \frac{c}{D}\right| \\
&\lesssim \frac{1}{b^{(l-1)m/l}}\left(a\left|\frac{1}{B} - \frac{1}{D}\right| + \frac{1}{D}|a-c|\right) \qquad (5.2)\\
&= \frac{a}{b^{(l-1)m/l}}\left|\frac{1}{B} - \frac{1}{D}\right| + \frac{|a-c|}{b^{(l-1)m/l}D}.
\end{aligned}
$$

Let I and II denote, respectively, the two summands of (5.2). Note that

$$d^{(m+l)/l} - b^{(m+l)/l} \leq \frac{m+l}{l}(d-b)d^{m/l}.$$

Because $a \leq b \leq d$, we easily see that

$$I \lesssim |b-d|\frac{a}{b^{(l-1)m/l}}\frac{d^{m/l}}{b^{(m+l)/l}d^{(m+l)/l}} \lesssim |b-d|\frac{1}{b^m d}.$$

Clearly, we have

$$II \lesssim |a-c|/b^m d.$$

Consequently,

$$I + II \lesssim (|a-c| + |b-d|)/b^m d$$

and the proof is complete. □

From this lemma, we easily get the following result.

PROPOSITION 5.2. *Let $\alpha > -1$. Then*

$$|\widetilde{R}_\alpha(z,w)| \lesssim \frac{|z-z_0|}{|z-\overline{w}|^{n+\alpha}|z_0-\overline{w}|} + \frac{|z-z_0|}{|z-\overline{w}||z_0-\overline{w}|^{n+\alpha}}$$

for all $z, w \in \mathbf{H}$.

Proof. From the definition of $P_z(w)$ and Lemma 5.1, we easily get that for each nonnegative integer k,

$$
\begin{aligned}
|D^{k+1}P_z(w) - D^{k+1}P_{z_0}(w)| &\lesssim \sum_{j=0}^{k+2}\left|\frac{(z_n+w_n)^j}{|z-\overline{w}|^{n+k+j}} - \frac{(1+w_n)^j}{|z_0-\overline{w}|^{n+k+j}}\right| \\
&\lesssim \frac{|z-z_0|}{|z-\overline{w}|^{n+k}|z_0-\overline{w}|} + \frac{|z-z_0|}{|z-\overline{w}||z_0-\overline{w}|^{n+k}}. \qquad (5.3)
\end{aligned}
$$

This completes the proof of this theorem for α is a nonnegative integer. So assume that α is not an integer. Let $k = [\alpha]$. Then k is a nonnegative integer and $k-\alpha>0$. Hence we see from the definition of $\widetilde{R}_\alpha(z,w)$ and (5.3) that $|\widetilde{R}_\alpha(z,w)|$ is less than or equal to some constant times

$$\begin{aligned}
&\int_0^\infty \left|D^{k+1}P_z(w',w_n+t)-D^{k+1}P_{z_0}(w',w_n+t)\right|t^{k-\alpha-1}\,dt\\
&\lesssim |z-z_0|\int_0^\infty \left\{\frac{t^{k-\alpha-1}}{|z-(w',-w_n-t)|^{n+k}|z_0-(w',-w_n-t)|}\right.\\
&\quad+\left.\frac{t^{k-\alpha-1}}{|z-(w',-w_n-t)||z_0-(w',-w_n-t)|^{n+k}}\right\}dt\\
&\lesssim \frac{|z-z_0|}{|z_0-\overline{w}|}\int_0^\infty \frac{t^{k-\alpha-1}}{(|z-\overline{w}|+t)^{n+k}}\,dt + \frac{|z-z_0|}{|z-\overline{w}|}\int_0^\infty \frac{t^{k-\alpha-1}}{(|z_0-\overline{w}|+t)^{n+k}}\,dt\\
&\approx \frac{|z-z_0|}{|z-\overline{w}|^{n+\alpha}|z_0-\overline{w}|}+\frac{|z-z_0|}{|z-\overline{w}||z_0-\overline{w}|^{n+\alpha}},
\end{aligned}$$

where we used the change of variable $t\mapsto |z-\overline{w}|t$ and $t\mapsto|z_0-\overline{w}|t$. Therefore the proof is complete. □

From Proposition 5.2, we easily see that $\widetilde{R}_\alpha(z,\cdot)\in L^1_\alpha(\mathbf{H})$ for each fixed $z\in\mathbf{H}$. Thus, we can define $\widetilde{\Pi}_\alpha$ on $L^\infty(\mathbf{H})$ by

$$\widetilde{\Pi}_\alpha f(z)=\int_{\mathbf{H}} f(w)\widetilde{R}_\alpha(z,w)\,dV_\alpha(w) \tag{5.4}$$

for $f\in L^\infty(\mathbf{H})$. Furthermore by passing the Laplacian through the integral in (5.4), we see that $\widetilde{\Pi}_\alpha f$ is harmonic on $\mathbf{H}$. Therefore to complete the proof of the following Proposition, we only need to show that $\widetilde{\Pi}_\alpha f\in\widetilde{\mathcal{B}}$ for $f\in L^\infty(\mathbf{H})$ with an appropriate norm bound.

Proposition 5.3. *Let $\alpha>-1$. Then $\widetilde{\Pi}_\alpha$ is a bounded linear map of $L^\infty(\mathbf{H})$ into $\widetilde{\mathcal{B}}$.*

Proof. Let $f\in L^\infty(\mathbf{H})$. Then, it is easily seen that $\widetilde{\Pi}_\alpha f(z_0)=0$. Note that for each j and for every $z,w\in\mathbf{H}$, $D_{z_j}\widetilde{R}_\alpha(z,w)=D_{z_j}R_\alpha(z,w)$. Therefore we have from (2.5) and Proposition 2.2 that for any j and for every $z\in\mathbf{H}$,

$$\begin{aligned}
\left|z_nD_{z_j}\widetilde{\Pi}_\alpha f(z)\right| &= z_n\left|\int_{\mathbf{H}} f(w)\left[D_{z_j}\widetilde{R}_\alpha(z,w)\right]\,w_n^\alpha\,dw\right|\\
&\lesssim \|f\|_\infty z_n\int_{\mathbf{H}}\frac{w_n^\alpha}{|z-\overline{w}|^{n+\alpha+1}}\,dw\\
&\lesssim \|f\|_\infty.
\end{aligned}$$

This shows that $\widetilde{\Pi}_\alpha f\in\widetilde{\mathcal{B}}$ with $\|\widetilde{\Pi}_\alpha f\|_{\mathcal{B}}\lesssim\|f\|_\infty$, as desired. □

We estimate some integral in the following lemma which is used in Proposition 5.5.

LEMMA 5.4. *Let*

$$I_\alpha(a) = \int_0^\infty \left\{ \frac{t^\alpha}{(a+t)^{\alpha+1}(1+t)} + \frac{t^\alpha}{(a+t)(1+t)^{\alpha+1}} \right\} dt$$

for $\alpha > -1$, $a > 0$ *and let* $\lambda \in (0,1)$ *with* $\alpha + \lambda > 0$. *Then* $I_\alpha(a) \lesssim a^{\lambda-1}$ *if* $0 < a \le 1$ *and* $I_\alpha(a) \lesssim a^{-\lambda}$ *if* $a > 1$.

Proof. If $0 < a \le 1$, then

$$I_\alpha(a) \approx \int_0^\infty \frac{t^\alpha}{(a+t)^{\alpha+1}(1+t)}\, dt \lesssim \int_0^\infty \frac{t^\alpha}{a^{1-\lambda} t^{\alpha+\lambda}(1+t)}\, dt \ \approx\ a^{\lambda-1}.$$

If $a > 1$, then

$$I_\alpha(a) \approx \int_0^\infty \frac{t^\alpha}{(1+t)^{\alpha+1}(a+t)}\, dt \lesssim \int_0^\infty \frac{t^\alpha}{t^{\alpha+\lambda}(a+t)}\, dt \ \approx\ a^{-\lambda}.$$

Therefore the proof is complete. □

From Lemma 5.4, we get the following proposition easily which is used in Proposition 5.7 and Proposition 5.11 to guarantee switching the order of integrations.

PROPOSITION 5.5. *Let* $\alpha > -1$ *and let* $u \in \mathcal{S}_{n+\alpha+1}$. *Then*

$$\int_{\mathbf{H}} \int_{\mathbf{H}} |u(z) \widetilde{R}_\alpha(z,w)|\, dV_\alpha(w)\, dV_\alpha(z) < \infty.$$

Proof. Because $u \in \mathcal{S}_{n+\alpha+1}$, we see that $|u(z)| \lesssim |z - \overline{z}_0|^{-(n+\alpha+1)}$ on $\mathbf{H}$. Thus we obtain from Proposition 5.2 and polar coordinates that

$$\begin{aligned}
&\int_{\mathbf{H}} |u(z) \widetilde{R}_\alpha(z,w)|\, dV_\alpha(w) \\
&\lesssim \frac{1}{|z-\overline{z}_0|^{n+\alpha}} \int_{\mathbf{H}} \left\{ \frac{w_n^\alpha}{|z-\overline{w}|^{n+\alpha}(1+w_n)} + \frac{w_n^\alpha}{(z_n+w_n)|z_0-\overline{w}|^{n+\alpha}} \right\} dw \\
&\lesssim \frac{1}{|z-\overline{z}_0|^{n+\alpha}} \left(\int_0^\infty \frac{w_n^\alpha}{(1+w_n)} \int_0^\infty \frac{r^{n-2}}{\big(r+(z_n+w_n)\big)^{n+\alpha}}\, dr\, dw_n \right.
\end{aligned}$$

$$+\int_0^\infty \frac{w_n^\alpha}{(z_n+w_n)}\int_0^\infty \frac{r^{n-2}}{\big(r+(1+w_n)\big)^{n+\alpha}}\,dr\,dw_n\Bigg)$$

$$\approx \frac{1}{|z-\overline{z}_0|^{n+\alpha}}\int_0^\infty \left\{\frac{w_n^\alpha}{(z_n+w_n)^{\alpha+1}(1+w_n)}\right.$$

$$\left. + \frac{w_n^\alpha}{(z_n+w_n)(1+w_n)^{\alpha+1}}\right\}dw_n.$$

Choose $\lambda \in (0,1)$ satisfying $\alpha+\lambda>0$. Then we get from Lemma 5.4 and polar coordinates that

$$\int_{\mathbf{H}}\int_{\mathbf{H}} |u(z)\widetilde{R}_\alpha(z,w)|\,dV_\alpha(w)\,dV_\alpha(z)$$

$$\lesssim \int_0^1\int_{\partial\mathbf{H}} \frac{1}{\big(|z'|+(1+z_n)\big)^{n+\alpha}}\,dz'\,z_n^{\alpha+\lambda-1}\,dz_n$$

$$+\int_1^\infty\int_{\partial\mathbf{H}} \frac{1}{\big(|z'|+(1+z_n)\big)^{n+\alpha}}\,dz'\,z_n^{\alpha-\lambda}\,dz_n$$

$$\lesssim \int_0^1 \frac{z_n^{\alpha+\lambda-1}}{(1+z_n)^{\alpha+1}}\,dz_n + \int_1^\infty \frac{z_n^{\alpha-\lambda}}{(1+z_n)^{\alpha+1}}\,dz_n$$

$$<\infty,$$

because $0<\lambda<1$ and $\alpha+\lambda>0$. This completes the proof. □

It is well known that any Bloch function grows at most like a logarithmic function. More precisely if $v \in \widetilde{\mathcal{B}}$, then

$$|v(z',z_n)| \le 2\|v\|_{\mathcal{B}}\big(1+|\log z_n|+2\log(1+|z'|)\big). \tag{5.5}$$

We can check that there are functions $u \in b_\alpha^1$ and $v \in \widetilde{\mathcal{B}}$ such that $uv \notin L_\alpha^1(\mathbf{H})$. However if $u \in \mathcal{S}_{n+\alpha+1}$, then $uv \in L_\alpha^1(\mathbf{H})$ for any $v \in \widetilde{\mathcal{B}}$. We show this in the following lemma.

LEMMA 5.6. *If $\alpha>-1$, then for any $u \in \mathcal{S}_{n+\alpha+1}$ and for every $v \in \widetilde{\mathcal{B}}$,*

$$\int_{\mathbf{H}} |u(z)v(z)|\,dV_\alpha(z) < \infty.$$

Proof. Let $u \in \mathcal{S}_{n+\alpha+1}$ and let $v \in \widetilde{\mathcal{B}}$. Then we know that

$$|u(z)| \lesssim \frac{1}{|z-\overline{z}_0|^{n+\alpha+1}}$$

on $\mathbf{H}$. Hence we get from (5.5) that

$$\int_{\mathbf{H}} |u(z)v(z)|\,dV_\alpha(z) \lesssim \int_{\mathbf{H}} \frac{z_n^\alpha\big(1+|\log z_n|+\log(1+|z'|)\big)}{|z-\overline{z}_0|^{n+\alpha+1}}\,dz.$$

Note that

$$\log(1+|z'|) \lesssim |z'|^{1/2} \lesssim |z|^{1/2}.$$

Because $\alpha+1>0$, we can choose λ satisfying $0<\lambda<\alpha+1$. Then we have

$$|\log z_n| \lesssim z_n^{1/2}+z_n^{-\lambda} \lesssim |z|^{1/2}+z_n^{-\lambda}.$$

Therefore after applying Proposition 2.2, we see that

$$\begin{aligned}\int_{\mathbf{H}} |u(z)v(z)|\, dV_\alpha(z) &\lesssim \int_{\mathbf{H}} \frac{z_n^\alpha + z_n^{\alpha-\lambda}}{|z-\overline{z}_0|^{n+\alpha+1}}\, dz + \int_{\mathbf{H}} \frac{z_n^\alpha}{|z-\overline{z}_0|^{n+\alpha+1/2}}\, dz \\ &< \infty.\end{aligned}$$

This completes the proof. □

In the following proposition, we show that for each bounded linear functional defined on b^1_α, there is a Bloch function which induces this functional.

PROPOSITION 5.7. *Let $\alpha>-1$. If $\Lambda \in (b^1_\alpha)^*$, then there is $v \in \widetilde{\mathcal{B}}$ such that*

$$\Lambda(u) = \int_{\mathbf{H}} u(z)v(z)\, dV_\alpha(z)$$

for all $u \in \mathcal{S}_{n+\alpha+1}$. Moreover, $\|v\|_{\mathcal{B}} \lesssim \|\Lambda\|$.

Proof. Let $\Lambda \in (b^1_\alpha)^*$. Then we know from Hahn-Banach Theorem, Riesz representation Theorem and Theorem 2.5 that there is a function $f \in (L^1_\alpha(\mathbf{H}))^* = L^\infty(\mathbf{H})$ such that

$$\Lambda(u) = \int_{\mathbf{H}} u(z)f(z)\, dV_\alpha(z)$$

for all $u \in b^1_\alpha$ with $\|\Lambda\| = \|f\|_\infty$. Let $v = \widetilde{\Pi}_\alpha f$. Then we know from Propostion 5.3 that $v \in \widetilde{\mathcal{B}}$ and $\|v\|_{\mathcal{B}} \lesssim \|f\|_\infty = \|\Lambda\|$. If $u \in \mathcal{S}_{n+\alpha+1}$, then we get from Proposition 5.5 and b^1_α-cancellation property, (5.1) that

$$\begin{aligned}\int_{\mathbf{H}} u(z)v(z)\, dV_\alpha(z) &= \int_{\mathbf{H}} u(z) \int_{\mathbf{H}} f(w)\widetilde{R}_\alpha(z,w)\, dV_\alpha(w)\, dV_\alpha(z) \\ &= \int_{\mathbf{H}} f(w) \int_{\mathbf{H}} u(z)\widetilde{R}_\alpha(z,w)\, dV_\alpha(z)\, dV_\alpha(w) \\ &= \int_{\mathbf{H}} f(w) \int_{\mathbf{H}} u(z)R_\alpha(w,z)\, dV_\alpha(z)\, dV_\alpha(w) \\ &= \Lambda(u),\end{aligned}$$

and this completes the proof. □

We will use the following lemma to prove Theorem 5.9 and Theorem 5.13.

LEMMA 5.8. *If* $v \in \widetilde{\mathcal{B}}$, *then for* $\gamma > 0$

$$\|w_n^\gamma \mathcal{D}^\gamma v\|_\infty \lesssim \|v\|_{\mathcal{B}}.$$

Proof. If γ is a positive integer, then we know from (5.13) of [10] that

$$|D^\gamma v(w)| \lesssim \frac{\|v\|_{\mathcal{B}}}{w_n^\gamma}. \tag{5.6}$$

Assume that γ is not an integer. Let $k = [\gamma]$. Then k is a positive integer and $k - \gamma > 0$. Therefore we see from (5.6) that

$$\begin{aligned} |\mathcal{D}^\gamma v(w)| &\lesssim \int_0^\infty |D^k v(w', w_n + t)| t^{k-\gamma-1}\, dt \\ &\lesssim \int_0^\infty \frac{t^{k-\gamma-1}}{(w_n+t)^k}\, dt\, \|v\|_{\mathcal{B}} \\ &\lesssim \frac{\|v\|_{\mathcal{B}}}{w_n^\gamma}, \end{aligned} \tag{5.7}$$

after applying change of variable $t \mapsto w_n t$. This completes the proof. □

Now, we extend the domain of $\widetilde{\Pi}_\alpha$ to the set of all functions f for which the integrand in (5.4) belongs to $L^1_\alpha(\mathbf{H})$. Then we know from Lemma 5.6 that $\widetilde{\Pi}_\alpha v$ is well defined for every $v \in \widetilde{\mathcal{B}}$.

THEOREM 5.9. *Let* $\alpha > -1$ *and let* $\gamma \geq 0$. *If* $v \in \widetilde{\mathcal{B}}$, *then for each* $z \in \mathbf{H}$

$$\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)(z) = \frac{C_\alpha}{C_{\alpha+\gamma}} v(z).$$

Proof. Fix $z \in \mathbf{H}$. We know from Lemma 5.6 and Lemma 5.8 that $\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)(z)$ is well defined for any range $\gamma \geq 0$. Let $\delta > 0$. Because

$$w_n|\nabla v_\delta(w)| \leq (w_n + \delta)|\nabla v(w', w_n + \delta)| \leq \|v\|_{\mathcal{B}}$$

for $w \in \mathbf{H}$, we see that $v_\delta \in \mathcal{B}$. However v_δ need not be in $\widetilde{\mathcal{B}}$.

Suppose that α and γ are not integers. Let $k = [\alpha]$. Then $\gamma > 0$ and $k - \alpha > 0$. Therefore after applying the change of variable $t \mapsto t w_n$, we see that $\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v_\delta)(z)$ becomes

$$\frac{C_\alpha}{\Gamma(k-\alpha)} \int_{\mathbf{H}} \int_0^\infty \left[D^{k+1}\widetilde{P}_z(w', (1+t)w_n)\right] t^{k-\alpha-1}\, dt\, \left[\mathcal{D}^\gamma v_\delta(w)\right] w_n^{\gamma+k}\, dw,$$

where $\widetilde{P}_z = P_z - P_{z_0}$. Note from (5.3) and (5.7) that

$$\left|D^{k+1}\widetilde{P}_z(w)\right| \lesssim |z_0 - \overline{w}|^{-(n+k+1)}, \quad |\mathcal{D}^\gamma v_\delta(w)| \lesssim (1 + w_n)^{-\gamma}. \tag{5.8}$$

We can check with this estimate (5.8) that we can apply Lemma 4.6 to the integral above. After switching the order of integrations above, we see from change of variable $s \mapsto sw_n$ that $\Gamma(k-\alpha)/C_\alpha \widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v_\delta)(z)$ becomes

$$\int_0^\infty \int_{\mathbf{H}} \left[D^{k+1}\widetilde{P}_z(w',(1+t)w_n)\right] \left[\mathcal{D}^\gamma v_\delta(w)\right] w_n^{\gamma+k}\, dw\, t^{k-\alpha-1}\, dt$$

$$(5.9) \quad = \frac{1}{\Gamma(m-\gamma)} \int_0^\infty \int_0^\infty \int_{\mathbf{H}} \left[D^{k+1}\widetilde{P}_z(w',(1+t)w_n)\right] \times \left[D^m v_\delta(w',(1+s)w_n)\right] w_n^{m+k}\, dw\, s^{m-\gamma-1}\, ds\, t^{k-\alpha-1}\, dt,$$

where $m = \lceil\gamma\rceil$. Note that $\gamma > 0$ implies $m \geq 1$. The estimate (5.8) also allows us to integrate by parts $(k+1)$-times in the inner integral of (5.9). Therefore we see that

$$(5.10) \quad \int_{\mathbf{H}} \left[D^{k+1}\widetilde{P}_z(w',(1+t)w_n)\right]\left[D^m v_\delta(w',(1+s)w_n)\right] w_n^{m+k}\, dw = \frac{(-1)^{k+1}}{(1+t)^{k+1}} \sum_{j=0}^{k+1} C(k+1,j)(1+s)^j \frac{(m+k)!}{(m+j-1)!}$$

$$(5.11) \quad \times \int_{\mathbf{H}} \widetilde{P}_z(w',(1+t)w_n)\left[D^{m+j} v_\delta(w',(1+s)w_n)\right] w_n^{m+j-1}\, dw.$$

Note that $v_\delta(z) - v_\delta(z_0) \in \widetilde{\mathcal{B}}$ as a function of $z \in \mathbf{H}$. Therefore we know from (5.6) that for each positive integer l,

$$|D^l v_\delta(z)| = \left|D^l\left(v_\delta(z) - v_\delta(z_0)\right)\right| \lesssim \frac{1}{(z_n+\delta)^l}$$

for every $z \in \mathbf{H}$. This shows that $D^l v_\delta$ is bounded and harmonic on $\overline{\mathbf{H}}$ for each positive integer l. Hence, we see that the integral in (5.11) becomes

$$(5.12) \quad \int_0^\infty \left\{\left[D^{m+j} v_\delta(z+(0,(2+t+s)w_n))\right] - \left[D^{m+j} v_\delta(z_0+(0,(2+t+s)w_n))\right]\right\} w_n^{m+j-1}\, dw_n.$$

The estimate (5.6) implies, after applying integration by parts $m+j-1$ times in (5.12), that (5.10) becomes

$$\frac{(-1)^{m+k+1}(m+k)!}{(2+t+s)^{m+k+1}} \left(v_\delta(z) - v_\delta(z_0)\right).$$

Consequently, $\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}_{w_n}^\gamma v_\delta)(z)$ is equal to

$$\left(\frac{(-1)^{m+k+1}(m+k)!C_\alpha}{\Gamma(k-\alpha)\Gamma(m-\gamma)} \int_0^\infty \int_0^\infty \frac{s^{m-\gamma-1}t^{k-\alpha-1}}{(2+t+s)^{m+k+1}}\, ds\, dt\right) \times \big(v_\delta(z) - v_\delta(z_0)\big).$$

We see that the quantity in parenthesis above becomes $C_\alpha / C_{\alpha+\gamma}$. Therefore we get

$$\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v_\delta)(z) = \frac{C_\alpha}{C_{\alpha+\gamma}} \big(v_\delta(z) - v_\delta(z_0)\big).$$

We can make our usual limiting argument for v_δ to obtain the desired result for an arbitrary function $v \in \widetilde{\mathcal{B}}$ from Lemma 5.8 and Proposition 5.3. The remaining cases can be proved similarly using the estimate (5.5) if necessary ($\gamma = 0$ case). Therefore the proof is complete. □

From the above, we have $\widetilde{\Pi}_\alpha v(z) = v(z)$ for every $v \in \widetilde{\mathcal{B}}$ and for each $z \in \mathbf{H}$. The following result follows directly from Proposition 5.3 and Theorem 5.9.

COROLLARY 5.10. *For $\alpha > -1$, $\widetilde{\Pi}_\alpha$ is a bounded linear map from $L^\infty(\mathbf{H})$ onto $\widetilde{\mathcal{B}}$.*

Now we show in the following proposition that every Bloch function induces a bounded linear functional on $\mathcal{S}_{n+\alpha+1}$, hence on b_α^1.

PROPOSITION 5.11. *If $\alpha > -1$ and $v \in \widetilde{\mathcal{B}}$, then the map Λ defined by*

$$\Lambda(u) = \int_{\mathbf{H}} u(z)v(z)\, dV_\alpha(z)$$

is a bounded linear functional on $\mathcal{S}_{n+\alpha+1}$ with respect to L_α^1-norm. Moreover, we have $\|\Lambda\| \lesssim \|v\|_{\mathcal{B}}$.

Proof. We know from Lemma 5.6 that $uv \in L_\alpha^1(\mathbf{H})$ whenever $u \in \mathcal{S}_{n+\alpha+1}$. We also know from Theorem 5.9 that

$$v(z) = \frac{C_{2\alpha+1}}{C_\alpha} \widetilde{\Pi}_\alpha(w_n^{\alpha+1}\mathcal{D}^{\alpha+1}v)(z).$$

Therefore we have

$$\Lambda(u) = \frac{C_{2\alpha+1}}{C_\alpha} \int_{\mathbf{H}} u(z) \int_{\mathbf{H}} w_n^{\alpha+1}\big[\mathcal{D}^{\alpha+1}v(w)\big] \widetilde{R}_\alpha(z,w)\, dV_\alpha(w)\, dV_\alpha(z).$$

Note from Lemma 5.8 that $\|w_n^{\alpha+1}\mathcal{D}^{\alpha+1}v\|_\infty \lesssim \|v\|_{\mathcal{B}}$. Then we know from Proposition 5.5 that we can switch the order of integrations above.

Thus,

$$\begin{aligned}\Lambda(u) &= \frac{C_{2\alpha+1}}{C_\alpha}\int_{\mathbf{H}} w_n^{\alpha+1}\big[\mathcal{D}^{\alpha+1}v(w)\big]\int_{\mathbf{H}} u(z)\widetilde{R}_\alpha(z,w)\,dV_\alpha(z)\,dV_\alpha(w)\\ &= \frac{C_{2\alpha+1}}{C_\alpha}\int_{\mathbf{H}} w_n^{\alpha+1}\big[\mathcal{D}^{\alpha+1}v(w)\big]u(w)\,dV_\alpha(w),\end{aligned}$$

where we used b^1_α-cancellation property (5.1). This shows that

$$\|\Lambda(u)\| \lesssim \|v\|_{\mathcal{B}}\|u\|_{L^1_\alpha(\mathbf{H})},$$

as desired and the proof is complete. □

By combining Proposition 5.7 and Proposition 5.11, we get the following duality result easily.

THEOREM 5.12. $(b^1_\alpha)^* \cong \widetilde{\mathcal{B}}$.

Proof. Define a map $\Phi : \widetilde{\mathcal{B}} \to (b^1_\alpha)^*$ by $\Phi(v) = \Lambda_v$, where

$$\Lambda_v(u) = \int_{\mathbf{H}} u(z)v(z)\,dV_\alpha(z)$$

for $u \in \mathcal{S}_{n+\alpha+1}$. Then we know from Proposition 5.11 that the linear map Φ is a well-defined bounded map with $\|\Lambda_v\| \lesssim \|v\|_{\mathcal{B}}$. We also know from Proposition 5.7 that Φ is onto and $\|v\|_{\mathcal{B}} \lesssim \|\Lambda_v\|$. Consequently, $\|v\|_{\mathcal{B}} \approx \|\Lambda_v\|$.

Suppose Λ_v is the zero functional on b^1_α for some $v \in \widetilde{\mathcal{B}}$. Note that for each fixed $z \in \mathbf{H}$, $\widetilde{R}_\alpha(z,\cdot) \in \mathcal{S}_{n+\alpha+1}$ by Proposition 5.2. Then Theorem 5.9 with $\gamma = 0$ case implies that

$$0 \;=\; \Lambda_v\big(\widetilde{R}_\alpha(z,\cdot)\big) \;=\; \widetilde{\Pi}_\alpha v(z) \;=\; v(z)$$

for all $z \in \mathbf{H}$. This shows that Φ is an one-to-one map. Therefore the proof is complete. □

In the following theorem, we show the corresponding result of the case $p = \infty$ to Theorem 4.8.

THEOREM 5.13. *If $\gamma \geq 0$, then $\|v\|_{\mathcal{B}} \approx \|w_n^\gamma \mathcal{D}^\gamma v\|_\infty$ as v ranges over all functions in $\widetilde{\mathcal{B}}$.*

Proof. We may assume that $\gamma > 0$. Let $v \in \widetilde{\mathcal{B}}$. Then we know from Lemma 5.8 that $w_n^\gamma \mathcal{D}^\gamma v \in L^\infty(\mathbf{H})$. Thus we see from Theorem 5.9, Proposition 5.3 and then Lemma 5.8 that

$$\|v\|_{\mathcal{B}} \approx \|\widetilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)\|_{\mathcal{B}} \lesssim \|w_n^\gamma D^\gamma v\|_\infty \lesssim \|v\|_{\mathcal{B}}.$$

Therefore the proof is complete. □

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