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PSEUDO-SYMMETRIC CONTACT 3-MANIFOLDS

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Dedicated to Professor Masami Sekizawa on the occasion of his sixtieth birthday

ABSTRACT. Contact Homogeneous 3-manifolds are pseudo-symmetric spaces of constant type. All Sasakian 3-manifolds are pseudo-symmetric spaces of constant type.

0. Introduction

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$, where R is the Riemannian curvature tensor and $R \cdot R$ is the derivative of R by R (see section 1). Obviously, locally symmetric spaces are semi-symmetric.

As a generalization of the semi-symmetry, R. Deszcz[13] introduced the notion of *pseudo-symmetry*. A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if there exists a function L such that $R(X, Y) \cdot R = L\{(X \wedge Y) \cdot R\}$ for all vector fields X and Y on M. Here $X \wedge Y$ is the endomorphism field defined by

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y.$$

A pseudo-symmetric space (M, g) is said to be *proper* if M is not semi-symmetric. In particular, a pseudo-symmetric space is called a *pseudo-symmetric space of constant type* if L is a constant. Semi-symmetric spaces are pseudo-symmetric spaces of constant type with $L = 0$. Three-dimensional pseudo-symmetric spaces of constant type have been studied extensively by O. Kowalski and M. Sekizawa[20]–[23]. N. Hashimoto and M. Sekizawa classified 3-dimensional conformally flat pseudo-symmetric spaces of constant type [15].

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As is well-known, for a Riemannian 3-manifold, its Riemannian curvature is determined by the Ricci curvature. In fact, the Riemannian curvature tensor R is expressed as

$$R(X,Y)Z = S(Y,Z)X - S(Z,X)Y \\ + g(Y,Z)QX - g(Z,X)QY - \frac{\mathrm{s}}{2}(X \wedge Y)Z,$$

where S is the Ricci tensor, Q is the corresponding Ricci operator and s is the scalar curvature. This fundamental fact implies that, in 3-dimensional Riemannian geometry, the constancy of the sectional curvature is equivalent to the Einstein condition, i.e., $\rho_1 = \rho_2 = \rho_3$ for the eigenvalues $\{\rho_j\}$ of the Ricci tensor. Moreover, the pseudo-symmetry is equivalent to the condition: the eigenvalues ρ_1, ρ_2, ρ_3 of the Ricci tensor satisfies $\rho_1 = \rho_2$ (up to numeration) in 3-dimension. Thus the pseudo-symmetry is a natural generalization of the constant curvature property in 3-dimension.

It is well known that the maximum dimension of the isometry group is 6 in 3-dimensional Riemannian geometry. The maximum dimension is attained by spaces of constant curvature. There is no Riemannian 3-manifold with 5-dimensional isometry group. Riemannian 3-manifolds with 4-dimensional isometry group are homogeneous. Moreover, such spaces are locally isometric to one of the following spaces; the special unitary group SU(2), the Heisenberg group $\mathbb{H}_3$, the special linear group SL(2, **R**), product spaces $S^2 \times \mathbf{R}$ or $H^2 \times \mathbf{R}$. These three Lie groups appear in the several classification tables, eg., naturally reductive Riemannian homogeneous 3-manifolds [33], 3-dimensional Sasakian space forms [6], 3-dimensional D'Atri spaces [18], or the model geometries in the sense of W. M. Thurston[32]. It is straightforward to check that every Riemannian 3-manifold with 4-dimensional isometry group is a pseudo-symmetric space of constant type. (See Appendix.)

In this article, we concentrate on the pseudo-symmetry of contact Riemannian 3-manifolds. In section 2, we shall show that every Sasakian 3-manifold is pseudo-symmetric of constant type. Next, in section 3, we shall investigate non-Sasakian contact homogeneous 3-manifolds. Our main result is that all the contact homogeneous 3-manifolds are pseudo-symmetric spaces of constant type. Furthermore, we exhibit explicit examples of 3-dimensional proper pseudo-symmetric spaces of constant type.

1. Preliminaries

Let (M, g) be a Riemannian manifold with its Levi-Civita connection ∇. Denote by R the Riemannian curvature tensor of M:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Here $\mathfrak{X}(M)$ is the Lie algebra of all vector fields on M. A tensor field F of type $(1,3)$;

$$F : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is said to be *curvature-like* provided that F has the symmetric properties of R. For example,

$$(1.1) \qquad (X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y, \quad X, Y \in \mathfrak{X}(M)$$

defines a curvature-like tensor field on M. Note that the curvature tensor R of a Riemannian manifold (M, g) of constant curvature c satisfies the formula $R(X,Y) = c(X \wedge Y)$.

As is well known, a curvature-like tensor field F acts on the algebra $\mathcal{T}_s^1(M)$ of all tensor fields on M of type $(1, s)$ as a derivation ([25], p.44):

$$\begin{aligned}&(F \cdot P)(X_1, \cdots, X_s; Y, X)\\ &= F(X,Y)\{P(X_1, \cdots, X_s)\} - \sum_{j=1}^{s} P(X_1, \cdots, F(X,Y)X_j, \cdots, X_s),\end{aligned}$$

where $X_1, \cdots, X_s \in \mathfrak{X}(M)$, $P \in \mathcal{T}_s^1(M)$. The derivative $F \cdot P$ of P by F is a tensor field of type $(1, s+2)$.

For a tensor field P of type $(1, s)$, we denote by $\mathcal{Q}(g, P)$ the derivative of P with respect to the curvature-like tensor defined by (1.1);

$$\begin{aligned}\mathcal{Q}(g,P)(X_1, \cdots, X_s; Y, X) = {}& (X \wedge Y)P(X_1, \cdots, X_s)\\ &- \sum_{j=1}^{s} P(X_1, \cdots, (X \wedge Y)X_j, \cdots, X_s).\end{aligned}$$

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$. Obviously, locally symmetric spaces ($\nabla R = 0$) are semi-symmetric.

R. Deszcz[13] introduced the notion of a pseudo-symmetric space. A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if

$$R \cdot R = L\ Q(g, R)$$

for some function L. In particular, if L is constant, M is called a *pseudo-symmetric space of constant type* [21]. A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric.

For Riemannian 3-manifolds, the following characterizations of the pseudo-symmetry are known (cf. [12, 21, 22]).

PROPOSITION 1.1. *A Riemannian 3-manifold (M, g) is pseudo-symmetric if and only if it is quasi-Einstein. Namely, there exists a one-form ω such that the Ricci tensor field S has the form:*

$$S = a\ g + b\ \omega \otimes \omega,$$

where a and b are functions.

PROPOSITION 1.2. *Let (M, g) be a Riemannian 3-manifold. Then (M, g) is a pseudo-symmetric space of constant type if and only if there exists a one-form ω such that the Ricci tensor S is expressed as $S = a\ g + b\omega \otimes \omega$, where a is a function and b is a constant.*

REMARK 1.3. The preceding proposition can be rephrased as follows (see [21], Proposition 0.1):

A Riemannian 3-manifold is a pseudo-symmetric space of constant type with $R \cdot R = L\ Q(g, R)$ if and only if the eigenvalues of the Ricci tensor locally satisfy the following relations (up to numeration):

$$\rho_1 = \rho_2, \quad \rho_3 = 2L(\text{constant}).$$

2. Contact manifolds

Let M be a 3-dimensional manifold. A *contact form* is a one-form η such that $\eta \wedge d\eta \neq 0$ on M. A 3-manifold M together with a contact form η is called a *contact 3-manifold.* The *Reeb vector field* ξ is the unique vector field satisfying $\eta(\xi) = 1,\ d\eta(\xi, \cdot) = 0$.

On a contact 3-manifold (M,η), there exists a $(1,1)$-tensor field φ and a Riemannian metric g such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.1}$$

$$g(X, \varphi Y) = d\eta(X,Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

The structure (φ, ξ, η, g) is called the associated contact Riemannian structure of (M,η). A contact 3-manifold together with its associated contact Riemannian structure is called a *contact Riemannian 3-manifold.* A contact Riemannian 3-manifold M satisfies the following formula [31]:

$$(\nabla_X \varphi)Y = g(X + \mathrm{h}X, Y)\xi - \eta(Y)(X + \mathrm{h}X), \quad X, Y \in \mathfrak{X}(M). \tag{2.3}$$

Here h is an endomorphism field defined by $\mathrm{h} = 1/2 L_\xi \varphi$. For later use, we introduce the tensor field τ of type $(0,2)$ by $\tau = L_\xi g$.

The *Webster scalar curvature* W of a contact Riemannian 3-manifold is defined by

$$W = \frac{1}{8}(\mathrm{s} - S(\xi,\xi) + 4).$$

The *torsion invariant* of M introduced by S. S. Chern and R. S. Hamilton [9] is the square norm $|\tau|^2$ of τ. This invariant can be computed as:

$$|\tau|^2 = -2S(\xi,\xi) + 4.$$

A contact Riemannian 3-manifold is said to be an *η-Einstein manifold* if the Ricci operator Q has the form:

$$Q = \alpha I + \beta \eta \otimes \xi \tag{2.4}$$

for some functions α and β.

By the definition, η-Einstein contact Riemannian 3-manifolds are pseudo-symmetric.

Definition 2.1. [3] A contact Riemannian manifold M is said to be a *contact (κ,μ)-space* if there exist real constants κ and μ such that

$$\begin{aligned} R(X,Y)\xi = {} & \kappa\{\eta(Y)X - \eta(X)Y\} \\ & + \mu\{\eta(Y)\mathrm{h}X - \eta(X)\mathrm{h}Y\}, \quad X, Y \in \mathfrak{X}(M). \end{aligned} \tag{2.5}$$

A contact 3-manifold $(M, \varphi, \xi, \eta, g)$ is called a *Sasakian manifold* if it satisfies

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.6}$$

for all $X, Y \in \mathfrak{X}(M)$.

The formulae (2.3) and (2.6) imply that a contact Riemannian 3-manifold is Sasakian if and only if its Reeb vector field ξ is a Killing vector field (cf. [29]).

We easily check that Sasakian manifolds are contact (κ, μ)-spaces with $\kappa = 1$ and $\mathrm{h} = 0$.

Sasakian 3-manifolds have some remarkable properties. For instance, the Ricci operator Q commutes with φ, i.e., $Q\varphi = \varphi Q$. Moreover Q has the form

$$Q = \alpha I + \beta \eta \otimes \xi, \quad \alpha = \frac{\mathrm{s}}{2} - 1, \ \beta = 3 - \frac{\mathrm{s}}{2}, \tag{2.7}$$

where s is the scalar curvature. Thus the principal Ricci curvatures are

$$\rho_1 = \rho_2 = \frac{\mathrm{s}}{2} - 1, \ \rho_3 = 2.$$

Hence Sasakian 3-manifolds are pseudo-symmetric spaces of constant type.

A plane section Π_x at a point x of a contact Riemannian 3-manifold is called a *holomorphic plane* if it is invariant under φ_x. The sectional curvature function of holomorphic planes is called the *holomorphic sectional curvature.*

A 3-dimensional contact Riemannian manifold with constant holomorphic sectional curvature is called a 3-dimensional *contact Riemannian space form* [10]. In particular, Sasakian 3-manifolds of constant holomorphic sectional curvature are called 3-dimensional *Sasakian space forms.*

Three-dimensional contact Riemannian space forms are locally homogeneous. (See [10]).

The first named author obtained the following result.

PROPOSITION 2.2. [10] *All 3-dimensional non-Sasakian contact (κ, μ)-spaces have constant holomorphic sectional curvature $-(\kappa + \mu)$.*

The formula (2.7) implies the following

PROPOSITION 2.3. *A Sasakian 3-manifold is of constant holomorphic sectional curvature if and only if it has constant scalar curvature. In particular, a Sasakian 3-manifold which is a homogeneous Riemannian 3-manifold is of constant holomorphic sectional curvature.*

Simply connected and complete 3-dimensional Sasakian space forms are classified as follows:

PROPOSITION 2.4. [6] *Simply connected and complete 3-dimensional Sasakian space forms $\mathcal{M}^3(c)$ of constant holomorphic sectional curvature c are isomorphic to one of the following unimodular Lie groups with left invariant Sasakian structures: the special unitary group SU(2) for $c > -3$, the Heisenberg group for $c = -3$, or the universal covering group $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ of the special linear group $\mathrm{SL}(2,\mathbf{R})$ for $c < -3$. The Sasakian space form $\mathcal{M}^3(1)$ is the unit 3-sphere S^3 with the canonical Sasakian structure.*

As a Riemannian 3-manifold, 3-dimensional Sasakian space form is a naturally reductive homogeneous space. The naturally reductive homogeneous representations of the above model spaces (except S^3) are given by

$$\mathrm{SU}(2) \ltimes \mathrm{U}(1)/\mathrm{U}(1), \quad \mathbb{H}_3 \ltimes \mathrm{SO}(2)/\mathrm{SO}(2), \quad \widetilde{\mathrm{SL}}(2,\mathbf{R}) \ltimes \mathrm{SO}(2)/\mathrm{SO}(2).$$

Next, we recall the following results due to D. E. Blair, Th. Kouforgiorgos, and R. Sharma:

PROPOSITION 2.5. [4] *Let M be a contact Riemannian 3-manifold. Then the following three conditions are mutually equivalent:*
(i) *M is η-Einstein;*
(ii) *$Q\varphi = \varphi Q$;*
(iii) *M is a contact $(\kappa, 0)$-space with $\kappa \leq 1$.*

PROPOSITION 2.6. [4] *Let M be a contact Riemannian 3-manifold. Then M satisfies $Q\varphi = \varphi Q$ if and only if M is either* (i) *a Sasakian 3-manifold,* (ii) *a flat contact Riemannian 3-manifold, or* (iii) *a non-Sasakian contact Riemannian space form of constant holomorphic sectional curvature $-\kappa$ and constant ξ-sectional curvature κ. In the third case, $\kappa < 1$.*

These propositions imply the following result.

COROLLARY 2.7. *Contact Riemannian 3-manifolds such that $Q\varphi = \varphi Q$ are pseudo-symmetric. In particular, every Sasakian 3-manifold is a pseudo-symmetric space of constant type.*

For explicit Sasakian structure of $\mathrm{SL}(2,\mathbf{R})$, we refer to [17].

REMARK 2.8. A Sasakian manifold (of general dimension) is semi-symmetric if and only if is of constant curvature 1 ([28]). Hence every Sasakian 3-manifold, other than space of constant curvature 1, is "proper" pseudo-symmetric space. S. Tanno[30] showed that conformally flat Sasakian 3-manifolds are of constant curvature 1. Thus there are no Sasakian examples in the classification table of conformally flat proper pseudo-symmetric spaces of constant type due to N. Hashimoto and M. Sekizawa[15].

3. Non-Sasakian contact homogeneous 3-manifolds

In this section, we study the pseudo-symmetry of contact homogeneous Riemannian 3-manifolds. A contact Riemannian 3-manifold is said to be *homogeneous* if there exists a connected Lie group G acting transitively as a group of isometries on it which preserve the contact form.

D. Perrone has proven that simply connected contact homogeneous Riemannian 3-manifolds are Lie groups together with left invariant contact Riemannian structures. Moreover such homogeneous spaces are classified by the Webster scalar curvature W and the torsion invariant $|\tau|^2$ as follows:

PROPOSITION 3.1. [27] *Let (M,η,g) be a simply connected contact homogeneous Riemannian 3-manifold. Then M is a Lie group G together with a left invariant contact Riemannian structure (η, g).*

If G is unimodular, then G is one of the following:

(1) *the Heisenberg group* $\mathbb{H}_3$ *if* $W = |\tau| = 0$;
(2) $\mathrm{SU}(2)$ *if* $4\sqrt{2}W > |\tau|$;
(3) $\widetilde{E}(2)$ *if* $4\sqrt{2}W = |\tau| > 0$;
(4) $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ *if* $-|\tau| \neq 4\sqrt{2}W < |\tau|$;
(5) $E(1,1)$ *if* $4\sqrt{2}W = -|\tau| < 0$.

The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ with commutation relation:

$$[e_1,e_2] = 2e_3,\ [e_2,e_3] = c_2 e_1,\ [e_3,e_1] = c_3 e_2. \tag{3.1}$$

If G is non-unimodular, then the Lie algebra $\mathfrak{g}$ of G satisfies the commutation relations:

$$[e_1, e_2] = \alpha e_2 + 2e_3, \ [e_2, e_3] = 0, \ [e_3, e_1] = \gamma e_2,$$

where $e_3 = \xi$, $e_1, e_2 \in \operatorname{Ker} \eta$, $e_2 = \varphi e_1$, $\alpha \neq 0$ and $4\sqrt{2}W < |\tau|$. If $\gamma = 0$, then the structure is Sasakian ($\tau = 0$) and $W = -\alpha^2/4$.

Now we investigate the pseudo-symmetry condition on a unimodular Lie group G with a left invariant non-Sasakian contact Riemannian structure. Then, by Proposition 3.1, there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ which satisfies the commutation relations (3.1). By the assumption "non-Sasakian", the case that $c_2 = c_3$ is excluded.

By using the well-known Koszul formula, the connection coefficients $\{\Gamma_{ijk}\}$ of (M, g) are computed explicitly as follows:

$$(3.2) \qquad \begin{cases} \Gamma_{123} = \dfrac{1}{2}(c_3 - c_2 + 2), \\ \Gamma_{213} = \dfrac{1}{2}(c_3 - c_2 - 2), \\ \Gamma_{312} = \dfrac{1}{2}(c_3 + c_2 - 2), \\ \text{all others are zero.} \end{cases}$$

Here we used the convention: $\Gamma_{ijk} := g(\nabla_{e_i} e_j, e_k)$. Then, using (3.2), by straightforward computations we find

$$(3.3) \qquad \begin{aligned} R(e_1, e_2)e_2 &= \left(\frac{1}{4}(c_3 - c_2)^2 + (c_3 + c_2) - 3\right) e_1, \\ R(e_1, e_3)e_3 &= \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right) e_1, \\ R(e_2, e_1)e_1 &= \left(\frac{1}{4}(c_3 - c_2)^2 - 3 + c_3 + c_2\right) e_2, \\ R(e_2, e_3)e_3 &= \left(\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right) e_2, \\ R(e_3, e_1)e_1 &= \left(-\frac{1}{4}(c_3 - c_2)^2 - \frac{1}{2}({c_3}^2 - {c_2}^2) + 1 - c_2 + c_3\right) e_3, \\ R(e_3, e_2)e_2 &= \left(\frac{1}{4}(c_3 + c_2)^2 - {c_2}^2 + 1 + c_2 - c_3\right) e_3. \end{aligned}$$

By using (3.3) we get

$$Qe_1 = F_1 e_1,\ Qe_2 = F_2 e_2,\ Qe_3 = F_3 e_3, \tag{3.4}$$

where we put

$$\begin{aligned} F_1 &= -\frac{1}{2}(c_3^2 - c_2^2) - 2 + 2c_3,\\ F_2 &= \frac{1}{2}(c_3^2 - c_2^2) - 2 + 2c_2,\\ F_3 &= -\frac{1}{2}(c_3 - c_2)^2 + 2. \end{aligned} \tag{3.5}$$

The Webster curvature and torsion invariant are given by

$$W = \frac{1}{4}(c_2 + c_3),\quad |\tau|^2 = (c_2 - c_3)^2.$$

Let $\{\omega^1, \omega^2, \omega^3\}$ be the dual orthonormal basis of $\{e_1, e_2, e_3\}$. Now we suppose that G is pseudo-symmetric, i.e., $Q = aI + b\omega \otimes \zeta$ for some functions a and b. If $b = 0$, then G is of constant curvature 1 or 0 ([5]). In the former case, G is (locally) isomorphic to S^3 with canonical Sasakian structure, or equivalently, SU(2) with biinvariant Sasakian structure.

In the latter case, G is the locally isomorphic to the Euclidean motion group $E(2)$ with flat left invariant contact Riemannian structure. The flat left invariant contact Riemannian structure on $E(2)$ is given explicitly in Example 3.9 below (see also [16], Section 6).

Hereafter, we restrict our attention to the case $b \neq 0$. Then we may have the following three cases:

(1) $Q = aI + b\eta \otimes \xi$ (G is η-Einstein) and $F_1 = F_2 = a$, $F_3 = a + b$. In this case, we have the commutation relations:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = (2 - c_2)e_2.$$

The Ricci operator is given by $Q = \{-2(1-c_2)^2 + 2\}\eta \otimes \xi$ with $c_2 \neq 0, 2$ (by our assumption $b \neq 0$). By the Milnor's result ([24]) we see that G is locally isometric to SU(2) (or SO(3)) or SL(2, $\mathbf{R}$) (or O(1, 2)). In this case, $W = 1/2$ and $|\tau|^2 = 4(c_2 - 1)^2$.

(2) $Q = aI + b\omega^2 \otimes e_2$ and $F_1 = F_3 = a$, $F_2 = a + b$. This case has the following two possibilities:

(i): $$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = (c_2 + 2)e_2,$$

with Ricci operator $Q = 4c_2\omega^2 \otimes e_2$ with $c_2 \neq 0$ (by the assumption $b \neq 0$).

(ii): $$[e_1, e_2] = 2e_3,\ [e_2, e_3] = 2e_1,\ [e_3, e_1] = c_3 e_2,$$

with Ricci operator $Q = (-\frac{1}{2}c_3^2 + 2c_3)I + (c_3^2 - 2c_3)\omega^2 \otimes e_2$. Since we assumed that $b \neq 0$, we have $c_3 \neq 0$. Moreover $c_3 \neq 2$, because we assumed that G is non-Sasakian.

In the former subcase (i), the possible Lie algebras are $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbf{R})$, or $\mathfrak{e}(1,1)$. In the latter subcase (ii), $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbf{R})$. Hence G is locally isometric to SU(2) (or SO(3)), SL(2, **R**) (or O(1, 2)), or $E(1,1)$.

(3) $Q = aI + b\omega^1 \otimes e_1$ and $F_2 = F_3 = a$, $F_1 = a + b$; This case has the following two subcases:

(iii): $$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = (c_2 - 2)e_2,$$

with the Ricci operator $Q = 4(c_2 - 2)\omega^1 \otimes e_1$ with $c_2 \neq 2$, or

(iv): $$[e_1, e_2] = 2e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = 2e_2$$

with $Q = (-\frac{1}{2}c_2^2 + 2c_2)I + (c_2^2 - 2c_2)\omega^1 \otimes e_1$. Here $c_2 \neq 0, 2$ from the assumption, $b \neq 0$. Hence G is locally isometric to SU(2) (or SO(3)), SL(2, **R**) (or O(1, 2)) or $E(1,1)$.

Finally, we consider the non-unimodular Lie group G with left invariant (non-Sasakian) contact Riemannian structures. Then by Proposition 3.1, there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{g}$ such that

$$[e_1, e_2] = \alpha e_2 + 2e_3,\ [e_2, e_3] = 0,\ [e_3, e_1] = \gamma e_2, \tag{3.6}$$

where $\alpha \neq 0$. Moreover, G is Sasakian if and only if $\gamma = 0$. From (3.6), by using the Koszul formula we have

$$\begin{cases} \Gamma_{123} = \dfrac{\gamma + 2}{2} \\ \Gamma_{212} = -\alpha \\ \Gamma_{213} = \dfrac{\gamma - 2}{2} \\ \Gamma_{312} = \dfrac{\gamma - 2}{2} \\ \text{all others are zero.} \end{cases} \tag{3.7}$$

Then, by the definition of the curvature tensor, we have

$$\begin{aligned}
R(e_1, e_2)e_2 &= \left(\frac{\gamma^2 + 4\gamma - 12}{4} - \alpha^2\right) e_1, \\
R(e_1, e_3)e_3 &= \left(\frac{-3\gamma^2 + 4\gamma + 4}{4}\right) e_1, \\
R(e_2, e_1)e_1 &= \left(\frac{\gamma^2 + 4\gamma - 12}{4} - \alpha^2\right) e_2 + \alpha\gamma e_3, \\
R(e_2, e_3)e_3 &= \frac{(\gamma - 2)^2}{4} e_2, \\
R(e_3, e_1)e_1 &= \alpha\gamma e_2 + \left(\frac{-3\gamma^2 + 4\gamma + 4}{4}\right) e_3, \\
R(e_3, e_2)e_2 &= \frac{(\gamma - 2)^2}{4} e_3.
\end{aligned}$$

From these, we have the Ricci operator

$$Qe_1 = f_{11}e_1, \; Qe_2 = f_{22}e_2 + f_{32}e_3, \; Qe_3 = f_{23}e_2 + f_{33}e_3, \tag{3.8}$$

where we have put

$$\begin{aligned}
f_{11} &= \left(-\alpha^2 - 2 + 2\gamma - \frac{\gamma^2}{2}\right), \\
f_{22} &= \left(-\alpha^2 - 2 + \frac{\gamma^2}{2}\right), \\
f_{32} &= f_{23} = \alpha\gamma, \quad f_{33} = \left(2 - \frac{\gamma^2}{2}\right).
\end{aligned} \tag{3.9}$$

Then in a similar way as in the unimodular case, we have

$$Q = f_{11}I + (f_{22} + f_{33} - 2f_{11})\sqrt{2}(\omega^2 + \omega^3) \otimes \frac{1}{\sqrt{2}}(e_2 + e_3).$$

Hence, we have

THEOREM 3.2. *Every 3-dimensional unimodular and non-unimodular Lie group with special left-invariant contact Riemannian structure is a pseudo-symmetric space of constant type.*

Together with the classification(Proposition 3.1) of contact homogeneous 3-dimensional manifolds, we obtain

PROPOSITION 3.3. *Three-dimensional contact homogeneous Riemannian manifolds with special left invariant contact Lie group structure are pseudo-symmetric spaces of constant type.*

Also, in view of the classification of contact (κ,μ)-space in [3], we get

COROLLARY 3.4. *Three-dimensional non-Sasakian contact (κ,μ)-spaces with special left invariant contact Lie group structure are pseudo-symmetric spaces of constant type.*

REMARK 3.5. The Ricci operator Q of a 3-dimensional non-Sasakian contact (κ,μ)-space is given by [3]:

$$Q = -I + \mu \mathrm{h} + (2\kappa + \mu)\eta \otimes \xi.$$

EXAMPLE 3.6. (Minkowski motion group) Let $G = E(1,1)$ be the Minkowski motion group:

$$E(1,1) = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y, z \in \mathbf{R} \right\}$$

equipped with the following left invariant metric:

$$g_\lambda = e^{-2z}dx^2 + e^{2z}dy^2 + \lambda^2 dz^2,$$

where λ is a positive constant. Then $(E(1,1), g_\lambda)$ is a proper irreducible Riemannian 4-symmetric space. Note that $(E(1,1), g_1)$ is the model space Sol of 3-dimensional solvegeometry [32].

The Ricci tensor of this homogeneous space is given by

$$\rho_1 = \rho_2 = 0, \;\; \rho_3 = -2/\lambda^2.$$

Hence $(E(1,1), g_\lambda)$ is proper pseudo-symmetric space of constant type.

Under the homothetic change of the metric

$$g = \frac{1}{4}(e^{-2z}dx^2 + e^{2z}dy^2 + \lambda^2 dz^2),$$

we obtain a contact homogeneous 3-manifold $E(1,1)$ with left invariant contact Riemannian structure determined by the metric g and the contact form $\eta = \frac{1}{2}(e^z dx + e^{-z}dy)$. This contact homogeneous 3-manifold is a non-Sasakian contact space form. In particular, $(E(1,1), g, \eta)$ is a proper pseudo-symmetric space.

REMARK 3.7. The Lie algebra $\mathfrak{e}(1,1)$ of $E(1,1)$ is given explicitly by

$$\mathfrak{e}(1,1) = \left\{ \begin{pmatrix} w & 0 & u \\ 0 & -w & v \\ 0 & 0 & 1 \end{pmatrix} \middle| \; u, v, w \in \mathbf{R} \right\}$$

Take a basis

$$F_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of $\mathfrak{e}(1,1)$. Then the left translated vector fields of $\{F_1, F_2, F_3\}$ are given by

$$f_1 = e^z \frac{\partial}{\partial x}, \; f_2 = e^{-z} \frac{\partial}{\partial y}, \; f_3 = \frac{\partial}{\partial z}.$$

The dual coframe field is

$$\omega^1 = e^{-z} dx, \; \omega^2 = e^z dy, \; \omega^3 = dz.$$

Now we take the following left invariant vector fields u_1, u_2, u_3:

$$u_1 = \frac{1}{\sqrt{2}}(-f_1 + f_2), \; u_2 = \frac{1}{\sqrt{2}}(f_1 + f_2), \; u_3 = f_3.$$

This left invariant frame field satisfies the commutation relations:

$$[u_1, u_2] = 0, \; [u_2, u_3] = u_1, \; [u_3, u_1] = -u_2.$$

We equip a left invariant Riemannian metric on $E(1,1)$ such that $\{e_1, e_2, e_3\} := \{u_1/\lambda_1, u_2/\lambda_2, u_3/\lambda_3\}$ is orthonormal, where $\lambda_1, \lambda_2, \lambda_3$ are positive constants. The resulting Riemannian metric is

$$g_{(\lambda_1,\lambda_2,\lambda_3)} = \frac{\lambda_1^2}{2}(-\omega^1 + \omega^2)^2 + \frac{\lambda_2^2}{2}(\omega^1 + \omega^2)^2 + \lambda_3^2(\omega^3)^2.$$

Any left invariant metric on $E(1,1)$ is isometric with one of the following metric:

$$g_{(\lambda_1,\lambda_2,\lambda_1\lambda_2)} = \frac{\lambda_1^2}{2}(-\omega^1 + \omega^2)^2 + \frac{\lambda_2^2}{2}(\omega^1 + \omega^2)^2 + \frac{1}{\lambda_1^2\lambda_2^2}(\omega^3)^2,$$

with $\lambda_1 \geq \lambda_2 > 0$ (see [26], Proposition 2.3). Note that $g_{(1,1,\lambda)} = g_1 = 4g$.

The commutation relations of $\{e_1, e_2, e_3\}$ are

$$[e_1, e_2] = c_3 e_3, \ [e_2, e_3] = c_1 e_1, \ [e_3, e_1] = c_2 e_2$$

with $c_1 = 1/(\lambda_2\lambda_3), \ c_2 = -1/(\lambda_3\lambda_1), \ c_3 = 0$. It follows also that

$$\mu_1 = -\frac{\lambda_1 + \lambda_2}{2\lambda_1\lambda_2\lambda_3}, \ \mu_2 = \frac{\lambda_1 + \lambda_2}{2\lambda_1\lambda_2\lambda_3}, \ \mu_3 = \frac{\lambda_1 - \lambda_2}{2\lambda_1\lambda_2\lambda_3},$$

where we have put $\mu_i = 1/2(c_1 + c_2 + c_3) - c_i$ for $i = 1, 2, 3$. Then we have $\rho_1 = 2\mu_2\mu_3, \ \rho_2 = 2\mu_3\mu_1, \ \rho_3 = 2\mu_1\mu_2$ (cf. [24]). From these, we see that $\rho_1 = \rho_2$ if and only if $\lambda_1 = \lambda_2$. The other two cases $\rho_2 = \rho_3$ or $\rho_3 = \rho_1$ can not occur. Hence, we obtain the following result.

COROLLARY 3.8. *Any pseudo-symmetric left invariant Riemannian metrics on* $E(1,1)$ *is homothetic to the* 4*-symmetric metric* $e^{-2z}dx^2 + e^{2z}dy^2 + \lambda^2 dz^2$ *for some* $\lambda > 0$.

EXAMPLE 3.9. (Euclidean motion group) The Euclidean motion group $G = E(2)$ is given explicitly by the following matrix group:

$$E(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \ x, y, \theta \in \mathbf{R} \right\}.$$

Let $\widetilde{G}$ the universal covering group of $E(2)$. Then $\widetilde{G}$ is $\mathbf{R}^3(x, y, z)$ with multiplication:

$$(x, y, z) \cdot (x', y', z') = (x + \cos z \ x' - \sin z \ y', y + \sin z \ x' + \cos z \ y', z + z').$$

Take positive constants α, β and γ and a left invariant frame:

$$\begin{aligned} e_1 &= \beta \left(-\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y} \right), \\ e_2 &= \gamma \frac{\partial}{\partial z}, \\ e_3 &= \alpha \left(\cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y} \right) = \xi. \end{aligned}$$

Then this frame satisfies the following commutation relations:

$$[e_1, e_2] = c_1 e_3, \quad [e_2, e_3] = c_2 e_1, \quad [e_3, e_1] = 0$$

with $c_1 = \beta\gamma/\alpha$, $c_2 = \alpha\gamma/\beta$. The left invariant Riemannian metric determined by the condition $\{e_1, e_2, e_3\}$ is orthonormal is given by (cf. [26]):

$$\begin{aligned} g_{\alpha,\beta,\gamma} = &\ \left(\alpha^{-2}\cos^2 z + \beta^{-2}\sin^2 z\right) dx^2 + \left(\alpha^{-2} - \beta^{-2}\right)\sin(2z)dxdy \\ &+ \left(\alpha^{-2}\sin^2 z + \beta^{-2}\cos^2 z\right) dy^2 + \gamma^{-2}dz^2. \end{aligned}$$

This family essentially exhausts all left invariant metrics on $\widetilde{G}$. See [26], Proposition 2.4.

The principal Ricci curvatures are given by

$$\rho_1 = -\frac{1}{2}(c_1+c_2)(c_1-c_2), \ \ \rho_2 = -\frac{1}{2}(c_1-c_2)^2, \ \ \rho_3 = \frac{1}{2}(c_1+c_2)(c_1-c_2).$$

Hence $\widetilde{G}$ is pseudo-symmetric if and only if $c_1 = c_2$. This condition is equivalent to $\alpha = \beta$, i.e., $\widetilde{G}$ is flat.

We may normalize $\{\alpha, \beta, \gamma\}$ so that $c_1 = 2$ (equivalently, $\alpha = \beta\gamma/2$). Under this normalization, $c := c_2 = \gamma^2/2$.

The associated contact Riemannian structure (η, ξ, φ) is given by

$$\eta = \alpha^{-1}(\cos z\ dx + \sin z\ dy), \quad \xi = e_1,$$

$$\varphi e_1 = 0, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2$$

COROLLARY 3.10. *Among all the left invariant Riemannian metrics on the universal covering $\widetilde{E}(2)$ of the Euclidean motion group, the flat metrics are only left invariant metrics which are pseudo-symmetric. Hence there are no proper pseudo-symmetric left invariant metric on $\widetilde{E}(2)$.*

There exist many homogeneous Riemannian 3-manifolds which are pseudo-symmetric. For instance, in this paper, we exhibit examples of contact homogeneous Riemannian 3-manifolds which are proper pseudo-symmetric spaces (of constant type).

On the other hand, O. Kowalski[19] gave examples of non-homogeneous pseudo-symmetric 3-spaces. Non-homogeneous Sasakian 3-manifolds provide examples of non-homogeneous pseudo-symmetric spaces.

In view of the results of this paper, one may raise the following question :

"*Are there examples of non-homogeneous, non-Sasakian, pseudo-symmetric contact manifolds ?*"

The classification due to O. Kowalski and M. Sekizawa[22, 23] motivates us to study the pseudo-symmetry of *confoliated 3-manifolds* (in the sense of Y. Eliashberg and W. Thurston[14]).

4. Appendix. Riemannian 3-manifolds with 4-dimensional isometry group

As we mentioned in Introduction, every Riemannian 3-manifold with 4-dimensional isometry group is a pseudo-symmetric space.

In this Appendix, we give a proof of this well known fundamental result for reader's convenience.

It is classically known that Riemannian 3-manifolds with 4-dimensional isometry group are homogeneous (E. Cartan[8]).

L. Bianchi[1] and E. Cartan[8] obtained the following two parameter family of homogeneous Riemannian metrics:

$$g_{\lambda,\mu} = \frac{dx^2+dy^2}{\{1+\mu(x^2+y^2)\}^2} + \left(dz + \frac{\lambda}{2}\,\frac{ydx-xdy}{1+\mu(x^2+y^2)}\right)^2.$$

The metric $g_{\lambda,\mu}$ is defined on the region:

$$\mathcal{D} = \{(x,y,z) \in \mathbf{R}^3(x,y,z) \mid 1+\mu(x^2+y^2) > 0\ \}.$$

Note that $\mathcal{D}$ is the whole $\mathbf{R}^3(x,y,z)$ for $\mu \geq 0$.

The Riemannian 3-manifold $(\mathcal{D}, g_{\lambda,\mu})$ is locally isometric to:

(1) $\lambda = \mu = 0$: Euclidean 3-space,
(2) $\lambda = 0$: Riemannian products $S^2 \times \mathbf{R}$ $(\mu > 0)$, or $H^2 \times \mathbf{R}$ $(\mu < 0)$,
(3) $\lambda \neq 0,\ \mu = 0$: Heisenberg group $\mathbb{H}_3$,
(4) $\lambda \neq 0,\ \mu > 0$: SU(2),
(5) $\lambda \neq 0,\ \mu < 0$: $\mathrm{SL}(2,\mathbf{R})$.

In particular, $\mathcal{D}$ is of constant positive curvature $\lambda^2/4$ if $4\mu = \lambda^2$ and $\lambda \neq 0$.

Moreover, every Riemannian 3-manifold with 4-dimensional isometry group is locally isometric to $\mathcal{D}$ for some λ, μ.

Take an orthonormal frame field

$$e_1 = \{1+\mu(x^2+y^2)\}\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z},$$
$$e_2 = \{1+\mu(x^2+y^2)\}\frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z},$$
$$e_3 = \frac{\partial}{\partial z}.$$

Then the Ricci tensor of $g_{\lambda,\mu}$ is given by $R_{11} = R_{22} = 4\mu - \lambda^2$, $R_{33} = \lambda^2/2$.

PROPOSITION A.1. *Every Riemannian 3-manifold with 4-dimensional isometry group is a pseudo-symmetric space of constant type.*

Direct computation shows that $R\cdot R = 0$ if and only if $\lambda^2(4\mu - \lambda^2) = 0$. This relation implies that $\mathcal{D}$ is semi-symmetric if and only if $\mathcal{D}$ is locally symmetric.

COROLLARY A.2. *A Riemannian 3-manifold with 4-dimensional isometry group is semi-symmetric if and only if it is locally symmetric and hence locally isometric to $S^2 \times \mathbf{R}$ or $H^2 \times \mathbf{R}$.*

Comparing the metrics as above and the classification of 3-dimensional D'Atri space due to O. Kowalski[18], we obtain

PROPOSITION A.3. *Three-dimensional D'Atri spaces are pseudo-symmetric spaces of constant type.*

REMARK A.4. The dual one-form ω^3 of e_3 is $\omega^3 = dz + (\lambda/2)(ydx - xdy)/\{1+\mu(x^2+y^2)\}$. This one-form is contact if and only if $\lambda \neq 0$. Now we assume that $\lambda \neq 0$. Take a contact form $\eta = \lambda\omega^3/2$. Then the associated Riemannian metric is $\hat{g} = \frac{\lambda^2}{4} g_{\lambda,\mu}$. The resulting contact Riemannian 3-manifold $(\mathcal{D}, \eta, \hat{g})$ is a Sasakian space form of constant holomorphic sectional curvature $-3 + 16\mu/\lambda^2$. (Compare with Proposition 3.2.)

Isometry groups of Riemannian 3-manifolds have dimension at most 6. A Riemannian 3-manifold has 6-dimensional isometry group if and only if it is of constant curvature. Moreover, there is no Riemannain 3-manifold with 5-dimensional isometry group.

COROLLARY A.5. *Riemannian 3-manifolds whose isometry groups have dimension ≥ 4 are pseudo-symmetric spaces of constant type.*

References

[1] L. Bianchi, *Lezioni di Geometrie Differenziale*, E. Spoerri Librao-Editore, 1894.

[2] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. **509** (1976), Springer-Verlag, Berlin-Heidelberg-New-York.

[3] D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. **91** (1995), 189–214.

[4] D. E. Blair, Th. Koufogiorgos, and R. Sharma, *A classification of 3-dimensional contact metric manifolds with* $Q\varphi = \varphi Q$, Kōdai Math. J. **13** (1990), 391–401.

[5] D. E. Blair and R. Sharma, *Three-dimensional locally symmetric contact metric manifolds*, Boll. Unione. Mat. Ital. Sez. A Mat. Soc. Cult. (8) **4** (1990), no. 3, 385–390.

[6] D. E. Blair and L. Vanhecke, *Symmetries and* φ*-symmetric spaces*, Tôhoku Math. J. **39** (1987), no. 2, 373–383.

[7] G. Calvaruso, D. Perrone, and L. Vanhecke, *Homogeneity on three-dimensional contact metric manifolds* , Israel J. Math. **114** (1999), 301–321.

[8] E. Cartan, *Leçon sur la geometrie des espaces de Riemann*, Second Edition, Gauthier-Villards, Paris, 1946.

[9] S. S. Chern and R. S. Hamilton, *On Riemannian metrics adapted to three-dimensional contact manifolds*, Lecture Notes in Math. **1111** (1985), Springer Verlag, 279–305.

[10] J. T. Cho, *A class of contact Riemannian manifolds whose associated CR-structure are integrable*, Publ. Math. Debrecen **63** (2003), 193–211.

[11] J. T. Cho and L. Vanhecke, *Classification of symmetric-like contact metric* (k,μ)*-spaces*, Publ. Math. Debrecen **62** (2003), 337–349.

[12] J. Deprez, R. Deszcz, and L. Verstraelen, *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math. **17** (1989), 51–65.

[13] R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. **44** (1992), 1–34.

[14] Y. Eliashberg and W. Thurston, *Confoliations*, AMS University Lecture Series **13** (1998).

[15] N. Hashimoto and M. Sekizawa, *Three-dimensional conformally flat pseudo-symmetric spaces of constant type*, Arch. Math. (Brno) **36** (2000), 279–286.

[16] J. Inoguchi, *Minimal surfaces in 3-dimensional solvable Lie groups*, Chinese Ann. Math. Ser. B **24** (2003), 73–84.

[17] J. Inoguchi, T. Kumamoto, N. Ohsugi, and Y. Suyama, *Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces III*, Fukuoka Univ. Sci. Rep. **30** (2000), 131–160.

[18] O. Kowalski, *Spaces with volume preserving symmetries and related classes of Riemannian manifolds*, Rend. Sem. Mat. Univ. Politec. Torino **64** (1983), 131–159.

[19] ———, *A classification of Riemannian 3-manifolds with constant principal Ricci curvatures* $\rho_1 = \rho_2 \neq \rho_3$, Nagoya Math. J. **132** (1993), 1–36.

[20] O. Kowalski and M. Sekizawa, *Local isometry classes of Riemannian 3-manifolds with constant Ricci eigenvalues* $\rho_1 = \rho_2 \neq \rho_3$, Arch. Math. (Brno) **32** (1996), 137–145.

[21] ———, *Three-dimensional Riemannian manifolds of c-conullity two*, Riemannian Manifolds of Conullity Two, Chapter 11, World Scientific, Singapore, 1996.

[22] ———, *Pseudo-symmetric spaces of constant type in dimension three-elliptic spaces*, Rend. Mat. Appl. (7) **17** (1997), 477–512.
[23] ———, *Pseudo-symmetric spaces of constant type in dimension three-non-elliptic spaces*, Bull. Tokyo Gakugei Univ. (4) **50** (1998), 1–28.
[24] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. Math. **21** (1976), 293–329.
[25] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, Orland, 1983.
[26] V. Patrangenaru, *Classifying* 3- *and* 4-*dimensional homogeneous Riemannian manifolds by Cartan triples* , Pacific J. Math. **173** (1996), 511–532.
[27] D. Perrone, *Homogeneous contact Riemannian three-manifolds*, Illinois J. Math. **13** (1997), 243–256.
[28] T. Takahashi, *Sasakian manifolds with pseudo-Riemannian metric*, Tôhoku Math. J. **21** (1969), 271–290.
[29] S. Tanno, *Sur une variété de K-contact métrique de dimension* 3, C. R. Math. Acad. Sci. Paris **263** (1966), A 317–A319.
[30] ———, *Locally symmetric K-contact Riemannian manifolds*, Proc. Japan Acad. Ser. A Math. Sci. **43** (1967), 581–583.
[31] ———, *Variational problems on contact Riemannian manifolds*, Trans. Amer. Math. Soc. **314** (1989), 349–379.
[32] W. M. Thurston, *Three-dimensional Geometry and Topology I*, Princeton Math. Ser., vol. 35 (S. Levy, eds.), 1997.
[33] F. Tricerri and L. Vanhecke, *Homogeneous Structures on Riemannian Manifolds*, London Math. Soc. Lecture Note Ser. vol. 83, Cambridge Univ. Press, London, 1983.

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