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ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF NEGATIVELY ORTHANT DEPENDENT RANDOM VARIABLES

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Abstract. For weighted sum of a sequence $\{X, X_n, n \geq 1\}$ of identically distributed, negatively orthant dependent random variables such that $|X|^r, r > 0$, has a finite moment generating function, a strong law of large numbers is established.

1. Introduction

The history and literature on the strong laws of large numbers is vast and rich as this concept is crucial in probability and statistical theory. The literature on concepts of negative dependence is much more limited but still very interesting. Lehmann[7] provided an extensive introductory overview of various concepts of positive and negative dependence in the bivariate case. Negative dependence has been particularly useful in obtaining strong laws of large numbers(see [3, 8, 9, 10]). The almost sure limiting law of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$, where $\{X, X_i, i \geq 1\}$ is a sequence of i.i.d. random variables with $EX = 0$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of weights, was investigated by many authors(see, Bai and Cheng[1], Chow and Lai[4]). For uniformly bounded weights $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ and i.i.d. random variables X_i with $EX = 0$, Teicher[11] obtained

$$(1) \qquad \lim_{n\to\infty} \sum_{i=1}^{n} a_{ni}X_i/b_n = 0 \text{ a.s.}$$

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at a rate $b_n = n^{1/\alpha}\log\ n$ for $1 < \alpha \leq 2$, and Chow and Lai[4] considered the case of $\sup_n(n^{-1}\sum_{i=1}^{n}|a_{ni}|^{\alpha}) < \infty$ for some $\alpha > 0$. A strong law of the form (1) with more general normalizing constants b_n was also obtained by Cuzick[5] under condition $\sup_n(n^{-1}\sum_{i=1}^{n}|a_{ni}|^{\alpha})^{\frac{1}{\alpha}} < \infty$ for some $1 < \alpha < \infty$. Recently, Bai and Cheng[1] derived the following strong law of large numbers by considering a standard case when $|X|^r$, $0 < r$, has a finite moment generating function: Let $\{X, X_i, i \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$ and

$$(2) \qquad E\{\exp(h|X|^r)\} < \infty \text{ for some } h > 0 \text{ and some } r > 0$$

and let $\{a_{ni}, 1 \leq i \leq n\}$ be an double array of real numbers such that, for $1 < \alpha < \infty$

$$(3) \qquad A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty,\ A_{\alpha,n}^{\alpha} = n^{-1}\sum_{i=1}^{n}|a_{ni}|^{\alpha}.$$

If (2) holds and (3) holds for $\alpha \in (0,2)$, then, for $0 < \alpha \leq 1$ and $b_n = n^{1/\alpha}(\log n)^{1/r}$

$$(4) \qquad \limsup_{n\to\infty}\left|\sum_{i=1}^{n} a_{ni}X_i/b_n\right| \leq h^{-1/r}A_\alpha \text{ a.s.},$$

moreover, for $1 < \alpha < 2$, $b_n = n^{1/\alpha}(\log\ n)^{(1/r)+r(\alpha-1)/\alpha(1+r)}$ and $EX = 0$, we have

$$(5) \qquad \lim_{n\to\infty}\sum_{i=1}^{n} a_{ni}X_i/b_n = 0 \text{ a.s.}$$

In this paper, we study the similar almost sure limiting behavior on the weighted sums of identically distributed and negatively orthant dependent(NOD) random variables of the form (1) under stronger condition on the moment generating function

$$(6) \qquad E\{\exp(h|X|^r)\} < \infty \text{ for any } h > 0 \text{ and any } r > 0.$$

2. Preliminaries

This section will contain some background materials on negative orthant dependence which will be used in obtaining the major strong law of large numbers in the next section.

DEFINITION 2.1.(Lehmann[7]) Random variables X and Y are negatively quadrant dependent(NQD) if

$$P\{X \le x, Y \le y\} \le P\{X \le x\}P\{Y \le y\} \tag{7}$$

for all $x, y \in \mathbb{R}$. A collection of random variables is said to be pairwise NQD if every pair of random variables in the collection satisfies (7). It is important to note that Definition 2.1 implies

$$P\{X > x, Y > y\} \le P\{X > x\}P\{Y > y\} \tag{8}$$

for all $x, y \in \mathbb{R}$. Moreover, it follows that (8) implies (7), and hence, they are equivalent for pairwise NQD.

DEFINITION 2.2.(Ebrahimi and Ghosh[6]) The random variables X_1, $X_2, \cdots$ are said to be
(a) lower negatively orthant dependent(LNOD) if for each n

$$P\{X_1 \le x_1, \cdots, X_n \le x_n\} \le \prod_{i=1}^{n} P\{X_i \le x_i\} \tag{9}$$

for all $x_1, \cdots, x_n \in \mathbb{R}$,
(b) upper negatively orthant dependent(UNOD) if for each n

$$P\{X_1 > x_1, \cdots, X_n > x_n\} \le \prod_{i=1}^{n} P\{X_i > x_i\} \tag{10}$$

for all $x_1, \cdots, x_n \in \mathbb{R}$,
(c) negatively orthant dependent (NOD) if both (9) and (10) hold.

REMARK. Ebrahimi and Ghosh[6] showed that (9) and (10) are not equivalent for $n \ge 3$. Consequently, the above definition is needed to define sequences of negatively dependent random variables.

The following properties are listed for reference in obtaining the main results in the next section. Detailed proofs can be found in [6] and [7].

LEMMA 2.3. *If $\{X_n, n \ge 1\}$ is a sequence of NOD random variables and $\{f_n, n \ge 1\}$ is a sequence of Borel functions all of which are monotone increasing* (or all monotone decreasing), *then $\{f_n(X_n), n \ge 1\}$ is a sequence of NOD random variables.*

THEOREM 2.4. *Let $X_1, X_2, \cdots, X_n$ be nonnegative random variables which are upper negatively orthant dependent. Then*

$$E\left(\prod_{i=1}^{n} X_i\right) \le \prod_{i=1}^{n} E(X_i) \tag{11}$$

3. Results

LEMMA 3.1. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NQD random variables satisfying* (6). *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise NOD random variables with $EX_{ni} = 0$ for $1 \leq i \leq n$ and $n \geq 1$, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive constants. Assume that, for $1 \leq i \leq n$, some $0 < \beta \leq r$ and some constant $C > 0$*

$$|a_{ni} X_{ni}| \leq C|X_i|^\beta / \log n \text{ a.s.} \tag{12}$$

and, for some sequence $\{u_n\}$ of positive constants such that $\lim_{n\to\infty} u_n = 0$, and some $\delta > 0$ and $1 < \alpha \leq 2$,

$$|X_{ni}|^\alpha \sum_{i=1}^{n} a_{ni}^\alpha \leq u_n \, |X_i|^\delta / (\log n)^{\alpha-1} \text{ a.s.} \tag{13}$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_{ni} X_{ni} = 0 \text{ a.s.} \tag{14}$$

Proof. From the inequality

$$e^x \leq 1 + x + \frac{1}{2} x^2 e^{|x|} \text{ for all } x \in \mathbb{R},$$

we have

$$E[\exp(ta_{ni}X_{ni})] \leq 1 + \frac{1}{2} t^2 a_{ni}^2 E\left[X_{ni}^2 \exp(ta_{ni}|X_{ni}|)\right] \text{ for any } t > 0.$$

Let $\epsilon > 0$ and put $t = 2$ (log $n)/\epsilon$. It follows from (12) and (13) and the fact that for some $\tau > 0$, any fixed $h > 0$, there exists a constant $D > 0$ such that inequality $|x|^2 \leq D(e^{h|x|^\beta})$ for all $x \in R$ that

$$\begin{aligned}
&E[\exp(ta_{ni}X_{ni})] \\
&\leq 1 + \frac{1}{2}\left(\frac{2}{\epsilon}\right)^2 (\log n)^2 a_{ni}^2 E\left[X_{ni}^2 \exp((2/\epsilon)(\log n) a_{ni}|X_{ni}|)\right] \\
&\leq 1 + \frac{2}{\epsilon^2} u_n (\log n)^{3-\alpha} \left(\frac{a_{ni}^\alpha}{\sum_{i=1}^n a_{ni}^\alpha}\right) \\
&\quad \times E[(C|X_i|^\beta / \log n)^{2-\alpha} |X_i|^\delta \exp(2/\epsilon) C|X_i|^\beta]
\end{aligned} \tag{15}$$

$$\leq 1+\frac{2}{\epsilon^2}u_n(\log n)\left(\frac{a_{ni}^\alpha}{\sum_{i=1}^n a_{ni}^\alpha}\right)E\left[\exp\left(\frac{2}{\epsilon}C'\left|X_i\right|^\beta\right)\right]$$
$$\leq 1+\frac{1}{2}(\log n)\left(\frac{a_{ni}^\alpha}{\sum_{i=1}^n a_{ni}^\alpha}\right)$$
$$\leq \exp\left\{\frac{1}{2}(\log n)\frac{a_{ni}^\alpha}{\sum_{i=1}^n a_{ni}^\alpha}\right\}$$

for all large n and some $C' > 0$ since $e^x > 1 + x$ for $x > 0$.

Next note that, for any $t > 0$,

(16)
$$E\exp\left(t\sum_{i=1}^n a_{ni}X_{ni}\right)=E\left(\prod_{i=1}^n \exp(ta_{ni}X_{ni})\right)\leq\prod_{i=1}^n E\exp(ta_{ni}X_{ni})$$

by Lemma 2.3 and Theorem 2.4. From the Markov inequality, (15) and (16) we obtain

$$\begin{aligned}P\left(\sum_{i=1}^n a_{ni}X_{ni}\geq\epsilon\right)&\leq e^{-t\epsilon}E\left[\exp\left(t\sum_{i=1}^n a_{ni}X_{ni}\right)\right]r\\ &\leq e^{-t\epsilon}\prod_{i=1}^n E\exp(ta_{ni}X_{ni})\\ &\leq e^{-2\log n}\prod_{i=1}^n \exp\left\{\frac{1}{2}\log n\frac{a_{ni}^\alpha}{\sum_{i=1}^n a_{ni}^\alpha}\right\}\\ &= n^{-3/2},\end{aligned}\tag{17}$$

which is summable. Since $-X_{ni}^{'}$s are NOD according to Lemma 2.3, by replacing X_{ni} with $-X_{ni}$, from the above statement we also have

$$P\left(\sum_{i=1}^n a_{ni}(-X_{ni})\geq\epsilon\right)\leq n^{-\frac{3}{2}} \text{ for all large } n. \tag{18}$$

Hence, by the Borel Cantelli lemma from (17) and (18) the result (14) follows. □

REMARK. Lemma 3.1 can be extended to the case where $\{a_{ni}\}$ is an array of real numbers.

THEOREM 3.2. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NOD random variables with $EX = 0$ and satisfying (6) and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive numbers such* *that* (3) *holds for some* $1 < \alpha \le 2$. *Then, for* $0 < r \le \frac{\alpha}{\alpha+1}$ *and* $b_n = n^{1/\alpha}(\log n)^{1/r}$,

$$\sum_{i=1}^{n} a_{ni}X_i/b_n \to 0 \text{ a.s.} \tag{19}$$

Proof. We first observe that

$$\begin{aligned}
E\left(\sum_{i=1}^{n} a_{ni}X_i/b_n\right)^2 &\le EX^2\sum_{i=1}^{n} a_{ni}^2/b_n^2 \\
&\le EX^2\left(\sum_{i=1}^{n} a_{ni}^{\alpha}\right)^{\frac{2}{\alpha}}/b_n^2 \\
&\le EX^2A_{\alpha,n}^2 n^{\frac{2}{\alpha}}/b_n^2 \ \to\ 0 \text{ as } n\ \to \infty.
\end{aligned}$$

It follows that $\sum_{i=1}^{n} a_{ni}X_i/b_n \ \to\ 0$ in probability.

Hence, by Theorem 3.2.1 in Stout[10], it suffices to prove that

$$\sum_{i=1}^{n} a_{ni}X_i^s/b_n \ \to\ 0 \text{ a.s.},$$

where $\{X_n^s\}$ is a symmetrized version of $\{X_n\}$. So we need only to prove the result for $\{X_n\}$ symmetric. Define

$$\begin{aligned}
X_{ni}^{'} = X_iI\left(|X_i| \le (\log n)^{1/r}\right) &- (\log n)^{1/r}I\left(X_i < -(\log n)^{1/r}\right) \\
&+ (\log n)^{1/r}I\left(X_i > (\log n)^{1/r}\right)
\end{aligned}$$

and

$$\begin{aligned}
X_{ni}^{''} &= X_i - X_{ni}' \\
&= \left(X_i - (\log n)^{1/r}\right)I\left(X_i > (\log n)^{1/r}\right) \\
&\quad + \left(X_i + (\log n)^{1/r}\right)I\left(X_i < -(\log n)^{1/r}\right),
\end{aligned}$$

where I stands for indicator function.

Note that both $X_{ni}^{'}$ and $X_{ni}^{''}$ are NOD by Lemma 2.3 and that

$$|X_{ni}^{''}| \le |X_i|I\left(|X_i| > (\log n)^{1/r}\right). \tag{20}$$

Since

$$E \exp |X|^r < \infty \Leftrightarrow \sum_{n=1}^{\infty} P\left(|X_n| > \log^{1/r} n\right) < \infty$$

we have

$$P(|X''_{ni}| > (\log i)^{1/r} \text{i.o}) = 0 \tag{21}$$

by the Borel-Cantelli lemma. It follows from (20) and (21) that

$$\begin{aligned} \sum_{i=1}^{n} |X''_{ni}| &\le \sum_{i=1}^{n} |X_i| I\left(|X_i| > (\log n)^{1/r}\right) \\ &\le \sum_{i=1}^{n} |X_i| I\left(|X_i| > (\log i)^{1/r}\right) \\ &< \infty, \end{aligned} \tag{22}$$

that is, $\sum_{i=1}^{n} |X''_{ni}|$ is bounded a.s. It follows that

$$\begin{aligned} b_n^{-1} \left| \sum_{i=1}^{n} a_{ni} X''_{ni} \right| &\le b_n^{-1} \sum_{i=1}^{n} \left| a_{ni} X''_{ni} \right| \\ &\le b_n^{-1} \max_{1 \le i \le n} |a_{ni}| \sum_{i=1}^{n} X''_{ni} \\ &\le b_n^{-1} \left(\sum_{i=1}^{n} |a_{ni}|^{\alpha} \right)^{\frac{1}{\alpha}} \sum_{i=1}^{n} \left| X''_{ni} \right| \\ &\le A_{\alpha,n} \sum_{i=1}^{n} \left| X''_{ni} \right| / (\log n)^{1/r} \to 0 \text{ a.s. as } n \to \infty. \end{aligned} \tag{23}$$

To complete the proof we will apply Lemma 3.1 to the random variable X'_{ni} and weight $b_n^{-1} a_{ni}$. Note that

$$\begin{aligned} \left| b_n^{-1} a_{ni} X'_{ni} \right| &\le b_n^{-1} |a_{ni}| (\log n)^{(1-r)/r} |X_i|^r \\ &\le b_n^{-1} \left(\sum_{i=1}^{n} |a_{ni}|^{\alpha} \right)^{\frac{1}{\alpha}} (\log n)^{(1-r)/r} |X_i|^r \\ &\le b_n^{-1} A_{\alpha,n} n^{1/\alpha} (\log n)^{(1-r)/r} |X_i|^r \\ &= A_{\alpha,n} |X_i|^r / \log n \end{aligned}$$

and

$$\left|X_{ni}^{'}\right|^{\alpha}\sum_{i=1}^{n}b_n^{-\alpha}\left|a_{ni}\right|^{\alpha}=\left|X_{ni}^{'}\right|^{\alpha}A_{\alpha,n}^{\alpha}/(\log n)^{\alpha/r}$$
$$\leq A_{\alpha,n}^{\alpha}\left|X_i\right|^{\alpha}/(\log n)^{\alpha/r},$$

which satisfy conditions (12) and (13) of Lemma 3.1. Hence, we have

$$\sum_{i=1}^{n}a_{ni}X_{ni}^{'}/b_n\to 0\text{ a.s.}\tag{24}$$

and the desired result follows by (23) and (24). □

REMARK. If (19) holds for any array $\{a_{ni}\}$ satisfying (3), then $EX = 0$ and (6) holds. The proof is similar to that of Bai and Cheng ([1], 108–109): Suppose (19) is true for any weights suquence satisfying (3). Choose, for each n, $a_{n1} = \cdot = a_{n,n-1} = 0$ and $a_{nn} = n^{1/\alpha}$. Then, by (19), we have $(\log n)^{-1/r}X_n \to 0$ a.s., which implies that $E\{\exp(h|X|^r) < \infty$.

The following theorem is a slight modification of Theorem 3.2. The theorem shows that if the norming constant b_n is stronger than that of Theorem 3.2, then condition (6) of Theorem 3.2 can be replaced by weaker condition (2).

THEOREM 3.3. *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NOD random variables satisfying $EX = 0$ and* (2) *and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying* (3) *for some $1 < \alpha \leq 2$. Then, for $0 < r < \frac{\alpha}{\alpha-1}$ and $b_n = n^{1/\alpha}(\log n)^{1/r+\beta}$ ($\beta > 0$),*

$$\sum_{i=1}^{n}a_{ni}X_i/b_n\to 0\text{ a.s.}$$

Proof. Define

$$\begin{aligned}X_{ni}^{'}&=X_iI\left(|X_i|\leq\left(h^{-1}\log n\right)^{1/r}\right)\\&-\left(h^{-1}\log\ n\right)^{1/r}I\left(X_i<-\left(h^{-1}\log n\right)^{1/r}\right)\\&+\left(h^{-1}\log\ n\right)^{1/r}I\left(X_i>(h^{-1}\log\ n)^{1/r}\right)\end{aligned}$$

and $X_{ni}^{''} = X_i - X_{ni}^{'}$ for $1 \leq i \leq n$ and $n \geq 1$. The rest of the proof is similar to that of Theorem 3.2 and is omitted. □

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