J. Korean Math. Soc. **42** (2005), No. 5, pp. 933–948

IDEALS AND SUBMODULES OF MULTIPLICATION MODULES

SANG CHEOL LEE*, SUNAH KIM, AND SANG-CHO CHUNG

ABSTRACT. Let R be a commutative ring with identity and let M be an R-module. Then M is called a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$. Let M be a non-zero multiplication R-module. Then we prove the following:

(1) there exists a bijection : $N(M) \cap V(\mathrm{ann}_R(M)) \longrightarrow \mathrm{Spec}_R(M)$ and in particular, there exists a bijection :

$$N(M) \cap \mathrm{Max}(R) \longrightarrow \mathrm{Max}_R(M),$$

(2) $N(M) \cap V(\mathrm{ann}_R(M)) = \mathrm{Supp}(M) \cap V(\mathrm{ann}_R(M))$, and
(3) for every ideal I of R,

$$(((\sqrt{I + \mathrm{ann}_R(M)})M) :_R M) = \cap_{P \in N(M) \cap V(\mathrm{ann}_R(M))} P.$$

The ideal $\theta(M) = \sum_{m \in M}(Rm :_R M)$ of R has proved useful in studying multiplication modules. We generalize this ideal to prove the following result: Let R be a commutative ring with identity, $P \in \mathrm{Spec}(R)$, and M a non-zero R-module satisfying

(1) M is a finitely generated multiplication module,
(2) PM is a multiplication module, and
(3) $P^n M \neq P^{n+1} M$ for every positive integer n,

then $\cap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M)) \in V(\mathrm{ann}_R(M)) = \mathrm{Supp}(M) \subseteq N(M)$.

1. Introduction

Throughout this paper, we consider only commutative rings with identity and modules which are unitary. Let R be a commutative ring

Received March 17, 2004.
2000 Mathematics Subject Classification: 13E15, 13A15, 16D10.
Key words and phrases: prime submodules, maximal submodules, finitely generated modules, multiplication modules.
*The first author was supported by research funds of Chonbuk National University.

and M an R-module. Then $\mathrm{Spec}(R)$ denotes the set of all prime ideals of R and $\mathrm{Spec}_R(M)$ denotes the set of all prime submodules of M. Obviously, $\mathrm{Spec}_R(R) = \mathrm{Spec}(R)$. If N is a submodule of M, then $(N :_R M)$ is defined by $\{r \in R \mid rM \subseteq N\}$. In particular, $(0 :_R M)$ is called the *annihilator* of M and is denoted by $\mathrm{ann}_R(M)$. There are three subsets of $\mathrm{Spec}(R)$ which depend on M:

(1) $N(M) = \{P \in \mathrm{Spec}(R) \mid PM \neq M\}$,
(2) $V(\mathrm{ann}_R(M)) = \{P \in \mathrm{Spec}(R) \mid \mathrm{ann}_R(M) \subseteq P\}$,
(3) $\mathrm{Supp}(M) = \{P \in \mathrm{Spec}(R) \mid M_P \neq 0\}$.

$\mathrm{Max}(R)$ denotes the set of all maximal ideals of R and $\mathrm{Max}_R(M)$ denotes the set of all maximal submodules of M. Clearly, $\mathrm{Max}_R(R) = \mathrm{Max}(R)$. By a *quasi-local ring*, we mean a commutative ring with a unique maximal ideal.

Let R be a commutative ring and let M be an R-module. Then a submodule N of M is said to be *extended* if $N = IM$ for some ideal I of R. M is called a *multiplication module* if every submodule of M is extended. For example, every proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is a multiplication module but the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ itself is not. We generalize [8, Theorem 6] as follows. If M is a non-zero multiplication module then there exists a bijection : $N(M) \cap V(\mathrm{ann}_R(M)) \longrightarrow \mathrm{Spec}_R(M)$ and in particular, there exists a bijection : $N(M) \cap \mathrm{Max}(R) \longrightarrow \mathrm{Max}_R(M)$.

In commutative ring theory, it is well-known that, for every non-zero finitely generated module over a commutative ring R,

$$\emptyset \neq V(\mathrm{ann}_R(M)) = \mathrm{Supp}(M).$$

In Section 2, we prove that if M is a non-zero multiplication module over a commutative ring R, then $N(M) \cap V(\mathrm{ann}_R(M)) = \mathrm{Supp}(M) \cap V(\mathrm{ann}_R(M))$.

In Section 3, we are concerned with relationships between the ideals of a commutative ring and the submodules of a multiplication module over the ring. A well-known result of commutative algebra saying that the radical of an ideal I of a commutative ring is the intersection of all prime ideals containing I is generalized to non-zero multiplication modules. Let R be a commutative ring and M an R-moodule. For an ideal I of R, we define the ideal $\theta(IM) = \sum_{x \in IM}(Rx :_R M)$ of R. This is a generalization of the ideal $\theta(M)$ of R which was introduced in [1] and recently, the ideal $\theta(M)$ was studied in [3]. Let R be a commutative ring with identity and let $P \in \mathrm{Spec}(R)$. If M is a non-zero R-module satisfying

(1) M is a finitely generated multiplication module,

(2) PM is a multiplication module, and
(3) $P^nM \neq P^{n+1}M$ for every positive integer n,

then we prove by making use of the notion of the ideal $\theta(M)$ of R that

$$\bigcap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M)) \in V(\mathrm{ann}_R(M)) = \mathrm{Supp}(M) \subseteq N(M).$$

Let R be a quasi-local ring with unique maximal ideal P. Let M be a non-zero R-module satisfying

(1) M is a finitely generated multiplication module,
(2) PM is a multiplication module, and
(3) $P^nM \neq P^{n+1}M$ for every positive integer n.

Then we prove that $R/\mathrm{ann}_R(M)$ is a discrete valuation domain. Finally, in particular, it is found under what conditions a Noetherian local ring is a discrete valuation domain.

Our first lemma gives three well-known results that will be used throughout this paper.

LEMMA 1.1. *Let R be a commutative ring and M an R-module.*

(1) *If M is a multiplication R-module, then it is locally cyclic.*
(2) *If M is a multiplication R-module, then*

$$\bigcap_{I\in\mathcal{I}}(IM) = \left(\bigcap_{I\in\mathcal{I}}(I + \mathrm{ann}_R(M))\right)M$$

for any non-empty collection $\mathcal{I}$ of ideals of R.

(3) *Let M be a non-zero multiplication R-module. Then*
    (i) *for every proper submodule N of M, there exists $K \in \mathrm{Max}_R(M)$ of M such that $N \subseteq K$, and*
    (ii) *$K \in \mathrm{Max}_R(M)$ if and only if there exists $P \in \mathrm{Max}(R)$ such that $K = PM \neq M$.*

Proof. (1) Let M be a multiplication R-module. Let P be any element of $\mathrm{Spec}(R)$. Then M_P is a multiplication R_P-module by [2, Corollary 3.5]. Since over a quasi-local ring every multiplication module is cyclic, M_P is cyclic. (2) follows from [5, Corollary 1.7]. (3) follows from [5, Theorem 2.5]. □

2. Prime spectra of multiplication modules

If M is a module over a commutative ring R, then for every submodule N of M, $(N :_R M) = \mathrm{ann}_R(M/N)$. The following lemma was motivated by definitions in [5, p.765] and [6, p.791].

LEMMA 2.1. *Let M be a non-zero R-module and let N be a submodule of M with $N \neq M$. Then the following statements are equivalent:*

(1) *$(N :_R K) = (N :_R M)$ for every submodule K of M such that $K \supsetneqq N$.*
(2) *If $ax \in N$, where $a \in R$ and $x \in M$, then $a \in (N :_R M)$ or $x \in N$.*

Proof. Assume (1). Assume $ax \in N$, where $a \in R$ and $x \in M$. Assume $x \notin N$. Then $N \subsetneqq N + Rx \subseteq M$. By (1), $(N :_R (N + Rx)) = (N :_R M)$. Since $ax \in N$, we have $a(N + Rx) = aN + Rax \subseteq N$. This shows that $a \in (N :_R (N + Rx))$. Hence, $a \in (N :_R M)$.

Conversely, assume (2). Let K be any submodule of M such that $K \supsetneqq N$. Then $K/N \subseteq M/N$ and so,

$$(N :_R K) = \mathrm{ann}_R(K/N) \supseteq \mathrm{ann}_R(M/N) = (N :_R M)$$

Let a be any element of $(N :_R K)$. Since $N \subsetneqq K$, we can find an element x of $K \backslash N$. Then $ax \in N$. Hence, by (2), $a \in (N :_R M)$. □

Let R be a commutative ring and let M be a non-zero R-module. Let N be a submodule of M. Then N is called a *prime submodule* of M if

(1) $N \neq M$ and
(2) N satisfies either (hence both) of the statements in Lemma 2.1.

Let R be a commutative ring and M an R-module. Then a submodule N of M is called an *extended submodule* if there exists an ideal I of R such that $N = IM$. M is called a *multiplication module* if every submodule of M is extended.

EXAMPLE 2.2. Consider the ring $\mathbb{Z}$ of integers. Let p be a fixed prime number. If we adapt the proof of the well-known fact that $\mathbb{Z}(p^\infty)$ is divisible, then we can get the following:

(1) the only proper, extended submodule of the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is 0, and
(2) every proper submodule of the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is a multiplication module but the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ itself is not.

Every finite-dimensional vector space with dimension greater than 1 cannot be a multiplication module. □

Compare the next result with [5, Corollary 1.7].

PROPOSITION 2.3. *Let R be a commutative ring and M an R-module. Then M is a multiplication module if and only if $\bigcap_{A\in\mathcal{A}} A = (\bigcap_{A\in\mathcal{A}}(A :_R M))M$ for any non-empty collection $\mathcal{A}$ of submodules of M.*

Proof. Assume that M is a multiplication module. Let $\mathcal{A}$ be any non-empty collection of submodules of M. Then

$$\bigcap_{A\in\mathcal{A}} A = ((\bigcap_{A\in\mathcal{A}} A) :_R M)M = (\bigcap_{A\in\mathcal{A}} (A :_R M))M$$

with the first equality following since M is a multiplication module and the second since residuation distributes over intersection.

Conversely, assume that $\bigcap_{A\in\mathcal{A}} A = (\bigcap_{A\in\mathcal{A}}(A :_R M))M$ for any non-empty collection $\mathcal{A}$ of submodules of M. Let N be any submodule of M. Then $\{N\}$ is a non-empty collection of a submodule of M. By our assumption, $N = (N :_R M)M$. Hence, M is a multiplication module. □

Let R be a ring. If M is a non-zero R-module, then $\operatorname{ann}_R(M) \neq R$. By Zorn's Lemma, $V(\operatorname{ann}_R(M)) \neq \emptyset$.

LEMMA 2.4. *Let R be a commutative ring. Let M be a non-zero multiplication module. Then*

(1) $(PM :_R M) = \begin{cases} P + \operatorname{ann}_R(M) & \text{if } P \in N(M) \\ R & \text{if } P \notin N(M) \end{cases}$

(2) *PM is an element of $\operatorname{Spec}_R(M)$ if $P \in N(M)$.*

Proof. (1) Clearly, $P+\operatorname{ann}_R(M) \subseteq (PM :_R M)$. Conversely, let a be any element of $(PM :_R M)$. Then $aM \subseteq PM$. Assume that $P \in N(M)$. Then we can take an element $x \in M\backslash PM$. Hence, $ax \in PM$.

M can be given $R/\operatorname{ann}_R(M)$-module structure as follows: for any $r \in R$ and $m \in M$, define $(r + \operatorname{ann}_R(M))m = rm$. Then the module structure is well-defined. M becomes an $R/\operatorname{ann}_R(M)$-module. Moreover, as an $R/\operatorname{ann}_R(M)$-module, M is a multiplication module. Since $ax \in PM$, we have $(a+\operatorname{ann}_R(M))x \in (P/\operatorname{ann}_R(M))M$. Further, since $x \notin PM$, we have $x \notin (P/\operatorname{ann}_R(M))M$. By [5, Lemma 2.10], we have $a + \operatorname{ann}_R(M) \in P/\operatorname{ann}_R(M)$. This implies $a \in P + \operatorname{ann}_R(M)$. Thus, $(PM :_R M) \subseteq P + \operatorname{ann}_R(M)$. Therefore, $(PM :_R M) = P + \operatorname{ann}_R(M)$.

Assume now that $PM = M$. Then $(PM :_R M) = (M :_R M) = R$.

(2) Let $ax \in PM$, where $a \in R$ and $x \in M$. Then as in the proof of (1), we can show that either $a \in P + \text{ann}_R(M)$ or $x \in PM$. If $a \in P + \text{ann}_R(M)$, then $a \in (PM :_R M)$. Thus, either $a \in (PM :_R M)$ or $x \in PM$. Hence, PM is a prime submodule of M if $PM \neq M$. □

The following result generalizes [8, Theorem 6. (*c*) ⇒ (*d*)] and [7, p.216, Property 1].

THEOREM 2.5. *Let R be a commutative ring. Let M be a non-zero multiplication module. Then there is a one-to-one order-preserving correspondence: $N(M) \bigcap V(\text{ann}_R(M)) \longrightarrow \text{Spec}_R(M)$*

Proof. Let $\mathcal{X} = N(M) \cap V(\text{ann}_R(M))$ and let $\mathcal{Y} = \text{Spec}_R(M)$. Define a map $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ by $\varphi(P) = PM$, where $P \in \mathcal{X}$. Then by Lemma 2.4(2), φ is well-defined. Now, define a map $\psi : \mathcal{Y} \longrightarrow \mathcal{X}$ by $\psi(N) = (N :_R M)$, where $N \in \mathcal{Y}$. Let N be any prime submodule of M. Then $\text{ann}_R(M/N)$ is a prime ideal of R and $\text{ann}_R(M) \subseteq \text{ann}_R(M/N)$ by definitions and hence $(N :_R M)$ is a prime ideal of R containing $\text{ann}_R(M)$. Further, since M is a multiplication module, we have $(N :_R M)M = N \neq M$. Hence, ψ is well-defined.

Let P be any element of $\mathcal{X}$. Then by Lemma 2.4(1),

$$(\psi \circ \varphi)(P) = \psi(\varphi(P)) = \psi(PM) = (PM :_R M) = P.$$

Hence, $\psi \circ \varphi = 1_{\mathcal{X}}$. Thus, φ is one-to-one.

Let N be any element of $\mathcal{Y}$. Then since M is a multiplication module,

$$(\varphi \circ \psi)(N) = \varphi(\psi(N)) = \varphi(N :_R M) = (N :_R M)M = N$$

Hence, $\varphi \circ \psi = 1_{\mathcal{Y}}$. Thus, φ is onto. Therefore, φ is a one-to-one correspondence between $\mathcal{X}$ and $\mathcal{Y}$. Moreover, it is clear that φ is order-preserving. □

If M is a non-zero multiplication module over a commutative ring R, then it follows from Theorem 2.5 that every prime submodule of M is of the form PM, where $P \in N(M) \cap V(\text{ann}_R(M))$.

LEMMA 2.6. *Let R be a commutative ring and M a non-zero module. Then $N(M) \cap \text{Max}(R) \subseteq V(\text{ann}_R(M))$.*

Proof. Assume that P is a maximal ideal of R such that $PM \neq M$. Suppose $\mathrm{ann}_R(M) \nsubseteq P$. Then $P + \mathrm{ann}_R(M) = R$. Hence,

$$M = RM = (P + \mathrm{ann}_R(M))M \subseteq PM + (\mathrm{ann}_R(M))M = PM,$$

and so $M = PM$. This contradiction shows that $\mathrm{ann}_R(M) \subseteq P$. □

Let R be a commutative ring and let M be a non-zero multiplication module. Then by Lemma 1.1 or [8, Theorem 2 (4)], $\mathrm{Max}_R(M) \neq \emptyset$. Compare the following result with [8, Theorem 2 (1)].

COROLLARY 2.7. *Let R be a commutative ring and M a non-zero multiplication module. Then there is a one-to-one order-preserving correspondence : $N(M) \cap Max(R) \longrightarrow Max_R(M)$.*

Proof. Let $\mathcal{X} = N(M) \cap V(\mathrm{ann}_R(M))$ and let $\mathcal{Y} = \mathrm{Spec}_R(M)$. Define a map $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ by $\varphi(P) = PM$, where $P \in \mathcal{X}$. Then by the proof of Theorem 2.5, φ is a one-to-one correspondence. Let $\mathcal{X}' = N(M) \cap \mathrm{Max}(R)$ and let $\mathcal{Y}' = \mathrm{Max}_R(M)$. Since every maximal ideal of R is prime, it follows from Lemma 2.6 that $\mathcal{X}' \subseteq \mathcal{X}$. We can now consider the restriction of φ to $\mathcal{X}'$ $\varphi|_{\mathcal{X}'} : \mathcal{X}' \longrightarrow \mathcal{Y}$. Then since φ is one-to-one, so is $\varphi|_{\mathcal{X}'}$.

Let P be a maximal ideal of R such that $M \neq PM$. Then by Lemma 1.1, there is a maximal submodule K of M such that $PM \subseteq K$. Hence, $P \subseteq PM :_R M \subseteq K :_R M \neq R$ and so $P = K :_R M$. Thus, $K = (K :_R M)M = PM$. This shows that PM is a maximal submodule of M. Therefore, in particular, $\mathrm{Im}(\varphi|_{\mathcal{X}'}) \subseteq \mathcal{Y}'$. Further, it follows from Lemma 1.1 that $\mathcal{Y}' \subseteq \mathrm{Im}(\varphi|_{\mathcal{X}'})$. Hence, $\mathrm{Im}(\varphi|_{\mathcal{X}'}) = \mathcal{Y}'$. Thus, $\varphi|_{\mathcal{X}'} : \mathcal{X}' \longrightarrow \mathcal{Y}'$ is a one-to-one correspondence. Moreover, it is clear that $\varphi|_{\mathcal{X}'}$ is order-preserving. □

If M is a non-zero multiplication module over a commutative ring R, then it follows from Corollary 2.7 that every maximal submodule of M is of the form PM where $P \in N(M) \cap \mathrm{Max}(R)$.

3. Multiplication modules

Let I be an ideal of a commutative ring R. Recall from [6, p.792] that an R-module M is said to be *I-torsion* if for each $m \in M$ there exists an element $i \in I$ such that $(1 - i)m = 0$.

Let I be an ideal of R and M a finitely generated R-module. Then it follows from standard determinant argument that M is I-torsion if and only if $M = IM$.

LEMMA 3.1. *Let I be an ideal of R and M a multiplication R-module. Then M is I-torsion if and only if $M = IM$.*

Proof. Adapt the proof of [10, p.229, Lemma 6] to show this. □

Let P be a maximal ideal of a commutative ring R. Recall [10, p.223] that an R-module M is said to be *P-cyclic* if there exists an element $x \in M$ and an element $p \in P$ such that $(1-p)M \subseteq Rx$.

DEFINITION 3.2. Let I be an ideal of a commutative ring R. An R-module M is said to be *I-cyclic* if there exists a maximal ideal P of R containing I such that M is P-cyclic.

Every R-module is R-torsion but no R-module is R-cyclic.

Let P be a maximal ideal of a commutative ring R. Let M be an R-module. Then we remark that M is P-cyclic when we regard P as an ideal if and only if it is P-cyclic when we regard P as a maximal ideal.

PROPOSITION 3.3. *Let R be a commutative ring and M an R-module. Then the following statements are equivalent.*

(1) *For every proper ideal I of R, M is I-cyclic.*
(2) *For every maximal ideal P of R, M is P-cyclic.*

Proof. Assume (1). Let P be any maximal ideal of R. Then P is a proper ideal of R. By (1), there exists a maximal ideal Q of R with $Q \supseteq P$ such that M is Q-cyclic. Since P is maximal, we must have $Q = P$. Hence, M is P-cyclic.

Assume (2). Let I be any proper ideal of R. There exists a maximal ideal P of R such that $P \supseteq I$. By (2), M is P-cyclic. Thus, M is I-cyclic. □

THEOREM 3.4. *Let R be a commutative ring and let M be a non-zero R-module. Then the following statements are equivalent.*

(1) *M is a multiplication module.*
(2) *For every ideal I of R either M is I-torsion or M is I-cyclic.*
(3) *For every maximal ideal P of R either M is P-torsion or M is P-cyclic.*

Proof. Assume (1). Let I be any ideal of R. Then $M = IM$ or $M \neq IM$.

Assume that $M = IM$. Then by Lemma 3.1, M is I-torsion.

Assume now that $M \neq IM$. Then by Lemma 1.1, there is a maximal submodule K of M such that $IM \subseteq K$. Further, by Lemma 1.1, there is a maximal ideal P of R such that $K = PM$. Since $PM \neq M$, it follows from Lemma 2.6 that $\text{ann}_R(M) \subseteq P$. Hence, by Lemma 2.4, $(PM :_R M) = P$. Thus, $I \subseteq (IM :_R M) \subseteq (K :_R M) = (PM :_R M) = P$. Since $PM \subsetneqq M$, we can take an element $x \in M \backslash PM$. By (1), there exists an ideal J of R such that $Rx = JM$. If J were a subset of P, then x would be an element of PM since $x \in Rx = JM \subseteq PM$. Hence, $J \not\subseteq P$. Since P is maximal, we have $P + J = R$. There exists an element $p \in P$ such that $1 - p \in J$. Further, $(1 - p)M \subseteq JM = Rx$. Hence, M is P-cyclic. This shows that M is I-cyclic. Therefore, (2) follows.

It follows from the remark just prior to Proposition 3.3 that (2) implies (3).

Finally, it follows from [5, Theorem 1.2] that (3) implies (1). □

THEOREM 3.5. *Let R be a commutative ring and M a non-zero multiplication R-module. Then*

(1) *$Supp(M) \subseteq N(M)$.*
(2) *$N(M) \cap V(ann_R(M)) = Supp(M) \cap V(ann_R(M))$.*

Proof. (1) There are two ways to prove this.

Method I. Use Lemma 3.1 to show this.

Method II. Assume that P is a prime ideal of R and M is a non-zero multiplication module with $M = PM$. By Lemma 1.1, M_P is cyclic. Further, $M_P = PR_PM_P$. By Nakayama's Lemma, $M_P = 0$.

(2) By (1), it suffices to prove

$$N(M) \cap V(\text{ann}_R(M)) \subseteq \text{Supp}(M) \cap V(\text{ann}_R(M)).$$

Assume that $P \in N(M) \cap V(\text{ann}_R(M))$. By Lemma 3.1, M is not P-torsion. By Theorem 3.4, M is P-cyclic. Hence, there exists an element $x \in M$ and an element $p \in P$ such that $(1 - p)M \subseteq Rx$. Then $x/1$ is a non-zero element of M_P. For, otherwise there exists an element $s \in R \backslash P$ such that $sx = 0$; hence

$$s(1 - p)M \subseteq s(Rx) = (sR)x = (Rs)x = R(sx) = 0$$

and so $s(1 - p) \in \text{ann}_R(M) \subseteq P$, a contradiction. Therefore, $M_P \neq 0$.□

4. Ideals and submodules of multiplication modules.

In this section we will be concerned with relationships between the ideals of a commutative ring and the submodules of a non-zero multiplication module over the commutative ring.

PROPOSITION 4.1. *Let R be a commutative ring and M a non-zero multiplication module. Then the following statements hold.*

(1) *For every ideal I of R with $M \neq IM$, there exists a maximal ideal P of R containing $I + \mathrm{ann}_R(M)$ such that PM is a maximal submodule of M.*
(2) *If P is a prime ideal of R containing $\mathrm{ann}_R(M)$ such that $M \neq PM$, then $P + J = R$ for every ideal J of R with $M = JM$.*
(3) *For every ideal I of R with $M \neq IM$ and for every ideal J of R with $M = JM$, there exists a maximal ideal P of R containing $I + \mathrm{ann}_R(M)$ such that $P + J = R$ and PM is a maximal submodule of M.*

Proof. (1) Let I be any ideal of R with $M \neq IM$. Then by Lemma 1.1, there is a maximal submodule K of M such that $IM \subseteq K$. Further, by Lemma 1.1, there is a maximal ideal P of R such that $K = PM$. Since $PM \neq M$, it follows Lemma 2.6 that $\mathrm{ann}_R(M) \subseteq P$. Suppose that $I \not\subseteq P$. Then $I + P = R$. Since $IM \subseteq K = PM$, it then follows that

$$M = RM = (I + P)M \subseteq IM + PM = PM.$$

Hence, $M = PM$. This contradiction shows that $I \subseteq P$. Thus, $I + \mathrm{ann}_R(M) \subseteq P$.

(2) Let P be any prime ideal of R containing $\mathrm{ann}_R(M)$ such that $M \neq PM$. Let J be any ideal of R with $M = JM$. Then there exists an element $x \in M \backslash PM$. Further, since M is a multiplication module and $M = JM$, it follows from Lemma 3.1 that M is J-torsion. Hence, there exists an element $j \in J$ such that $(1 - j)x = 0$. Further, $(1 - j)x = 0 \in PM$. By Lemma 2.4(2), PM is a prime submodule of M. Hence, $1 - j \in P$. Therefore, $P + J = R$.

(3) follows from (1) and (2). □

Given an ideal I of a commutative ring R, the *radical* of I, denoted by $\sqrt{I}$, is defined by $\{r \in R \mid r^n \in I$ for some positive integer $n\}$. It is well-known that if I is an ideal of a commutative ring R, then $\sqrt{I} = \bigcap_{P \in V(I)} P$. We will generalize this.

THEOREM 4.2. *Let R be a commutative ring. Let M be a non-zero multiplication module. Then for every ideal I of R,*

$$\left(\left(\left(\sqrt{I + \mathrm{ann}_R(M)}\right) M\right) :_R M\right) = \bigcap_{P \in V(I + \mathrm{ann}_R(M)) \bigcap N(M)} P.$$

Proof. Let I be any ideal of R. Assume that $IM = M$. Then

$$R = (M :_R M) = (IM :_R M) \subseteq (((\sqrt{I + \text{ann}_R(M)})M) :_R M).$$

Hence, $(((\sqrt{I + \text{ann}_R(M)})M) :_R M) = R$. Let $\mathcal{A} = V(I + \text{ann}_R(M)) \cap N(M)$. Then $\mathcal{A} = \emptyset$. For, otherwise there exists a prime ideal P of R containing $I + \text{ann}_R(M) \subseteq P$ and $PM \neq M$. Then

$$M = IM = (I + \text{ann}_R(M))M \subseteq PM \subsetneqq M,$$

a contradiction. Hence, $\bigcap_{p\in\mathcal{A}} P = R$. Therefore,

$$\left(\left(\left(\sqrt{I + \text{ann}_R(M)}\right) M\right) :_R M\right) = \bigcap_{p\in\mathcal{A}} P.$$

Now, assume $IM \neq M$. Then $I + \text{ann}_R(M) \neq R$. There exists a prime ideal Q of R such that $I + \text{ann}_R(M) \subseteq Q$. Let $\mathcal{P} = V(I + \text{ann}_R(M))$. Then $Q \in \mathcal{P}$. In particular, $\mathcal{P} \neq \emptyset$. Then it is easy to show that

$$\left(\left(\bigcap_{P\in\mathcal{P}} (PM)\right) :_R M\right) = \bigcap_{P\in\mathcal{P}} (PM :_R M).$$

By Proposition 4.1(1), $\mathcal{A} \neq \emptyset$. Let $\mathcal{B} = V(I + \text{ann}_R(M)) \cap (\text{Spec}(R) \setminus N(M))$. Then $\mathcal{P} = \mathcal{A} \cup \mathcal{B}$. Hence, by Lemma 1.1 and Lemma 2.4(1), we have

$$\begin{aligned}
&\left(\left(\left(\sqrt{I + \text{ann}_R(M)}\right) M\right) :_R M\right) \\
&= \left(\left(\left(\bigcap_{P\in\mathcal{P}} P\right) M\right) :_R M\right) \\
&= \left(\left(\bigcap_{P\in\mathcal{P}} (PM)\right) :_R M\right) \\
&= \bigcap_{P\in\mathcal{P}} \left(PM :_R M\right) \\
&= \left(\bigcap_{P\in\mathcal{A}} \left(PM :_R M\right)\right) \cap \left(\bigcap_{P\in\mathcal{B}} \left(PM :_R M\right)\right) \\
&= \bigcap_{P\in\mathcal{A}} P.
\end{aligned}$$

□

COROLLARY 4.3. *If M is a non-zero faithfully flat multiplication module over a commutative ring R, then for every ideal I of R,*

$$\left(\left(\left(\sqrt{I+\mathrm{ann}_R(M)}\right)M\right):_R M\right)=\sqrt{I+\mathrm{ann}_R(M)}.$$

Proof. Let I be any ideal of R. Then with the same notations as in the proof of Theorem 4.2,

$$\sqrt{I+\mathrm{ann}_R(M)}=\bigcap_{P\in\mathcal{P}}P=\Big(\bigcap_{P\in\mathcal{A}}P\Big)\bigcap\Big(\bigcap_{P\in\mathcal{B}}P\Big).$$

If M is faithfully flat, it follows from [9, Theorem 7.2] that $\mathcal{B}=\emptyset$. Hence, by Theorem 4.2,

$$\sqrt{I+\mathrm{ann}_R(M)}=\bigcap_{P\in\mathcal{A}}P=\left(\left(\left(\sqrt{I+\mathrm{ann}_R(M)}M:_R M\right).\qquad\square$$

For any ideal I of R, let $I^0M=M$ and $I^\infty M=\bigcap_{n=1}^{\infty}(I^nM)$. [6, p.791, Lemma 3.1 (ii)] can be recast as follows.

LEMMA 4.4. *Let R be a commutative ring and P an ideal of R. Let M be an R-module such that PM is a multiplication module. Then for any submodule N of PM, either $N\subseteq P^\infty M$ or there exists a positive integer k and k ideals $I_0, I_1, \cdots, I_{k-1}$ of R with $I_0\not\subseteq P$, $I_1\not\subseteq P^2, \cdots$, $I_{k-1}\not\subseteq P^k$ such that*

$$N=I_0P^kM=I_1P^{k-1}M=\cdots=I_{k-1}PM.$$

Proof. Assume that N is a submodule of PM such that $N\not\subseteq P^\infty M$. Then there exists a positive integer k such that $N\subseteq P^kM$ but $N\not\subseteq P^{k+1}M$. Since for each $i\in\{0,1,\cdots,k-1\}$, $N\subseteq P^kM\subseteq P^{k-i}M$ and by [6, Lemma 3.1(i)] $P^{k-i}M$ is a multiplication module, we have, for each $i\in\{0,1,\cdots,k-1\}$, $N=(N:_R P^{k-i}M)P^{k-i}M$. Further, $(N:_R P^{k-i}M)\supseteq \mathrm{ann}_R(P^{k-i}M)$ implies $(N:_R P^{k-i}M)+\mathrm{ann}_R(P^{k-i}M)=N:_R P^{k-i}M$. Hence, it follows from Lemma 1.1 and the modular law that for each $i \in \{0, 1, \cdots, k-1\}$,

$$\begin{aligned} N &= N \cap P^k M \\ &= N \cap (P^i P^{k-i} M) \\ &= ((N :_R P^{k-i} M) P^{k-i} M) \cap (P^i P^{k-i} M) \\ &= (((N :_R P^{k-i} M) + \operatorname{ann}_R(P^{k-i} M)) \\ &\qquad \cap (P^i + \operatorname{ann}_R(P^{k-i} M))) P^{k-i} M \\ &= ((N :_R P^{k-i} M) \cap (P^i + \operatorname{ann}_R(P^{k-i} M))) P^{k-i} M \\ &= (((N :_R P^{k-i} M) \cap P^i) + \operatorname{ann}_R(P^{k-i} M)) P^{k-i} M \\ &= ((N :_R P^{k-i} M) \cap P^i) P^{k-i} M \end{aligned}$$

Now, for each $i \in \{0, 1, \cdots, k-1\}$, let $I_i = (N :_R P^{k-i} M) \cap P^i$. Then

$$N = I_0 P^k M = I_1 P^{k-1} M = \cdots = I_{k-1} PM.$$

Further, since $N \nsubseteq P^{k+1} M$, we get $I_0 \nsubseteq P$, $I_1 \nsubseteq P^2$, $\cdots$, $I_{k-1} \nsubseteq P^k$, as required. □

Let R be a commutative ring and M an R-module. The ideal $\theta(M) = \sum_{m \in M} (Rm :_R M)$ of R has proved useful in studying multiplication modules. We generalize this ideal as follows: $\theta(IM) = \sum_{x \in IM} (Rx :_R M)$ for an ideal of a commutative ring R and an R-module M. It is always true that $I\theta(M) \subseteq \theta(IM)$ for every ideal I of a commutative ring R and for every module M over the ring R. If M is a multiplication module over a commutative ring R, then for every ideal I of R,

$$\begin{aligned} IM &= \sum_{x \in IM} Rx \\ &= \sum_{x \in IM} ((Rx :_R M) M) \\ &= \Big(\sum_{x \in IM} (Rx :_R M) \Big) M \\ &= \theta(IM) M \end{aligned}$$

and $IM = (IM :_R M) M$. Hence, we have the following result.

LEMMA 4.5. *Let R be a commutative ring and M a multiplication R-module. Then the following conditions are equivalent:*

(1) *M is finitely generated, and*
(2) *for every ideal I of R, $\theta(IM) = (IM :_R M) = I + \mathrm{ann}_R(M)$.*

Proof. (1) $\Rightarrow$ (2) follows from [10, Theorem 9 Corollary].

(2) $\Rightarrow$ (1). (2) gives $\theta(M) = R$. Hence, it follows from [3, Corollary 2.2] that M is finitely generated. □

THEOREM 4.6. *Let R be a commutative ring and let P be a maximal ideal of R. Let M be a non-zero R-module satisfying*

(1) *M is a finitely generated multiplication module,*
(2) *PM is a multiplication module, and*
(3) *$P^nM \neq P^{n+1}M$ for every positive integer n.*

Then $\bigcap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M)) \in V(\mathrm{ann}_R(M)) = Supp(M) \subseteq N(M)$.

Proof. By [6, Corollary 3.2], $P^{\infty}M$ is a prime submodule of M. By the statement just prior to Lemma 2.6, there exists a prime ideal Q of R containing $\mathrm{ann}_R(M)$ with $QM \neq M$ such that $P^{\infty}M = QM$. It suffices to prove that $Q = \bigcap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M))$.

By Lemma 1.1, we have

$$QM = P^{\infty}M = \bigcap_{n=1}^{\infty}(P^nM) = \left(\bigcap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M))\right)M.$$

Hence, by Lemma 4.5, we have

$$\begin{aligned} Q = \theta(QM) &= \theta\left(\left(\left(\bigcap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M)\right)\right)M\right) \\ &= \bigcap_{n=1}^{\infty}(P^n + \mathrm{ann}_R(M)), \end{aligned}$$

as required. □

Note that intersection of powers of multiplication ideals are considered in [4, Theorem 2.2]. [4, Theorem 4.1] says: Let (R, P) be a quasi-local ring whose maximal ideal P is finitely generated. Then R is Noetherian if and only if for every finitely generated ideal I of R, $\bigcap_{n=1}^{\infty}(P^n + I) = I$. Therefore, by Theorem 4.6, we have the following result.

COROLLARY 4.7. *Let R be a Noetherian local ring with unique maximal ideal P. Let M be a non-zero R-module satisfying*

(1) *M is a multiplication module,*
(2) *PM is a multiplication module, and*
(3) *$P^n M \neq P^{n+1} M$ for every positive integer n.*

Then $R/\text{ann}_R(M)$ is a discrete valuation domain.

Proof. Over a quasi-local ring a multiplication module is cyclic. So $M = R/\text{ann}_R(M)$. Now $PM = P/\text{ann}_R(M)$ is principal so $R/\text{ann}_R(M)$ is a PIR. Then (3) gives that $R/\text{ann}_R(M)$ is a DVR. Further, by Theorem 4.6, $R/\text{ann}_R(M)$ is an integral domain. □

Notice that if a module over a commutative ring satisfies the assumptions of Corollary 4.7, then it is Noetherian module but not Artinian.

COROLLARY 4.8. *Let R be a Noetherian local ring with unique maximal ideal P satisfying*

(1) *P is a multiplication ideal of R and*
(2) *$P^n \neq P^{n+1}$ for every positive integer n.*

Then R is a discrete valuation domain.

References

[1] D. D. Anderson, *Some Remarks on Multiplication Ideals*, Math. Japon. **25** (1980), 463–469.

[2] ———, *Some Remarks on Multiplication Modules II*, Comm. Algebra **28** (2000), no. 5, 2577–2583.

[3] D. D. Anderson and Yousef Al-Shaniafi, *Multiplication Modules and the Ideal theta(M)*, Comm. Algebra **30** (2002), no. 7, 3383–3390.

[4] D. D. Anderson, J. Matijevic, and Nichols, *The Krull Intersection Theorem II*, Pacific J. Math. **66** (1976), no. 1, 15–22.

[5] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), no. 4, 755–779.

[6] ———, *Multiplication Modules and Theorems of Mori and Mott*, Comm. Algebra **16** (1988), no. 4, 781–796.

[7] C. P. Lu, *M-radicals of submodules*, Math. Japan. **34** (1989), no. 2, 211–219.

[8] ———, *Spectra of Modules*, Comm. Algebra **23** (1995), no. 10, 3741–3752.

[9] Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1989.

[10] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math. **50** (1988), 223–235.

Sang Cheol Lee
Department of Mathematics Education
Chonbuk National University
Chonju 561-756, Korea
E-mail: scl@chonbuk.ac.kr

Sunah Kim
Department of Mathematics
Chosun University
Kwangju 501-759, Korea
E-mail: sakim@mail.chosun.ac.kr

Sang-Cho Chung
Department of Mathematics
Chungnam National University
Taejon 305-764, Korea
E-mail: scchung@math.cnu.ac.kr