STRASSEN'S FUNCTIONAL LIL
FOR d-DIMENSIONAL SELF-SIMILAR
GAUSSIAN PROCESS IN HÖLDER NORM

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ABSTRACT. In this paper, based on large deviation probabilities on Gaussian random vectors, we obtain Strassen's functional LIL for d-dimensional self-similar Gaussian process in Hölder norm via estimating large deviation probabilities for d-dimensional self-similar Gaussian process in Hölder norm.

1. Introduction and results

The functional law of the iterated logarithm (LIL) for a Brownian motion (BM), fractional Brownian motions (FBM), Ornstein-Uhlendbeck (OU) processes and related Gaussian processes have been studied in various direction. In particular, Strassen[16] proved that as $t \uparrow \infty$, $\left\{ \frac{X(st)}{(2tLLt)^{1/2}} ; 0 \le s \le 1 \right\}$ is a.s. relatively compact in $C[0, 1]$, with cluster set equal to the unit ball in a reproducing kernel Hilbert space (RKHS) connected with a Brownian motion under the sup-norm. The result was extended to fractional Brownian motions under the sup-norm in Oodaira[14], and also Monrad and Rootzén[13] refined the Strassen law of the iterated logarithm for fractional Brownian motions under the sup-norm. Using Schilder's theorem giving large deviation estimates for the Brownian motion with the sup-norm replaced by any Hölder norm with exponent $\alpha < 1/2$, Baldi, Ben Arous and Kerkyacharian[2] investigated the Strassen law of the iterated logarithm for Brownian motion under Hölder norm. In this paper, based on large deviation probabilities on

Gaussian random vectors, we obtain Strassen's functional LIL for d-dimensional self-similar Gaussian process in Hölder norm via estimating large deviation probabilities for d-dimensional self-similar Gaussian process in Hölder norm.

Let us denote by $C_0[0,1]^d$ the Banach space of all continuous functions $f = (f_1, \cdots, f_d) : [0,1] \to R^d$ with value zero at the origin, and let φ be a strictly positive function on $[0,1]$ with $\varphi(0) = 0$ satisfying

(a) $\varphi(t)$ is non-decreasing and $\varphi(2t) \leq A\varphi(t)$ for $0 \leq t \leq 1/2$,

(b) $\int_0^\delta \varphi(t)/t\, dt \leq L\varphi(\delta)$ for $0 < \delta \leq 1$,

(c) $\delta \int_\delta^1 \varphi(t)/t^2\, dt \leq M\varphi(\delta)$ for $0 < \delta \leq 1$,

where A, M, L are positive constants. For $f \in C_0[0,1]^d$, define the Hölder norm

$$\|f\|_{\varphi,r,v} = \sup_{r \leq s < t \leq v} \max_{1 \leq i \leq d} \frac{|f_i(t) - f_i(s)|}{\varphi(t-s)}, \quad \|f\|_\varphi = \|f\|_{\varphi,0,1}.$$

For every $\delta > 0$ let

$$m_f(\delta) = \sup_{\substack{0 \leq s < t \leq 1 \\ t-s \leq \delta}} \max_{1 \leq i \leq d} \frac{|f_i(t) - f_i(s)|}{\varphi(t-s)}.$$

Then the modulus of continuity for f is $\varphi(\delta)m_f(\delta)$. We shall denote by $C_0^\varphi[0,1]^d$ the subspace of $C_0[0,1]^d$ of all functions such that $\lim_{\delta \to 0} m_f(\delta) = 0$. It is clear from [5, 6] that $C_0^\varphi[0,1]^d$ is a closed subspace of $C_0[0,1]^d$, so that it is a separable Banach space endowed with the Hölder norm $\|\cdot\|_\varphi$, whereas $C_0[0,1]^d$ not.

Let $\{X_j(t); 0 \leq t < \infty\}$, $j = 1, \cdots, d$ be independent real-valued centered self-similar Gaussian processes with $X_j(0) = 0$ and $E\{X_j(t) - X_j(s)\}^2 = \sigma_j^2(|t-s|)$, where $\sigma_j(t)$ are positive nondecreasing continuous functions of $t > 0$. Assume that $X_j(t)$ has continuous covariance function

$$(1.1) \quad R_j(s,t) = E\{X_j(s)X_j(t)\} = \int_R \left(e^{is\lambda} - 1\right)\left(e^{-it\lambda} - 1\right)\Delta_j(d\lambda),$$

where the symmetric spectral measure Δ_j satisfies

$$\int_R \frac{\lambda^2}{1+\lambda^2} \Delta_j(d\lambda) < \infty.$$

There exists a centered, complex-valued Gaussian random measure $W_j(d\lambda)$ such that

$$(1.2) \qquad X_j(t) = \int_R (e^{it\lambda} - 1) W_j(d\lambda).$$

The measures W_j and Δ_j are related by the identity $E\{W_j(E)\overline{W_j(F)}\} = \Delta_j(E \cap F)$ for all Borel sets E and F in R and $W_j(-E) = \overline{W_j(E)}$. Further assume that for $j = 1, \cdots, d$,

   (i) $\sigma_j(t)$ are regularly varying functions of $t > 0$ with exponent γ_j at ∞ for some $0 < \gamma_j < 1$,

   (ii) $\lim_{t \to 0} \sigma_j(t) \sqrt{\log(1/t)}/\varphi(t) = 0$,

   (iii) $\sigma_j(t) - \sigma_j(s) \le \sigma_j(t - s)$ for $0 \le s \le t < \infty$.

Let $X^d(t) = (X_1(t), \cdots, X_d(t)) \in R^d, t \in [0, \infty)$, be a d-dimensional self-similar Gaussian process. Noting that the modulus of continuity of X_j is $\sqrt{2\sigma_j^2(h)\log(1/h)}$ and using condition (ii), the sample paths of X_j are φ-Hölder continuous. Hence we may consider $X^d(t)$ as a random variable taking values in $C_0^\varphi[0,1]^d$.

Let $H(R) = \otimes_{j=1}^d H(R_j) \subset C_0^\varphi[0,1]^d$ be the RKHS with reproducing kernel (r.k.) function $R(s,t) = \otimes_{j=1}^d R_j(s,t)$, endowed with the inner product

$$\langle f, g \rangle_{H(R)} = \sum_{j=1}^d \langle f_j, g_j \rangle_{H(R_j)},$$

where $H(R_j)$ is the RKHS corresponding to r.k. function $R_j(s,t) = E\{X_j(s)X_j(t)\}$, $0 \le s, t \le 1$, and let us define the set K by

$$K = \left\{ h \in H(R) : \|h\|_{H(R)} \le 1 \right\},$$

where $\| \cdot \|_{H(R)}$ denotes the norm of $H(R)$, then $H(R)$ is the RKHS corresponding to the centered Gaussian measure μ on the separable Banach space $C_0^\varphi[0,1]^d$ induced by $\{X^d(t); t \ge 0\}$, and the set K is the unit ball of $H(R)$.

Throughout this paper for $\varepsilon > 0$, define

$$K_\varepsilon = \left\{ g \in C_0^\varphi[0,1]^d : \inf_{f \in K} \|g - f\|_\varphi < \varepsilon \right\},$$

and for $0 < T < \infty$, define

$$Z_T(x) = \eta_T^{-1} X^d(xT), \quad 0 \le x \le 1,$$

where

$$\eta_T = \left\{ 2\sigma^{*2}(T) \log\log T \right\}^{1/2} \quad \text{and} \quad \sigma^*(T) = \max_{1 \le j \le d} \sigma_j(T).$$

Our main theorem is as follows:

THEOREM 1.1. *With probability one* $\{Z_T(x); 0 \le x \le 1, T \ge 3\}$ *(as* $T \to \infty$*) is relatively compact in* $C_0^\varphi[0,1]^d$, *and the set of its limit points is* K, *specifically, we have*

$$(1.3) \qquad \lim_{T \to \infty} \inf_{f \in K} \| Z_T(\cdot) - f(\cdot) \|_\varphi = 0 \quad \text{a.s.}$$

and for any $f \in K$

$$(1.4) \qquad \lim_{T \to \infty} \| Z_T(\cdot) - f(\cdot) \|_\varphi = 0 \quad \text{a.s.}$$

REMARK. *Let* $p \ge 1$, *for* $f = (f_1, \cdots, f_d) \in C_0^\varphi[0,1]^d$, *define norm* $\|f\|_{l^p} = \sup_{0 \le x \le 1} \left(\sum_{j=1}^d |f_j(x)|^p \right)^{1/p}$. *By the definition of the Hölder norm* $\| \cdot \|_\varphi$, *clearly, for any* $f \in C_0^\varphi[0,1]^d$, *we have* $\|f\|_{l^p} \le d\|f\|_\varphi$. *Thus, the convergence in norm* $\| \cdot \|_\varphi$ *implies the convergence in norm* $\| \cdot \|_{l^p}$.

COROLLARY 1.1. *Let* $p \ge 1$. *Then, with probability one* $\{\|Z_T(x)\|_{l^p}; 0 \le x \le 1, T \ge 3\}$ *(as* $T \to \infty$*) is relatively compact in* $C_0[0,1]^d$, *and the set of its limit points is* $\{\|f(x)\|_{l^p}; 0 \le x \le 1, f \in K\}$, *specifically, we have*

$$\lim_{T \to \infty} \inf_{f \in K} \sup_{0 \le x \le 1} \left| \|Z_T(x)\|_{l^p} - \|f(x)\|_{l^p} \right| = 0 \quad \text{a.s.}$$

and for any $f \in K$

$$\lim_{T \to \infty} \sup_{0 \le x \le 1} \left| \|Z_T(x)\|_{l^p} - \|f(x)\|_{l^p} \right| = 0 \quad \text{a.s.}$$

COROLLARY 1.2. *Let* $p \ge 1$. *Then we have*

$$\lim_{T \to \infty} \frac{\|X^d(T)\|_{l^p}}{\sqrt{2\sigma^{*2}(T) \log\log T}} = \begin{cases} d^{(2-p)/2p}, & \text{if } 1 \le p < 2 \\ 1, & \text{if } p \ge 2 \end{cases} \quad \text{a.s.}$$

2. Application to FBM

Throughout this section we let $X^d(t) = (X_1(t), \cdots, X_d(t)) \in R^d, t \in [0, \infty)$, be a d-dimensional FBM with $X^d(0) = 0$, i.e., $\{X_j(t); t \geq 0\}$, $j = 1, \cdots, d$, are independent real-valued Gaussian processes with mean zero, stationary increments, $X_j(0) = 0$ and covariance function

$$R_j(s,t) = E\{X_j(s)X_j(t)\} = \frac{1}{2}\big(|s|^{2\gamma_j} + |t|^{2\gamma_j} - |s-t|^{2\gamma_j}\big)$$

and representation

$$X_j(t) = \int_{\mathbb{R}^1} \frac{1}{k_j}\big\{\big|x-t\big|^{(2\gamma_j-1)/2} - \big|x\big|^{(2\gamma_j-1)/2}\big\}dB_j(x),$$

where

   (a) $0 < \gamma_j < 1$, $k_j^2 = \int_{\mathbb{R}^1}\big\{\big|x-1\big|^{(2\gamma_j-1)/2} - \big|x\big|^{(2\gamma_j-1)/2}\big\}^2 dx$,

   (b) $\big\{B_j(t), -\infty < t < \infty\big\}$ is a BM,

   (c) $\dfrac{1}{k_j}\big\{\big|x-t\big|^{(2\gamma_j-1)/2} - \big|x\big|^{(2\gamma_j-1)/2}\big\}$ is interpreted to be $I_{(0,t]}$ when $\gamma_j = 1/2$, i.e., X_j is a BM.

   Put

$$Y_T(x) = \frac{X^d(xT)}{\sqrt{2T^{2\gamma^*}\log\log T}}, \quad 0 \leq x \leq 1.$$

There are a lot of papers in the literature to investigate limit behaviors of BM and FBM under sup-norm or Hölder norm. Chen[4], Goodman and Kuelbs[7], Monrad and Rootzén[13] and Oodaira[14] studied functional LIL and their convergence rates for FBM in the sup-norm. Also Baldi and Roynette[3], Kuelbs and Li[11], Kuelbs, Li and Shao[12] and Wei[19] studied limit behavior of BM and FBM under Hölder norm. The following Theorem 2.1 is Strassen's functional LIL for FBM under Hölder norm and generalizes the related results for BM (see [2, 4]).

THEOREM 2.1. *Let K be defined as in Section 1 and let $\varphi(t) = t^\alpha$, $\|\cdot\|_\varphi = \|\cdot\|_\alpha$ and $0 < \alpha < \gamma_*$. Then we have*

$$\lim_{T\to\infty}\inf_{f\in K}\|Y_T(\cdot) - f(\cdot)\|_\alpha = 0 \quad \text{a.s.}$$

and for any $f \in K$

$$\lim_{T\to\infty}\|Y_T(\cdot) - f(\cdot)\|_\alpha = 0 \quad \text{a.s.},$$

where $\gamma_* = \min\{\gamma_j, j = 1, \cdots, d\}$.

REMARK. If $\gamma_j = 1/2$ ($j = 1, \cdots, d$), then Theorem 2.1 is the results of Bladi, Ben Arou and Kerkyacharian[2]. For any $f \in C_0^{\varphi}[0,1]^d$ with $\varphi(t) = t^{\alpha}$, we have $\|f\|_{\infty} \leq \|f\|_{\alpha}$. Since the convergence in $\| \cdot \|_{\alpha}$ implies the convergence in sup-norm $\| \cdot \|_{\infty}$, Theorem 2.1 generalizes Strassen's results[16].

3. Large deviation probabilities for Gaussian random vectors

Let B^* be the topological dual of B with norm $\| \cdot \|$ and X be a centered B-valued Gaussian random vector with law $\mu = \mathcal{L}(X)$. It is well-known that there is a unique Hilbert space $H_{\mu} \subset B$ (also call the RKHS generated by μ) such that μ is determined by considering the pair (B, H_{μ}) as an abstract Wiener space (see [8]). The Hilbert space H_{μ} can be described as the completion of the range of the mapping $S : B^* \to B$ defined by the Bochner integral

$$S_f = \int_B x f(x) d\mu(x), \quad f \in B^*,$$

where the completion is in the inner product norm

$$< S_f, S_g >_{\mu} = \int_B f(x)g(x)d\mu(x), \quad f, g \in B^*.$$

We use $\| \cdot \|_{\mu}$ to denote the inner product norm induced on H_{μ}, and for well known properties and various relationship between μ, H_{μ} and B refer to [9, 10]. Let $\{\alpha_k, k \geq 1\}$ be a sequence in B^* orthonormal in $L^2(\mu)$ and $\{\Delta_{\alpha_k}, k \geq 1\}$ is a CONS in H_{μ} defined by $\Delta_{\alpha_k} = \int_B x\alpha_k(x)d\mu(x)$, then the operators defined by

$$(3.1) \qquad \Pi_d(x) = \sum_{k=1}^{d} \alpha_k(x)\Delta_{\alpha_k} \quad \text{and} \quad Q_d(x) = x - \Pi_d(x)$$

are continuous mappings from B to B. Furthermore, when restricted to H_{μ}, Π_d and Q_d are orthogonal projections onto their ranges. It is also known [9] that for centered Gaussian measure μ, $\lim_{d\to\infty} \|Q_d(X)\| = 0$ with probability one, and

$$(3.2) \qquad\qquad E\|Q_d(X)\| \downarrow 0 \quad \text{as} \quad d \uparrow \infty.$$

The following Proposition 3.1 is a large deviation inequality on Gaussian random vectors (see [7, 17]).

PROPOSITION 3.1. *Let X be a centered Gaussian random vector with values in B. Let $Q_d(d \geq 1)$ be the linear operators of (3.1), and $U_a = \{f \in H_\mu : \|f\|_\mu \leq a\}$, where $a > 0$ and H_μ is the RKHS with $\mu = \mathcal{L}(X)$. Let $d_\lambda \, (\lambda \geq 1)$ be an integer such that*

$$(3.3) \qquad d_\lambda \geq \inf \left\{ m \geq 1 : E\|Q_m(X)\|/m \leq 2\tau \log \lambda/\lambda \right\},$$

and $\varepsilon_\lambda = \gamma d_\lambda \log \lambda/\lambda^2$ with some constant $\gamma > 3\tau$, where $\tau = \sup_{x \in U_a} \|x\|$. Then for any $\varepsilon \geq \varepsilon_\lambda$ (where ε may depend on λ)
(3.4)

$$P \left(\inf_{f \in U_a} \left\| \frac{X}{\lambda} - f \right\| \geq \varepsilon \right)$$
$$\leq \frac{1}{\sqrt{2\pi(d_\lambda + 1)}} \exp \left(-\frac{(a\lambda(1+\varepsilon))^2}{2} + \frac{d_\lambda - 1}{2} \log \frac{(a\lambda(1+\varepsilon))^2 e}{d_\lambda - 1} \right)$$

for any $\lambda \geq \lambda_0$ with some $\lambda_0 > 0$.

4. Proofs

To prove our theorem we need several lemmas. The following lemma 4.1 is a modified version of Proposition 3.1 on a d-dimensional Gaussian process in the Hölder norm.

LEMMA 4.1. *Let d_λ and ε_λ be defined as in Proposition 3.1 and K as in Section 1 with $B = C_0[0,1]^d$. Then for any $\varepsilon \geq \varepsilon_\lambda$ (where ε may depend on λ) we have*

$$(4.1) \qquad P \left\{ \inf_{f \in K} \left\| \frac{X^d(\cdot)}{\lambda} - f(\cdot) \right\|_\varphi \geq \varepsilon \right\}$$
$$\leq C \exp \left(-\frac{(\lambda(1+\varepsilon))^2}{2} + \frac{d_\lambda - 1}{2} \log \frac{(\lambda(1+\varepsilon))^2 e}{d_\lambda - 1} \right)$$

for any $\lambda > \lambda_0$ with some $\lambda_0 > 0$, where C is an absolute constant. Particularly, if ε is independent of λ, we have

$$(4.2) \qquad P \left\{ \inf_{f \in K} \left\| \frac{X^d(\cdot)}{\lambda} - f(\cdot) \right\|_\varphi \geq \varepsilon \right\} \leq C \exp \left(-\frac{(\lambda(1+\varepsilon))^2}{2} \right).$$

Proof. The proof is straightforward form Proposition 3.1. $\square$

LEMMA 4.2. *Let K be defined as in Section 1, then for every $\varepsilon > 0$ and $\theta > 0$ there exist Ω_0 with $P(\Omega_0) = 1$ and $n_0 = n_0(\omega)$, $\omega \in \Omega_0$ such that $Z_{\theta^n}(\cdot) \in K_\varepsilon$ for every $n \geq n_0$.*

Proof. Let $K_\varepsilon^c = \{g \in C_0^\varphi[0,1]^d : \inf_{f \in K} \|f - g\|_\varphi \geq \varepsilon\}$. By the Borel-Cantelli lemma, it is sufficient to prove that $P\{Z_{\theta^n} \in K_\varepsilon^c\}$ is summable. By Lemma 4.1, we have

$$P\{Z_{\theta^n} \in K_\varepsilon^c\} = P\{\inf_{f \in K} \|Z_{\theta^n} - f\|_\varphi \geq \varepsilon\}$$

$$\leq C \exp\left\{-\frac{2(1+\varepsilon)^2}{2} \log\log\theta^n\right\}$$

$$= C(n\log\theta)^{-(1+\varepsilon)^2}$$

and the right hand side is summable. $\square$

LEMMA 4.3. *Set*

$$Y_n = \sup_{\theta^n \leq u \leq \theta^{n+1}} \frac{\|X^d(u\cdot) - X^d(\theta^n\cdot)\|_\varphi}{\sqrt{2\sigma^{*2}(\theta^n)\log\log\theta^n}}.$$

For every $\varepsilon > 0$ there exists $\theta_\varepsilon > 1$ such that for every $1 < \theta < \theta_\varepsilon$ there exists $n_0 = n_0(\omega)$ such that $Y_n(\omega) \leq \varepsilon$ for every $n \geq n_0$.

Proof. Note that condition (ii) implies that there is $0 < c < \infty$ such that

(4.3) $\sigma_j(t)/\varphi(t) \leq c$

for any $0 < t \leq 1$ and by the scaling property of self-similar, the definition of $\|\cdot\|_\varphi$ and condition (iii) of $\sigma_j(\cdot)$,

$$\sup_{\theta^n \leq u \leq \theta^{n+1}} \|X^d(u\cdot) - X^d(\theta^n\cdot)\|_\varphi$$

$$= \sup_{\theta^n \leq u \leq \theta^{n+1}} \sup_{0 \leq s < t \leq 1} \max_{1 \leq j \leq d} \frac{|X_j(ut) - X_j(us) - (X_j(\theta^n t) - X_j(\theta^n s))|}{\varphi(|t-s|)}$$

$$= \sup_{\theta^n \leq u \leq \theta^{n+1}} \sup_{0 \leq s < t \leq 1} \max_{1 \leq j \leq d} \frac{(\sigma_j(t) - \sigma_j(s))|X_j(u) - X_j(\theta^n))|}{\varphi(|t-s|)}$$

$$\leq \sup_{\theta^n \leq u \leq \theta^{n+1}} \sup_{0 \leq s < t \leq 1} \max_{1 \leq j \leq d} \frac{\sigma_j(|t-s|)|X_j(u) - X_j(\theta^n))|}{\varphi(|t-s|)}$$

$$\leq c \sup_{\theta^n \leq u \leq \theta^{n+1}} \max_{1 \leq j \leq d} |X_j(u) - X_j(\theta^n)|.$$

Following the same lines of the proof of Lemma 2 in Ortega[15], we have that for any $\varepsilon > 0$ there exists a positive constant $C = C_\varepsilon$ such that

(4.4)
$$P\left\{\sup_{0\leq t\leq T}\sup_{0\leq s\leq h}|X_j(t+s) - X_j(t)| \geq x\sigma_j(h)\right\}$$
$$\leq C\frac{T}{h}\exp(-x^2/(2+\varepsilon))$$

for any T, $0 \leq h \leq T$ and $x \geq x_0$ with some $x_0 > 0$. So we have, by (4.4)

$$P\left\{\sup_{\theta^n\leq u\leq\theta^{n+1}}\frac{\|X^d(u\cdot) - X^d(\theta^n\cdot)\|_\varphi}{\sqrt{2\sigma^{*2}(\theta^n)\log\log\theta^n}} \geq \varepsilon\right\}$$

$$\leq P\left\{\sup_{\theta^n\leq u\leq\theta^{n+1}}\max_{1\leq j\leq d}\frac{|X_j(u) - X_j(\theta^n)|}{\sqrt{2\sigma^{*2}(\theta^n)\log\log\theta^n}} \geq \varepsilon/c\right\}$$

$$\leq P\left\{\sup_{1\leq u/\theta^n\leq\theta}\max_{1\leq j\leq d}\frac{|X_j(u/\theta^n) - X_j(1)|}{\sqrt{2\log\log\theta^n}} \geq \varepsilon/c\right\}$$

$$\leq P\left\{\sup_{1\leq z\leq\theta}\max_{1\leq j\leq d}\frac{|X_j(z) - X_j(1)|}{\sigma_j(\theta-1)} \geq \frac{\varepsilon}{c\sigma^*(\theta-1)}\sqrt{2\log\log\theta^n}\right\}$$

$$\leq \sum_{j=1}^{d}P\left\{\sup_{0\leq s\leq\theta-1}\frac{|X_j(1+s) - X_j(1)|}{\sigma_j(\theta-1)} \geq \frac{\varepsilon}{c\sigma^*(\theta-1)}\sqrt{2\log\log\theta^n}\right\}$$

$$\leq C_d\frac{1}{\theta-1}\exp\left\{-\frac{2\varepsilon^2}{2+\varepsilon}\frac{1}{c^2\sigma^{*2}(\theta-1)}\log\log\theta^n\right\}$$

$$\leq C_d\frac{1}{\theta-1}(n\log\theta)^{-\varepsilon^2/(2c^2\sigma^{*2}(\theta-1))}.$$

Hence, given $\varepsilon > 0$ we can take $\theta > 1$ such that $\varepsilon^2/(2c^2\sigma^{*2}(\theta-1)) > 1$, so we have
$$\sum_{n\geq 1}P\{Y_n \geq \varepsilon\} < \infty$$

and by the Borel-Cantelli lemma, the proof is complete.   □

We shall also need the following lemma (cf. [13]).

LEMMA 4.4. *Let V be a convex, symmetric, measurable subset of B. Then for all $f \in H_\mu$*

$$\mu(f + V) \geq \mu(V) \exp \left\{ -\frac{1}{2} \|f\|_\mu^2 \right\}.$$

PROOF OF THEOREM 1.1.

Proof of (1.3). Put $T_n = \theta^n$ with $\theta > 1$. For large T, there exists T_n such that $T_n \leq T \leq T_{n+1}$. We have

$$
\begin{aligned}
&\inf_{f \in K} \|Z_T(\cdot) - f(\cdot)\|_\varphi \\
&\leq \sup_{T_n \leq T \leq T_{n+1}} \|Z_T(\cdot) - Z_{T_n}(\cdot)\|_\varphi + \inf_{f \in K} \|Z_{T_n}(\cdot) - f(\cdot)\|_\varphi \\
&\leq \sup_{T_n \leq T \leq T_{n+1}} \frac{\|X^d(T\cdot) - X^d(T_n\cdot)\|_\varphi}{\sqrt{2\sigma^{*2}(T_n) \log \log T_n}} \\
&\quad + \sup_{T_n \leq T \leq T_{n+1}} \left| \frac{1}{\eta_T} - \frac{1}{\eta_{T_n}} \right| \|X^d(T_n\cdot)\|_\varphi + \inf_{f \in K} \|Z_{T_n}(\cdot) - f(\cdot)\|_\varphi.
\end{aligned}
$$

(4.5)

Now, for given $\varepsilon > 0$ if $\theta > 1$ is near enough to one, by Lemma 4.3, we have

$$(4.6) \qquad \limsup_{n \to \infty} \sup_{T_n \leq T \leq T_{n+1}} \frac{\|X^d(T\cdot) - X^d(T_n\cdot)\|_\varphi}{\sqrt{2\sigma^{*2}(T_n) \log \log T_n}} \leq \varepsilon \quad \text{a.s.}$$

As for the second term in the right hand of (4.5), $\|X^d(T_n\cdot)\|_\varphi / \eta_{T_n} \to 1$ a.s. as $n \to \infty$ by Lemma 4.2 and, by condition (i) of $\sigma_j(\cdot)$ and the property of a slowly varying function

$$\lim_{n \to \infty} \eta_{T_n} \left| \frac{1}{\eta_{T_n}} - \frac{1}{\eta_{T_{n+1}}} \right| \leq 1 - \theta^{-2\gamma^*}.$$

Thus if θ is chosen to be close enough to 1, then we have

$$(4.7) \qquad \lim_{n \to \infty} \sup_{T_n \leq T \leq T_{n+1}} \left| \frac{1}{\eta_T} - \frac{1}{\eta_{T_n}} \right| \|X^d(T_n\cdot)\|_\varphi \leq \varepsilon \quad \text{a.s.}$$

Combining (4.5) with (4.6), (4.7) and Lemma 4.2, we obtain

$$\lim_{T \to \infty} \inf_{f \in K} \|Z_T(\cdot) - f(\cdot)\|_\varphi = 0 \quad \text{a.s.} \qquad \square$$

Proof of (1.4). Let $f \in K$. For $\theta > 1$, also let $T_n = \theta^n$ and $T_n \leq T \leq T_{n+1}$. We have

(4.8) $\|Z_T(\cdot) - f(\cdot)\|_\varphi \leq \sup_{T_n \leq T \leq T_{n+1}} \|Z_T(\cdot) - Z_{T_n}(\cdot)\|_\varphi + \|Z_{T_n}(\cdot) - f(\cdot)\|_\varphi.$

By (4.6) and (4.7), for given $\varepsilon > 0$ and θ near enough to 1, we have

(4.9) $\limsup_{n \to \infty} \sup_{T_n \leq T \leq T_{n+1}} \|Z_T(\cdot) - Z_{T_n}(\cdot)\|_\varphi \leq \varepsilon$ a.s.

It is sufficient to prove that

(4.10) $\limsup_{n \to \infty} \|Z_{T_n}(\cdot) - f(\cdot)\|_\varphi = 0$ a.s.

From (1.1) and (1.2), for $j = 1, \cdots, d$,

(4.11) $\sigma_j^2(h) = 2 \int_R (1 - \cos(h\lambda))\Delta_j(d\lambda), \quad 0 < h < 1.$

Then there exists a constant $K > 0$ such that for all $0 < h < 1$

(4.12) $\int_{|\lambda| \geq 1/h} \Delta_j(d\lambda) \leq K\sigma_j^2(h)$

and

(4.13). $\int_{|\lambda| \leq 1/h} |\lambda|^2 \Delta_j(d\lambda) \leq Kh^{-2}\sigma_j^2(h)$ (see [13] and [18])

Let $d_n = n^{n+1-r}$, $s_n = n^{-n/2}$ with $n \geq 1$ and $0 < r < 1$. Define for $n = 1, 2, \cdots$ and $0 \leq t \leq 1$

(4.14)
$$X_i^{(n)}(t) = \int_{|\lambda| \in (d_{n-1}, d_n]} (e^{it\lambda} - 1)W_j(d\lambda), \quad \tilde{X}_i^{(n)}(t) = X_j(t) - X_j^{(n)}(t),$$

$$\tilde{Y}_j^{(n)}(h, t) = \int_{|\lambda| \notin (hd_{n-1}, hd_n]} (e^{iht\lambda} - 1)W_j(d\lambda) \text{ and}$$

$$Y^{(n)}(t) = (X_1^{(n)}(t), \cdots, X_d^{(n)}(t)), \quad \tilde{Y}^{(n)}(t) = (\tilde{X}_1^{(n)}(t), \cdots, \tilde{X}_d^{(n)}(t)).$$

Then $\{X_j^{(n)}(t)\}$, $n = 1, 2, \cdots$ are independent and, by self-similarity of X_i, for any $h > 0$,

(4.15) $$\sigma_j(h)\tilde{X}_j^{(n)}(\cdot) \overset{\mathcal{D}}{=} \tilde{Y}_j^{(n)}(h, \cdot).$$

By the standard Borel-Cantelli augments, (4.10) follows if we prove that

$$(4.16) \qquad \sum_{n=1}^{\infty} P\left\{ \left\| \frac{Y^{(n)}(T_n \cdot)}{\eta_{T_n}} - f(\cdot) \right\|_{\varphi} \leq \varepsilon \right\} = \infty$$

and

$$(4.17) \qquad \sum_{n=1}^{\infty} P\left\{ \left\| \frac{\tilde{Y}^{(n)}(T_n \cdot)}{\eta_{T_n}} \right\|_{\varphi} \geq \varepsilon \right\} < \infty$$

since the events in (4.16) are independent. By Lemma 4.4, we have, for any $\varepsilon > 0$,

$$P\left\{ \left\| \frac{Y^{(n)}(T_n \cdot)}{\eta_{T_n}} - f(\cdot) \right\|_{\varphi} \leq \varepsilon \right\}$$

$$= P\left\{ \left\| \frac{Y^{(n)}(T_n \cdot)}{\sigma^*(T_n)} - f(\cdot)\sqrt{2\log\log T_n} \right\|_{\varphi} \leq \varepsilon\sqrt{2\log\log T_n} \right\}$$

$$\geq \exp\left\{ -\|f\|_{\mu}^2 \log\log T_n \right\} P\left\{ \left\| \frac{Y^{(n)}(T_n \cdot)}{\sigma^*(T_n)} \right\|_{\varphi} \leq \varepsilon\sqrt{2\log\log T_n} \right\}$$

$$\geq C \exp\left\{ -\log\log T_n \right\} = C(n\log\theta)^{-1}$$

for large enough n and C is a constant, so we obtain (4.16). From condition (i) of σ_j, there is $0 < \nu < 1$ such that

$$(4.18) \qquad \sigma_j(lh) \leq 2l^{1-\nu}\sigma_j(h)$$

for small $0 < h < 1$ and l with $1 \leq l \leq 1/h$. By (4.12), (4.13), (4.18) and nondecreasity of σ_j, we have

$$\mathrm{Var}\big(\tilde{Y}_j^{(n)}(s_n, t)\big) = 2\int_{|\lambda| \notin (s_n d_{n-1}, s_n d_n]} (1 - \cos(s_n t\lambda))\Delta_j(d\lambda)$$

$$\leq 2\int_{|\lambda| \leq s_n d_{n-1}} s_n^2 t^2 \lambda^2 \Delta_j(d\lambda) + 4\int_{|\lambda| \geq s_n d_n} \Delta_j(d\lambda)$$

$$\leq K s_n^4 d_{n-1}^2 \sigma_j^2\big(s_n/(s_n^2 d_{n-1})\big) + K\sigma_j^2\big(s_n/(s_n^2 d_n)\big)$$

$$\leq K n^{-2\nu r}\sigma_j^2(s_n) + K\sigma_j^2\big(s_n/(s_n^2 d_{n-1})\big)$$

$$\leq K\big(n^{-2\nu r} + n^{-2(1-\nu)(1-r)}\big)\sigma_j^2(s_n)$$

$$\leq K n^{-\delta}\sigma_j^2(s_n)$$

for large $n \geq 1$ and $0 \leq t \leq 1$, where $\delta = \min\{2(1-\nu)(1-r), 2\nu r\}$ and K is a constant which differs from lines to lines. Therefore, for any $s_n > 0$ and for every $0 \leq |t - s| \leq h \leq 1$,

$$\mathrm{Var}\big(\tilde{Y}_j^{(n)}(s_n, t) - \tilde{Y}_j^{(n)}(s_n, s)\big) \leq \tilde{\sigma}_j^{(n)}(h)^2,$$

where $\tilde{\sigma}_j^{(n)}(h)^2 = \min\{\sigma_j^2(h), Kn^{-\delta}\sigma_j^2(s_n)\}$. Since X_j has self-similarity, i.e., for any $h > 0$, $X_j(h\cdot) \overset{D}{=} \sigma_j(h)X_j(\cdot)$, it is clear that, from (4.14), $\tilde{X}_j^{(n)}(h\cdot) \overset{D}{=} \sigma_j(h)\tilde{X}_j^{(n)}(\cdot)$. By (4.15), (4.4) and (4.3), we have

$$P\left\{\left\|\frac{\tilde{Y}^{(n)}(T_n\cdot)}{\eta_{T_n}}\right\|_{\varphi} \geq \varepsilon\right\}$$

$$\leq P\left\{\max_{1\leq j\leq d}\sup_{0\leq x<y\leq 1}\frac{\left|\tilde{X}_j^{(n)}(y) - \tilde{X}_j^{(n)}(x)\right|}{\varphi(y-x)} \geq \varepsilon\sqrt{2\log\log T_n}\right\}$$

$$\leq \sum_{1\leq j\leq d} P\left\{\sup_{0\leq x<y\leq 1}\frac{(\sigma_j(y) - \sigma_j(x))\left|\tilde{X}_j^{(n)}(1)\right|}{\varphi(y-x)} \geq \varepsilon\sqrt{2\log\log T_n}\right\}$$

$$\leq \sum_{1\leq j\leq d} P\left\{\sup_{0\leq x<y\leq 1}\frac{\sigma_j(y-x)\left|\tilde{X}_j^{(n)}(1)\right|}{\varphi(y-x)} \geq \varepsilon\sqrt{2\log\log T_n}\right\}$$

$$\leq \sum_{1\leq j\leq d} P\left\{\frac{\left|\tilde{Y}_j^{(n)}(s_n, 1)\right|}{\sqrt{Kn^{-\delta}\sigma_j^2(s_n)}} \geq \frac{\varepsilon}{c\sqrt{Kn^{-\delta}}}\sqrt{2\log\log T_n}\right\}$$

$$\leq C\exp\left(-\frac{2\varepsilon^2}{2+\varepsilon}\frac{n^{\delta}}{c^2 K}\log\log T_n\right)$$

$$\leq C\exp\left(-\frac{\varepsilon^2}{2c^2 K}n^{\delta}\log(n\log\theta)\right)$$

$$= C\,(n\log\theta)^{-\frac{\varepsilon^2}{2c^2 K}n^{\delta}}$$

for large enough n, and we obtain (4.17). Hence (1.4) is proved. $\qquad\square$

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