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DILATIONS FOR POLYNOMIALLY BOUNDED OPERATORS

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Abstract. We discuss a certain geometric property $X_{\theta,\gamma}$ of dual algebras generated by a polynomially bounded operator and property $(\mathbb{A}_{\aleph_0,\aleph_0})$; these are central to the study of $\aleph_0 \times \aleph_0$-systems of simultaneous equations of weak*-continuous linear functionals on a dual algebra. In particular, we prove that if $T \in \mathbb{A}^M$ satisfies a certain sequential property, then $T \in \mathbb{A}^M_{\aleph_0}(\mathcal{H})$ if and only if the algebra $\mathcal{A}_T$ has property $X_{0,1/M}$, which is an improvement of Li-Pearcy theorem in [8].

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual* (*operator*) *algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. Note that the ultraweak operator topology coincides with the weak* topology on $\mathcal{L}(\mathcal{H})$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be *polynomially bounded* if there exists a positive number M such that for every polynomial p, $\|p(T)\| \leq M \sup_{|\lambda|\leq 1} |p(\lambda)|$. It is well-known that every contraction in $\mathcal{L}(\mathcal{H})$ is polynomially bounded. Concerning the converse implication, P. Halmos[6] posed the question as to whether each polynomially bounded operator is similar to a contraction operator. In [11], G. Pisier gave a polynomially bounded operator on l^2 which is not similar to a contraction, and K. Davidson and V. Paulsen[3] provided a class of examples of polynomially bounded operators which are not similar to contractions which includes examples due to Pisier. So there is strong motivation for the study of dual algebras generated by polynomially bounded operators. As one of such studies, W. Li and C. Pearcy in [7]

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and [8] studied polynomially bounded operators by developing a theory of the dual algebras generated by such operators that parallels, as much as possible, the theory of dual algebras generated by a contraction and gave an open problem concerning an abstract geometric criterion for membership in a certain class $\mathbb{A}^M_{\aleph_0}$. In this paper we will discuss this problem. In particular, we prove that if $T \in \mathbb{A}^M$ satisfies a certain sequential property, then $T \in \mathbb{A}^M_{\aleph_0}(\mathcal{H})$ if and only if the algebra $\mathcal{A}_T$ has property $X_{0,1/M}$.

Before we start the work, we recall some definitions and terminology concerning the theory of dual algebras (cf. [1]). The notation employed herein agrees with that in [1], [7], [8], and [12]. Let $\mathcal{C}_1(\mathcal{H})$ be the Banach space of trace class operators on $\mathcal{H}$ equipped with the trace norm. If $\mathcal{A}$ is a dual algebra, then it follows from [3] that $\mathcal{A}$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$, where ${}^{\perp}\mathcal{A}$ is the preannihilator in $\mathcal{C}_1(\mathcal{H})$ of $\mathcal{A}$, under the pairing $\langle T, [L]_{\mathcal{A}}\rangle = \text{trace}(TL)$, $T \in \mathcal{A}$, $[L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}$. The Banach space $\mathcal{Q}_{\mathcal{A}}$ is called a *predual* of $\mathcal{A}$. We write $[L]$ for $[L]_{\mathcal{A}}$, and $\|[L]\|$ for $\|[L]\|_{\mathcal{A}}$, when there is no possibility of confusion. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\mathcal{A}_T$ the dual algebra generated by T and denote by $\mathcal{Q}_T$ the predual space $\mathcal{Q}_{\mathcal{A}_T}$ of $\mathcal{A}_T$. For x and y in $\mathcal{H}$, we define $(x \otimes y)(u) = (u, y)x$, for all $u \in \mathcal{H}$. Suppose m and n are any cardinal numbers such that $1 \le m, n \le \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}]$, $0 \le i < m$, $0 \le j < n$, where $\{[L_{ij}]\}_{\substack{0 \le i < m \\ 0 \le j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i\}_{0 \le i < m}$, $\{y_j\}_{0 \le j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$.

We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$, $\mathbb{T}$ for the boundary of $\mathbb{D}$, and $\mathbb{N}$ for the set of natural numbers. Recall that an operator is absolutely continuous if its maximal unitary direct summand is absent or has spectral measure absolutely continuous with respect to Lebesgue measure.

Let $\text{PB}(\mathcal{H}) = \text{PB}$ be the set of all polynomially bounded operators T in $\mathcal{L}(\mathcal{H})$. If $T \in \text{PB}$, then there exists a smallest positive number $M \ge 1$ such that for every polynomial p, $\|p(T)\| \le M \sup_{\lambda \in \mathbb{D}} |p(\lambda)|$. We write $\text{PB}^M(\mathcal{H})$ or PB^M for the set of all T in $\text{PB}(\mathcal{H})$ for which M is the smallest such constant. The class of all absolutely continuous operators in $\text{PB}(\mathcal{H})$ will be denoted by $\text{ACPB}(\mathcal{H})$, and we write $\text{ACPB}^M(\mathcal{H}) = \text{PB}^M(\mathcal{H}) \cap \text{ACPB}(\mathcal{H})$. For f in $H^\infty(\mathbb{T})$ write $\hat{f}$ for the function defined by $\hat{f}(e^{it}) = \overline{f(\overline{e^{it}})}$.

THEOREM 1.1. [7, Theorem 4.1] *Let $T \in ACPB^M(\mathcal{H})$ for some $M \geq 1$. Then there is a unique norm-continuous algebra homomorphism $\Phi_T : H^\infty(\mathbb{T}) \longrightarrow \mathcal{A}_T$ such that*

(a) $\Phi_T(1) = I_{\mathcal{H}}$, $\Phi_T(\xi) = T$, *and if* $\sigma(T) \subset \mathbb{D}$, *then* $\Phi_T(f) = \widetilde{f}(T)$, *where $\widetilde{f}$ is the analytic extension of f to $\mathbb{D}$ and $\widetilde{f}(T)$ is defined by the Riesz-Dunford functional calculus,*
(b) $\|\Phi_T\| = M$,
(c) $\Phi_{T^*}(f) = \Phi_T(\hat{f})^*$, $f \in H^\infty(\mathbb{T})$,
(d) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*
(e) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*
(f) *there exists a bounded, linear, one-to-one map $\phi_T : \mathcal{Q}_T \longrightarrow L^1/H_0^1$ such that $\phi_T^* = \Phi_T$, and*
(g) *if Φ_T is bounded below, then Φ_T is an invertible isomorphism of H^∞ onto $\mathcal{A}_T$ that is also a weak* homeomorphism between H^∞ and $\mathcal{A}_T$, and ϕ_T is an invertible linear transformation of $\mathcal{Q}_T$ onto L^1/H_0^1.*

We usually write $f(T)$ for $\Phi_T(f)$.

The class $\mathbb{A}^M(\mathcal{H})$ consists of all those M-polynomially bounded operators T in ACPB^M for which $\|h\| \leq \|\Phi_T(h)\| \leq M\|h\|_\infty$, $h \in H^\infty$. Furthermore, if m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$, we denote by $\mathbb{A}^M_{m,n} = \mathbb{A}^M_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}^M(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a dual algebra and θ is a nonnegative real number. We denote by $\mathcal{X}_\theta(\mathcal{A})$ the set of all $[L]$ in $\mathcal{Q}_\mathcal{A}$ such that there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ of vectors from the closed unit ball of $\mathcal{H}$ satisfying $\limsup_{i\to\infty} \|[x_i \otimes y_i] - [L]\| \leq \theta$ and $\|[x_i \otimes z]\| + \|[z \otimes y_i]\| \to 0$, for all z in $\mathcal{H}$. For $0 \leq \theta < \gamma$, the dual algebra $\mathcal{A}$ is said to have property $X_{\theta,\gamma}$ if the closed absolutely convex hull $\overline{\mathrm{aco}}(\mathcal{X}_\theta(\mathcal{A}))$ of the set $\mathcal{X}_\theta(\mathcal{A})$ contains the closed ball $B_{0,\gamma}$ of radius γ centered at the origin in $\mathcal{Q}_\mathcal{A}$. Recall from [1, Theorem 7.1] that if $T \in \mathbb{A}(\mathcal{H})$ and $\mathcal{A}_T$ has property $X_{\theta,\gamma}$ for some $0 \leq \theta < \gamma$, then $T \in \mathbb{A}_{\aleph_0}$, and the converse holds as well. But, perhaps surprisingly, the forward result does, but this converse does not, generalize to polynomially bounded operators (cf. [8, Proposition 2.3]). As an example of what can be shown, Li–Pearcy[8] proved using spectral theory that if $T \in \mathbb{A}^M_{\aleph_0}(\mathcal{H}) \cap C_{00}$ for some $M \geq 1$, then $\mathcal{A}_T$ has property $X_{0,1/M}$.

A brief outline of this work is as follows: in Section 2, we introduce some dilation lemmas used frequently in the work and discuss a sequential property **S** in a dual algebra. In Section 3, we investigate the sequential properties in several ways. In Section 4, we generalize the above theorem of Li–Pearcy using the sequential property **S**.

2. Preliminary lemmas and a sequential property

In this section, we will discuss some lemmas. First we will assemble some dilation theorems.

LEMMA 2.1. *Suppose that $T \in \mathbb{A}_n^M$ for $n \in \mathbb{N}$ and $M \geq 1$. Let A be a completely nonunitary contraction on $\mathcal{K}$ with $\dim \mathcal{K} < \infty$ and having an n-cyclic set. Then there exist $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that the compression $T_{\mathcal{M}\ominus\mathcal{N}}$ of T to $\mathcal{M} \ominus \mathcal{N}$ is similar to A.*

Proof. Since ϕ_T is onto, we can use the method of proof of [1, Theorem 4.12]. □

For each $\lambda \in \mathbb{D}$, let P_λ be the Poisson kernel function in L^1

$$(1) \qquad P_\lambda(e^{it}) = \frac{1-|\lambda|^2}{|1-\overline{\lambda}e^{it}|^2}, \quad e^{it} \in \mathbb{T}.$$

In particular, for a given operator $T \in \mathbb{A}^M$, let us denote $\phi_T^{-1}([P_\lambda])$ by $[C_\lambda]$. Then we have $\langle f(T), [C_\lambda]\rangle = \widetilde{f}(\lambda)$, $f \in H^\infty$, where $\widetilde{f}$ is the analytic extension of f to $\mathbb{D}$.

COROLLARY 2.2. *Suppose $T \in \mathbb{A}_1^M$ and $M \geq 1$. Then for any $\lambda \in \mathbb{D}$, there exists a sequence $\{x_k^\lambda\}_{k=1}^\infty$ of unit vectors in $\mathcal{H}$ which converges weakly to zero and satisfies $[C_\lambda]_T = [x_k^\lambda \otimes x_k^\lambda]_T$.*

Proof. For $\lambda_0 \in \mathbb{D}$ and any $m \in \mathbb{N}$, we consider an m by m Jordan block

$$(2) \qquad J_m = \begin{pmatrix} \lambda_0 & 1 & & & \\ & \lambda_0 & 1 & O & \\ & & \ddots & \ddots & \\ & O & & \ddots & 1 \\ & & & & \lambda_0 \end{pmatrix}$$

relative to the standard orthonormal basis on $\mathbb{C}^{(m)}$. Since J_m has a cyclic vector and is similar to a contraction on $\mathbb{C}^{(m)}$, Lemma 2.1 implies that there exist $\mathcal{M}$ and $\mathcal{N}$ in $\mathrm{Lat}(T)$ with $\mathcal{M} \supset \mathcal{N}$ such that $\dim \mathcal{M} \ominus \mathcal{N} = m$

and $T_{\mathcal{M}\ominus\mathcal{N}}$ is similar to J_m. Use the proof of [1, Theorem 6.6] to obtain the lemma. □

For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2. By using Lemma 2.1, the proof of [4, Theorem 1.1] and [1, Theorem 5.3], we can obtain the following.

THEOREM 2.3. *Suppose that* $T \in \mathbb{A}_n^M$ *for* $1 \le n \le \aleph_0$ *and* $M \ge 1$. *Let* A *be a normal operator on an* n*-dimensional Hilbert space whose matrix to some orthonormal basis is the diagonal matrix* $\operatorname{Diag}(\{\lambda_i\}_{1\le n\le\aleph_0})$ *with* λ_i *in* $\mathbb{D}$. *Then there exist* $\mathcal{M}$ *and* $\mathcal{N}$ *in* $\operatorname{Lat}(T)$ *with* $\mathcal{M} \supset \mathcal{N}$ *such that* $T_{\mathcal{M}\ominus\mathcal{N}} \cong A$.

REMARKS. According to the proof of [1, Proposition 6.5], if $T \in C_{\cdot 0} \cap \mathbb{A}_1^M$ and $M \ge 1$, then for a sequence $\{x_i\}$ converging weakly to 0 we have

$$\|[x_i \otimes z]\|_{\mathcal{A}_T} \to 0, \text{ for all } z \in \mathcal{H}.$$

Then if $T \in C_{00} \cap \mathbb{A}_1^M$ and $M \ge 1$, then we can prove that $\mathcal{A}_T$ has property $X_{0,1/M}$. The proof uses the dilation theory instead of the spectral theory approach used in [8, Theorem 2.14]. (Indeed, it follows from Corollary 2.2 that for $\lambda \in \mathbb{D}$, there exists a sequence $\{x_k^\lambda\}_{k=1}^\infty$ of unit vectors in $\mathcal{H}$, such that $\{x_k^\lambda\}_{k=1}^\infty \to 0$ weakly and $[C_\lambda]_T = [x_k^\lambda \otimes x_k^\lambda]_T$. Then it follows from the above remark that

$$\|[x_k^\lambda \otimes z]\|_{\mathcal{A}_T} + \|[z \otimes x_k^\lambda]\|_{\mathcal{A}_T} \to 0, \; z \in \mathcal{H}. \tag{3}$$

Hence $[C_\lambda] \in \mathcal{X}_0(\mathcal{A}_T)$, for all $\lambda \in \mathbb{D}$. Since

$$\mathcal{X}_0(\mathcal{A}_T) = \overline{\operatorname{aco}}\mathcal{X}_0(\mathcal{A}_T) = \operatorname{Ball}_{1/M}(\mathcal{Q}_T), \tag{4}$$

$\mathcal{A}_T$ has property $X_{0,1/M}$.)

Now we introduce a sequential property as follows.

DEFINITION 2.4. A dual algebra $\mathcal{A}$ has *property* **S** if, for any given vector x in $\mathcal{H}$, sequence $\{e_k\}_{k=1}^\infty$ of unit vectors in $\mathcal{H}$, and any $\epsilon > 0$, there exist $n \in \mathbb{N}$ and complex scalars $a_1, \dots, a_n$ such that

i) $\sum_{k=1}^n |a_k|^2 = 1$, and
ii) $\|[x \otimes \sum_{k=1}^n a_k e_k]\|_{\mathcal{A}} < \epsilon$.

If $\mathcal{A}$ is singly generated, so $\mathcal{A} = \mathcal{A}_T$ for some operator T, we shall also say that T has property **S**.

LEMMA 2.5. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be dual algebras with* $\mathcal{A} \subset \mathcal{B}$. *If* $\mathcal{B}$ *has property* **S**, *then* $\mathcal{A}$ *has property* **S**.

Proof. Let $x \in \mathcal{H}$ and let $\{e_k\}_{k=1}^{\infty}$ be a sequence of unit vectors in $\mathcal{H}$. Then for any $\epsilon > 0$, there exist $n \in \mathbb{N}$ and complex scalars $a_1, \ldots, a_n$ in $\mathbb{C}$ such that

$$\sum_{k=1}^{n} |a_k|^2 = 1, \text{ and } \left\| \left[x \otimes \sum_{k=1}^{n} a_k e_k \right] \right\|_{\mathcal{B}} < \epsilon. \tag{5}$$

Since ${}^{\perp}\mathcal{B} \subset {}^{\perp}\mathcal{A}$, according to the definition of norm $\|[L]\|_{\mathcal{A}}$ we have $\|[L]\|_{\mathcal{A}} \leq \|[L]\|_{\mathcal{B}}$ for any $L \in \mathcal{C}_1$. So we have that

$$\left\| \left[x \otimes \sum_{k=1}^{n} a_k e_k \right] \right\|_{\mathcal{A}} < \epsilon, \tag{6}$$

as desired. □

PROPOSITION 2.6. *Let T be a dilation of $\widetilde{T} \in \mathcal{L}(\widetilde{\mathcal{H}})$. If $\mathcal{A}_T$ has property* **S**, *then $\mathcal{A}_{\widetilde{T}}$ has property* **S**.

Proof. Use that $\|[x \otimes y]\|_{\mathcal{A}_T} \geq \|[x \otimes y]\|_{\mathcal{A}_{\widetilde{T}}}$ for all x and y in $\widetilde{\mathcal{H}}$. □

PROPOSITION 2.7. *If $T \in C_{0\cdot}$ and T is polynomially bounded, then $\mathcal{A}_T$ has property* **S**.

Proof. Let x, $\{e_k\}_{k=1}^{\infty} \subset \mathcal{H}$ and $\epsilon > 0$ be given. If $\{e_k\}_{k=1}^{\infty}$ has terms e_i, e_j $(i \neq j)$ with $e_i = e_j$, then

$$\left\| \left[x \otimes (\frac{1}{\sqrt{2}} e_i - \frac{1}{\sqrt{2}} e_j) \right] \right\| = 0 < \epsilon. \tag{7}$$

Otherwise, the sequence must be infinite. Hence it has a weakly convergent subsequence $\{e_{k_i}\}_{i=1}^{\infty}$ converging weakly to an element e in $\mathcal{H}$. Since $T \in C_{0\cdot}$ and is polynomially bounded, we have

$$\|[x \otimes (e_{k_i} - e)]\| \to 0 \quad (i \to \infty), \tag{8}$$

because $\{e_{k_i} - e\}$ is convergent weakly to zero. But then for any i, j sufficiently large, we have

(9)
$$\begin{aligned} &\left\| \left[x \otimes \left(\frac{1}{\sqrt{2}} e_{k_i} - \frac{1}{\sqrt{2}} e_{k_j} \right) \right] \right\| \\ &\qquad = \left\| \left[x \otimes \left(\frac{1}{\sqrt{2}} (e_{k_i} - e) - \frac{1}{\sqrt{2}} (e_{k_j} - e) \right) \right] \right\| \\ &\qquad \leq \left\| \left[x \otimes \left(\frac{1}{\sqrt{2}} (e_{k_i} - e) \right) \right] \right\| + \left\| \left[x \otimes \left(\frac{1}{\sqrt{2}} (e_{k_j} - e) \right) \right] \right\| \\ &\qquad < \epsilon. \end{aligned}$$

Hence the proof is complete. □

3. Consideration of sequential properties

We pause for a moment to investigate property $\mathbf{S}$ and some related properties before proceeding to applications. The sum in Definition 2.4 ii) may be written

$$\sum_{k=1}^{n} \overline{a_k}\,[x \otimes e_k]_{\mathcal{A}}\,, \tag{10}$$

and this leads to a definition suitable for a general Banach space:

DEFINITION 3.1. A subspace F of a (complex) Banach space X has *property* $\mathbf{S}_B$ if, for any bounded sequence $\{f_k\}_{k=1}^{\infty}$ in F and any $\epsilon > 0$, there exist n in $\mathbb{N}$ and complex scalars $a_1, a_2, \cdots, a_n$ such that

i) $\sum_{k=1}^{n} |a_k|^2 = 1$, and
ii) $\| \sum_{k=1}^{n} a_k f_k \| < \epsilon$.

(Indeed, further (possibly useful?) properties arise from using different norms in *i*) on the scalars a_k.) A little thought shows that any finite dimensional space has property $\mathbf{S}_B$ and that if X is any Banach space containing a copy of c_0, l^1, or L^1, X itself does not (characteristic functions provide counter examples). Thus "most" Banach spaces do not – does any infinite dimensional Banach space? It is natural in view of our intended applications to ask whether L^1/H_0^1 has property $\mathbf{S}_B$; we are grateful to Christopher Boyd for showing us the following argument and allowing us to include it here. Following Mujica [9], we define $G^{\infty}(\mathbb{D})$ as the space of all linear maps from $H^{\infty}(\mathbb{D})$ into $\mathbb{C}$, which when restricted to $B_{H^{\infty}(\mathbb{D})}$, the unit ball of $H^{\infty}(\mathbb{D})$, are continuous for the topology of uniform convergence on compact subsets of $\mathbb{D}$. We give $G^{\infty}(\mathbb{D})$ the topology of uniform convergence on $B_{H^{\infty}(\mathbb{D})}$. With this topology $G^{\infty}(\mathbb{D})$ becomes a Banach space (see [9]) and has the property that the Banach dual of $G^{\infty}(\mathbb{D})$ is isometrically isomorphic to $H^{\infty}(\mathbb{D})$. For $x \in \mathbb{D}$ we denote by the δ_x the map from $H^{\infty}(\mathbb{D})$ into $\mathbb{C}$ defined by $\delta_x(f) = f(x)$. From Corollary 4.12 of [9] it follows that $G^{\infty}(\mathbb{D})$ can be represented as the space of all l_1 sums of sequences δ_x's with x in $\mathbb{D}$. That is,

$$G^{\infty}(\mathbb{D}) = \left\{ \sum_{n=1}^{\infty} \lambda_n \delta_{x_n} : x_n \in \mathbb{D} \text{ and } \sum_{n=1}^{\infty} |\lambda_n| < \infty \right\}.$$

It follows from [5, Theorem V.5.4] (see also [10]) that $G^{\infty}(\mathbb{D})$ is isometrically isomorphic to L^1/H_0^1.

PROPOSITION 3.2. *The space L^1/H^1_0 does not have property $\mathbf{S}_B$.*

Proof. To see that L^1/H^1_0 does not have property $\mathcal{S}_B$ we use the idea of an interpolating sequence. A sequence $\{z_j\}_{j\in\mathbf{N}}$ in $\mathbb{D}$ is called a H^∞ interpolating sequence if for any $\{\alpha_j\}_{j\in\mathbf{N}}$ in l_∞ there is $f \in H^\infty(\mathbb{D})$ such that $f(z_j) = \alpha_j$. It is shown in [2] that a necessary and sufficient condition for $\{z_j\}_{j\in\mathbf{N}}$ to be an interpolating sequence is that the infimum of the Blaschke products satisfies

$$\inf_k \prod_{j=1,\ j\neq k}^{\infty} \left|\frac{z_k - z_j}{1-\bar{z}_j z_k}\right| = \delta > 0. \tag{11}$$

Furthermore, given any interpolating sequence $\{z_j\}_{j\in\mathbf{N}}$ there is a constant K (see [5]), called the interpolation constant of $\{z_j\}_{j\in\mathbf{N}}$, such that if $f(z_j) = \alpha_j, 1 \le j \le \infty$, then

$$\|f\| \le K\sup_j|\alpha_j|. \tag{12}$$

To show that $G^\infty(\mathbb{D})$ does not satisfy property $\mathcal{S}_B$, consider $\{\delta_{z_n}\}_{n\in\mathbf{N}}$ where $\{z_n\}_{n\in\mathbf{N}}$ is a H^∞ interpolating sequence. Clearly $\{\delta_{z_n}\}_{n\in\mathbf{N}}$ is bounded in $G^\infty(\mathbb{D})$. Given any positive integer n and any finite sequence of complex numbers $\{\alpha_k\}_{k=1}^n$ such that

$$\sum_{k=1}^n |\alpha_k|^2 = 1 \tag{13}$$

we can find $f \in H^\infty(\mathbb{D})$ with $\|f\| \le K$ and $f(z_j) = \bar{\alpha}_j, 1 \le j \le n$ (of course $|\alpha_j| \le 1$ for $1 \le j \le n$). Now

$$\begin{aligned} \|\alpha_1\delta_{z_1} &+ \alpha_2\delta_{z_2} + \cdots + \alpha_n\delta_{z_n}\| \\ &\ge \frac{1}{K}\,|\alpha_1 f(z_1) + \alpha_2 f(z_2) + \cdots + \alpha_n f(z_n)| \\ &= \frac{1}{K}\sum_{k=1}^n |\alpha_k|^2 \\ &= \frac{1}{K}. \end{aligned} \tag{14}$$

So clearly we cannot make

$$\|\alpha_1\delta_{z_1} + \alpha_2\delta_{z_2} + \cdots + \alpha_n\delta_{z_n}\| < \frac{1}{K} \tag{15}$$

for any choice of $\{\alpha_j\}_{j=1}^n$ and therefore the space L^1/H^1_0 cannot have property $\mathbf{S}_B$. □

Remark that it might be interesting to know which subsets of L^1/H_0^1 have property $\mathbf{S}_B$. An alternate route to generalizing Definition 2.4 is to return to the case of a dual algebra and its predual and change the allowable sequences of vectors. For example, we have the following definition.

DEFINITION 3.3. A dual algebra $\mathcal{A}$ has *property* $\mathbf{S}_b$ if, for any given vector x in $\mathcal{H}$, bounded sequence $\{f_k\}_{k=1}^{\infty}$ of vectors in $\mathcal{H}$, and any $\epsilon > 0$, there exist $n \in \mathbb{N}$ and complex scalars $a_1, \ldots, a_n$ such that

i) $\sum_{k=1}^{n} |a_k|^2 = 1$, and
ii) $\|[x \otimes \sum_{k=1}^{n} a_k f_k]\|_{\mathcal{A}} < \epsilon$.

We may make similar definitions in which the restriction on the sequence of vectors changes, and so may define properties $\mathbf{S}_w$ (the sequence assumed weakly convergent), $\mathbf{S}_u$ (the sequence to consist of unit vectors, yielding the original property $\mathbf{S}$), $\mathbf{S}_{o.n.}$ (the sequence assumed orthonormal), and so on. Again we abuse the language slightly to say that the operator T has these properties. Note that any operator on finite dimensional space has property $\mathbf{S}_b$. Observe also that the proof of Proposition 2.7 actually says that $T \in C_{0\cdot}$ and polynomially bounded implies $\mathcal{A}_T$ has property $\mathcal{S}_b$. We next show that the converse does not hold using the following lemma.

LEMMA 3.4. *Let T be an operator such that $\mathcal{A}_T$ has property $\mathbf{S}_b$, and let T' be any operator on $\mathbb{C}^n$. Then $\mathcal{A}_{T\oplus T'}$ has property $\mathbf{S}_b$.*

Proof. Let T act on $\mathcal{H}$ and T' on $\mathcal{H}' = \mathbb{C}^n$. Let $\epsilon > 0$ and suppose $\hat{x} \in \mathcal{H} \oplus \mathcal{H}'$ and

$$\{\hat{e}_k\}_{k=1}^{\infty} \subseteq \mathcal{H} \oplus \mathcal{H}' \tag{16}$$

a bounded sequence are given. Write vectors in $\mathcal{H} \oplus \mathcal{H}'$ as $\hat{\omega} = \omega \oplus \omega'$ with respect to the obvious decomposition. Observe that since T' acts on finite dimensional space it has property $(\mathbb{A}_{1,1})$ (see [1, Theorem 2.06]), so each element $[L]$ in $\mathcal{Q}_{T'}$ is of the form $[L] = [a \otimes b]$ for some a and b in $\mathcal{H}'$. It is then easy to see that $\mathcal{Q}_{T'}$ is spanned by $\{[v_i \otimes v_j] : 1 \leq i, j \leq n\}$ where $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{C}^n$. In particular, $\mathcal{Q}_{T'}$ is of finite dimension. In what follows we restrict our attention to the case $n = 2$ for ease of exposition; note that in this case $\mathcal{Q}_{T'}$ is of dimension at most four. Since $\{e_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence, by repeated use of the definition of property $\mathbf{S}_b$ we may find non-negative integers

$0 = N_0 < N_1 < \ldots < N_5$ and scalars $\alpha_1, \cdots, \alpha_{N_5}$ so that

$$\sum_{k=N_i+1}^{N_{i+1}} |\alpha_k|^2 = 1, \quad 0 \le i \le 4, \tag{17}$$

and

$$\left\| \left[x \otimes \sum_{k=N_i+1}^{N_{i+1}} \alpha_k e_k \right] \right\|_{\mathcal{A}_T} < \frac{\epsilon}{5}, \quad 0 \le i \le 4. \tag{18}$$

Observe that the collection

$$\left\{ \left[x' \otimes \sum_{k=N_i+1}^{N_{i+1}} \alpha_k e_k{}' \right] : 0 \le i \le 4 \right\} \tag{19}$$

is a linearly dependent set in $\mathcal{Q}_{T'}$. We may then choose scalars γ_i, $0 \le i \le 4$, not all zero, so that

$$\sum_{i=0}^{4} \gamma_i \left[x' \otimes \sum_{k=N_i+1}^{N_{i+1}} \alpha_k e_k{}' \right]_{\mathcal{A}_{T'}} = 0. \tag{20}$$

To ease the notation slightly, let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_4)$, so

$$\|\gamma\| = \sqrt{\sum_{j=0}^{4} |\gamma_j|^2}.$$

Define scalars as follows:

$$\beta_k = \frac{\overline{\gamma}_i \alpha_k}{\|\gamma\|}, \quad 0 \le i \le 4, \quad N_i + 1 \le k \le N_{i+1}. \tag{21}$$

Then

$$\begin{aligned} & \left\| \left[\hat{x} \otimes \left(\sum\nolimits_{k=1}^{N_5} \beta_k \hat{e}_k \right) \right] \right\|_{\mathcal{A}_{T \oplus T'}} \\ \le & \left\| \left[x \otimes \sum\nolimits_{k=1}^{N_5} \beta_k e_k \right] \right\|_{\mathcal{A}_T} + \left\| \left[x' \otimes \left(\sum\nolimits_{k=1}^{N_5} \beta_k e'_k \right) \right] \right\|_{\mathcal{A}_{T'}} \\ \le & \sum_{i=0}^{4} \frac{|\gamma_i|}{\|\gamma\|} \left\| \left[x \otimes \sum\nolimits_{k=N_i+1}^{N_{i+1}} \alpha_k e_k \right] \right\|_{\mathcal{A}_T} \end{aligned} \tag{22}$$

$$+\frac{1}{\|\gamma\|}\left\|\sum_{i=0}^{4}\gamma_i\left[x'\otimes\sum_{k=N_i+1}^{N_{i+1}}\alpha_k e_k{}'\right]\right\|_{\mathcal{A}_{T'}}$$
$$\leq \sum_{i=0}^{4}\frac{|\gamma_i|}{\sqrt{\sum_{j=0}^{4}|\gamma_j|^2}}\frac{\epsilon}{5}+0$$
$$< \epsilon,$$

where we have used (18) and (20). Further,

$$\begin{aligned}\sum_{k=1}^{N_5}|\beta_k|^2 &= \sum_{i=0}^{4}\frac{|\gamma_i|^2}{\sum_{j=0}^{4}|\gamma_j|^2}\sum_{k=N_i+1}^{N_{i+1}}|\alpha_k|^2\\ &= \sum_{i=0}^{4}\frac{|\gamma_i|^2}{\sum_{j=0}^{4}|\gamma_j|^2}\cdot 1\\ &= 1,\end{aligned}\tag{23}$$

where we have used (17). Thus we have a satisfactory sequence of scalars for ϵ, $\hat{x}$, and $\{\hat{e}_k\}_{k=1}^{\infty}$, and the proof is complete. □

Note that any operator T on a finite dimensional space has property $\mathbf{S}_b$, and so there are operators which are not in $C_{0\cdot}$ with property $\mathbf{S}_b$. With the aid of the previous lemma, we may do a little better.

COROLLARY 3.5. *There exist operators with property* $\mathbf{S}_b$ (and hence properties $\mathbf{S}_w$, $\mathbf{S}_u$ and $\mathbf{S}_{o.n.}$) *which act on infinite dimensional space and are not in* $C_{0\cdot}$.

Proof. Let T be any operator acting on infinite dimensional space with property $\mathbf{S}_b$ (e.g., $T \in C_{0\cdot}$ and polynomially bounded) and consider $T \oplus I_{\mathbb{C}}$. □

It is natural to consider direct sums of operators with the various properties. An attempt to prove the anticipated result leads to a realization that these properties are not at least obviously well behaved with respect to direct sums (indeed, even $T \oplus T$ presents difficulties). We can obtain the following.

PROPOSITION 3.6. *Suppose T and T' are such that $\mathcal{A}_T$ and $\mathcal{A}_{T'}$ have property $\mathbf{S}_b$. Then $\mathcal{A}_{T\oplus T'}$ has property $\mathbf{S}_{o.n.}$. In particular, if $\mathcal{A}_T$ has property $\mathbf{S}_b$ and $T' \in C_{0\cdot}$, then $\mathcal{A}_{T\oplus T'}$ has property $\mathbf{S}_{o.n.}$.*

Proof. Let $\hat{x} = x \oplus x'$, $\{\hat{e}_k\}_{k=1}^{\infty} = \{e_k \oplus e'_k\}_{k=1}^{\infty}$ an orthonormal sequence, and $\epsilon > 0$ be given. Observe that if $\{\gamma_k\}_{k=N}^{M}$ is any finite sequence of scalars satisfying $\sum |\gamma_k|^2 = 1$, then

$$\left\| \sum_{k=N}^{M} \gamma_k \hat{e}_k \right\| = 1, \tag{24}$$

and hence

$$\left\| \sum_{k=N}^{M} \gamma_k e_k \right\| \leq 1, \tag{25}$$

and

$$\left\| \sum_{k=N}^{M} \gamma_k e'_k \right\| \leq 1. \tag{26}$$

Let $\{\delta_n\}_{n=1}^{\infty}$ be some sequence with strictly positive entries such that

$$\sum_{n=1}^{\infty} \delta_n < \frac{\epsilon}{2}. \tag{27}$$

For convenience of notation, let $N_0 = 0$. By repeated use of the fact that $\mathcal{A}_T$ has property $\mathbf{S}_b$, and the observation that $\{e_k\}_{k=1}^{\infty}$ is bounded, we may produce an increasing sequence of positive integers $\{N_n\}_{n=1}^{\infty}$ and a sequence $\{\alpha_k\}_{k=1}^{\infty}$ of scalars satisfying

$$\sum_{k=N_{n-1}+1}^{N_n} |\alpha_k|^2 = 1, \quad n = 1, 2, \cdots, \tag{28}$$

and

$$\left\| \left[x \otimes \sum_{k=N_{n-1}+1}^{N_n} \alpha_k e_k \right] \right\|_{\mathcal{A}_T} < \delta_n, \quad n = 1, 2, \cdots. \tag{29}$$

To ease the notation, let

$$y_n = \sum_{k=N_{n-1}+1}^{N_n} \alpha_k e_k, \quad n = 1, 2, \cdots \tag{30}$$

and

$$z'_n = \sum_{k=N_{n-1}+1}^{N_n} \alpha_k e'_k, \quad n = 1, 2, \cdots. \tag{31}$$

Using our initial observation, the z_n form a bounded sequence. Since $\mathcal{A}_{T'}$ has property $\mathbf{S}_b$, there exist scalars $\{\beta_n\}_{n=1}^M$ such that

$$\sum_{n=1}^{M} |\beta_n|^2 = 1 \tag{32}$$

and

$$\left\| \left[x' \otimes \sum_{n=1}^{M} \beta_n z_n' \right] \right\|_{\mathcal{A}_{T'}} < \frac{\epsilon}{2}. \tag{33}$$

Then

(34)

$$\begin{aligned}
\left\| \left[\hat{x} \otimes \sum_{n=1}^{M} \beta_n (y_n \oplus z_n') \right] \right\|_{\mathcal{A}_{T\oplus T'}} &= \left\| \left[\hat{x} \otimes \sum_{n=1}^{M} \sum_{k=N_{n-1}+1}^{N_n} \beta_n \alpha_k \hat{e}_k \right] \right\|_{\mathcal{A}_{T\oplus T'}} \\
&\leq \left\| \left[x \otimes \sum_{n=1}^{M} \sum_{k=N_{n-1}+1}^{N_n} \beta_n \alpha_k e_k \right] \right\|_{\mathcal{A}_T} \\
&\quad + \left\| \left[x' \otimes \sum_{n=1}^{M} \sum\nolimits_{k=N_{n-1}+1}^{N_n} \beta_n \alpha_k e_k' \right] \right\|_{\mathcal{A}_{T'}} \\
&\leq \sum_{n=1}^{M} |\beta_n| \left\| \left[x \otimes \sum_{k=N_{n-1}+1}^{N_n} \alpha_k e_k \right] \right\|_{\mathcal{A}_T} \\
&\quad + \left\| \left[x' \otimes \sum_{n=1}^{M} \beta_n \sum_{k=N_{n-1}+1}^{N_n} \alpha_k e_k' \right] \right\|_{\mathcal{A}_{T'}} \\
&\leq \sum_{n=1}^{M} |\beta_n| \delta_n + \left\| \left[x' \otimes \sum_{n=1}^{M} \beta_n z_n' \right] \right\|_{\mathcal{A}_{T'}} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence the proof is complete. □

4. Two dual algebra properties

The following is the main theorem of this paper.

THEOREM 4.1. *Suppose $\mathcal{A}_T$ and $\mathcal{A}_{T^*}$ have property $\mathbf{S}_{o.n.}$. Assume that $T \in \mathbb{A}^M$. Then $T \in \mathbb{A}^M_{\aleph_0}(\mathcal{H})$ if and only if the algebra $\mathcal{A}_T$ has property $X_{0,1/M}$.*

(Recall it is known from [8] that the reverse implication holds in general.) Proposition 2.7, property $\mathbf{S}_{o.n.}$ *a priori* weaker than property $\mathbf{S}$, and Theorem 4.1 yield the following.

COROLLARY 4.2. *If $T \in \mathbb{A}^M_{\aleph_0}(\mathcal{H}) \cap C_{00}$, then the algebra $\mathcal{A}_T$ has property $X_{0,1/M}$.*

Proof of Theorem 4.1. According to Theorem 2.3, T dilates, to some semi-invariant subspace, a normal operator diagonal with respect to some orthonormal basis and with eigenvalues dense in the disk and each of infinite multiplicity. Write $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ with respect to the decomposition induced by this dilation. Fix (for the moment) $\lambda \in \mathbb{D}$ an eigenvalue of $T_2 \triangleq T_{\mathcal{H}_2}$, and let $\{e_i\}_{i=1}^{\infty}$ be a collection of orthonormal eigenvectors for T_2 associated with λ. Let us assemble some well known facts. First,

$$(35) \qquad [(0 \oplus v \oplus 0) \otimes (0 \oplus v \oplus 0)]_T = [C_\lambda], \quad v \in \bigvee\{e_i\}_{i=1}^{\infty}, \quad \|v\| = 1.$$

Second, since $T_2 \in C_{00}$,

(36)

$$\|[v_n \otimes w]\|_{\mathcal{A}_{T_2}} + \|[w \otimes v_n]\|_{\mathcal{A}_{T_2}} \to 0, \ w \in \mathcal{H}_2, \ \{v_n\}_{n=1}^{\infty} \subseteq \mathcal{H}_2, \ v_n \xrightarrow{w} 0,$$

and thus easily

$$(37) \ \|[(0\oplus v_n\oplus 0)\otimes(0\oplus w\oplus 0)]\|_{\mathcal{A}_T} \to 0, w \in \mathcal{H}_2, \{v_n\}_{n=1}^{\infty} \subseteq \mathcal{H}_2, v_n \xrightarrow{w} 0,$$

and

$$(38) \ \|[(0\oplus w\oplus 0)\otimes(0\oplus v_n\oplus 0)]\|_{\mathcal{A}_T} \to 0, w \in \mathcal{H}_2, \{v_n\}_{n=1}^{\infty} \subseteq \mathcal{H}_2, v_n \xrightarrow{w} 0.$$

Third,

$$(39) \qquad [(0 \oplus v \oplus 0) \otimes (0 \oplus 0 \oplus z)]_T = 0, \quad v \in \mathcal{H}_2, \quad z \in \mathcal{H}_3,$$

and

$$(40) \qquad [(w \oplus 0 \oplus 0) \otimes (0 \oplus v \oplus 0)]_T = 0, \quad v \in \mathcal{H}_2, \quad w \in \mathcal{H}_1.$$

Let $\{w_j\}_{j=1}^{\infty}$ be a dense subset of $\mathcal{H}_1$, and $\{z_j\}_{j=1}^{\infty}$ be a dense subset of $\mathcal{H}_3$. We shall construct a collection $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{H}_2$ of orthonormal vectors in $\bigvee_{i=1}^{\infty} e_i$ satisfying

$$(41) \qquad \lim_{n\to\infty} \|[(0 \oplus x_n \oplus 0) \otimes (w_j \oplus 0 \oplus 0)]\|_{\mathcal{A}_T} = 0, \quad j \in \mathbb{N},$$

and

$$(42)\qquad \lim_{n\to\infty} \|[(0\oplus 0\oplus z_j)\otimes(0\oplus x_n\oplus 0)]\|_{\mathcal{A}_T}=0,\quad j\in\mathbb{N}.$$

Suppose for a moment this has been done. Of course by (35) we get $[x_n\otimes x_n]=[C_\lambda]_T$ for each n. Equations (41) and (42), and the results in (37), (39), and (40), show that the sequence $\{0\oplus x_n\oplus 0\}_{n=1}^\infty$ satisfies both vanishing conditions for all vectors in the set $\{(w_j\oplus v\oplus z_k): v\in\mathcal{H}_2, j,k\in\mathbb{N}\}$ (which is clearly dense in $\mathcal{H}$). A standard argument then promotes these vanishing conditions to all vectors in $\mathcal{H}$, and then we obtain as in the Remarks in Section 2 that $[C_\lambda]\in\mathcal{X}_0(\mathcal{A}_T)$. Since we may perform an identical construction for any of the eigenvalues of T_2, we obtain via standard arguments $B_{0,1/M}(\mathcal{Q}_T)\supseteq\overline{\mathrm{aco}}\mathcal{X}_0(\mathcal{A}_T)$, and thus $\mathcal{A}_T$ has property $X_{0,1/M}$ as desired.

We embark now upon the construction of the $\{x_n\}_{n=1}^\infty$. The construction is both recursive and diagonal, so let us set some notation and definitions. The definition below indicates the sort of sequences we will need.

DEFINITION 4.3. Suppose $\{a_n\}_{n=1}^\infty$ is some sequence of vectors. We say that a sequence $\{b_i\}_{i=1}^\infty$ is *block supported* on the sequence $\{a_n\}_{n=1}^\infty$ if there exists an increasing sequence $\{n_i\}_{i=1}^\infty$ of positive integers so that
i) The vector b_1 is a linear combination of a_n satisfying $1\le n\le n_1$, and
ii) Each b_i, $i=2,\ldots$, is a linear combination of a_n satisfying $n_{i-1}+1\le n\le n_i$.

We make four observations for later use. First, if b is a sequence block supported on a and c is block supported on b then c is block supported on a. Second, if b is a sequence block supported on a where a is a sequence of pairwise orthogonal vectors, then b also is a sequence of pairwise orthogonal vectors. Third, if the sequence a consists of orthonormal vectors, then any b_n is a unit vector if and only if its coefficients with respect to a are absolutely square summable to one, and that any such unit vector sequence b (block supported on a) is actually again an orthonormal sequence. Fourth, call a subsequence $\{j(n)\}_{n=1}^\infty$ of the positive integers *integer thin* if

$$\sum_{n=N}^{\infty}\frac{1}{j(n)^2}<\frac{1}{N^2},\quad N=1,2,\ldots.$$

Call a subsequence $\{a_{j(n)}\}_{n=1}^\infty$ of the sequence of vectors a *thin* if $\{j(n)\}_{n=1}^\infty$ is integer thin. If b is any sequence of vectors block supported on a thin subsequence of a, then note that b is block supported on a.

Observe that each b_N is a linear combination of $a_{j(n)}$'s with $n \geq N$. Write

$$b_N = \sum_{n=N}^{\infty} \alpha_{j(n)}^N a_{j(n)}, \tag{43}$$

where all but finitely many of the $\alpha_{j(n)}^N$ are zero (for each N). Now suppose w is some fixed vector and a is a sequence of unit vectors chosen so that

$$\|[w \otimes a_n]\| < 1/n, \quad n \in \mathbb{N},$$

with b a sequence of vectors supported on a thin subsequence of a, and in addition each b_N is a linear combination of $a_{j(n)}$'s with coefficients absolutely square summable to 1. We have the following estimate, for each N, showing that the b_N "inherit" the good property of the a_n with respect to w (we omit temporarily subscripts $\mathcal{A}_T$ on the norms):

$$\begin{aligned}
\|[w \otimes b_N]\| &= \left\| \left[w \otimes \left(\sum_{n=N}^{\infty} \alpha_{j(n)}^N a_{j(n)} \right) \right] \right\| \\
&\leq \sum_{n=N}^{\infty} \left| \alpha_{j(n)}^N \right| \cdot \left\| [w \otimes a_{j(n)}] \right\| \\
&\leq \sum_{n=N}^{\infty} \left| \alpha_{j(n)}^N \right| \cdot \frac{1}{j(n)} \\
&\leq \left(\sum_{n=N}^{\infty} \left| \alpha_{j(n)}^N \right|^2 \right)^{1/2} \left(\sum_{n=N}^{\infty} \frac{1}{j(n)^2} \right)^{1/2} \\
&< 1/N.
\end{aligned} \tag{44}$$

A similar result obviously holds for subsequences block supported on thin subsequences of sequences a chosen so that $\|[a_n \otimes w]\|$ is small. We shall construct a collection of sequences $\{r_n^i\}_{n=1}^{\infty}$, one for each i in $\mathbb{N}$, and a collection of sequences $\{s_n^i\}_{n=1}^{\infty}$, one for each i in $\mathbb{N}$, satisfying the following:

(45)

(i) Each r_n^i and s_n^i is a unit vector in $\bigvee_{j=1}^{\infty} e_j$, $i, n \in \mathbb{N}$,

(ii) $\{r_n^1\}_{n=1}^{\infty}$ is block supported on $\{e_n\}_{n=1}^{\infty}$,

(iii) $\{r_n^i\}_{n=1}^{\infty}$ is block supported on a thin subsequence of $\{s_n^{i-1}\}_{n=1}^{\infty}$, $i \in \mathbb{N}$, $i \geq 2$,

(iv) $\{s_n^i\}_{n=1}^{\infty}$ is block supported on a thin subsequence of $\{r_n^i\}_{n=1}^{\infty}$, $i \in \mathbb{N}$,

(v) $\|[(0 \oplus 0 \oplus z_i) \otimes (0 \oplus r_n^i \oplus 0)]\|_{\mathcal{A}_T} < \frac{1}{n}$, $i, n \in \mathbb{N}$, and

(vi) $\|[(0 \oplus s_n^i \oplus 0) \otimes (w_i \oplus 0 \oplus 0)]\|_{\mathcal{A}_T} < \frac{1}{n}$, $i, n \in \mathbb{N}$.

Supposing these sequences have been constructed, let us show how to choose the $\{x_n\}_{n=1}^{\infty}$. We may write the r_n^i and s_n^i in the array

$$\begin{matrix} r_1^1 & r_2^1 & r_3^1 & \cdots \\ s_1^1 & s_2^1 & s_3^1 & \cdots \\ r_1^2 & r_2^2 & r_3^2 & \cdots \\ s_1^2 & s_2^2 & s_3^2 & \cdots \\ \vdots & & & \ddots \end{matrix}$$

and choose the x_n diagonally:

$$\begin{aligned} x_1 &= r_1^1, \\ x_2 &= s_2^1, \\ x_3 &= r_3^2, \\ \vdots &= \vdots \end{aligned} \tag{46}$$

Using the third observation after Definition 4.3, it is clear that $\{x_n\}_{n=1}^{\infty}$ is an orthonormal sequence. By (45-i) and (35) we have that

$$[x_n \otimes x_n] = [C_\lambda]_T, \quad n \in \mathbb{N}.$$

Then by repeated use of the fourth observation following Definition 4.3 one may show that

$$\|[(0 \oplus x_n \oplus 0) \otimes (w_i \oplus 0 \oplus 0)]\|_{\mathcal{A}_T} < \frac{1}{n}, \quad i, n \in \mathbb{N},\ i \leq n/2, \tag{47}$$

and

$$\|[(0 \oplus 0 \oplus z_i) \otimes (0 \oplus x_n \oplus 0)]\|_{\mathcal{A}_T} < \frac{1}{n}, \quad i, n \in \mathbb{N},\ i \leq n/2, \tag{48}$$

which obviously yield (41) and (42). Then, *modulo* the construction of the r_n^i and s_n^i, we are done.

We sketch the construction of the $\{r_n^i\}_{n=1}^\infty$ and $\{s_n^i\}_{n=1}^\infty$ and leave the details to the interested reader. Consider the collection

$$\{[(0 \oplus 0 \oplus z_1) \otimes (0 \oplus e_n \oplus 0)]_T : n \in \mathbb{N}\}.$$

If this collection is finite, it is easy to construct the $\{r_n^1\}_{n=1}^\infty$: find m_1 and m^1, distinct, so that

$$[(0 \oplus 0 \oplus z_1) \otimes (0 \oplus e_{m_1} \oplus 0)]_T = [(0 \oplus 0 \oplus z_1) \otimes (0 \oplus e_{m^1} \oplus 0)]_T,$$

where upon

$$r_1^1 = \frac{1}{\sqrt{2}} e_{m_1} + \frac{-1}{\sqrt{2}} e_{m^1}$$

is satisfactory. Repeating, it is easy to finish what is required for $i = 1$ in (45-i), (45-ii), and (45-v). If the collection $\mathcal{C} = \{[(0 \oplus 0 \oplus z_1) \otimes (0 \oplus e_n \oplus 0)]_T : n \in \mathbb{N}\}$ is infinite, we may use that $\mathcal{A}_T$ has property $\mathbf{S}_{o.n.}$ to deduce that there exist n_1 and scalars $a_1, \ldots, a_{n_1}$ so that

$$\sum_{i=1}^{n_1} |a_i|^2 = 1 \tag{49}$$

and

$$\begin{aligned} &\|a_1[(0 \oplus 0 \oplus z_1) \otimes (0 \oplus e_1 \oplus 0)] + \ldots \\ &\qquad + a_{n_1}[(0 \oplus 0 \oplus z_1) \otimes (0 \oplus e_{n_1} \oplus 0)]]\|_{\mathcal{A}_T} \\ &\qquad < 1. \end{aligned} \tag{50}$$

It is clear from (49) that

$$r_1^1 = \sum_{j=1}^{n_1} \overline{a_j} e_j \tag{51}$$

is a unit vector and from (50) that

$$\|[(0 \oplus 0 \oplus z_1) \otimes (0 \oplus r_1^1 \oplus 0)]\|_{\mathcal{A}_T} < 1. \tag{52}$$

Thus this choice for r_1^1 is satisfactory for $i = n = 1$ in (45-i), (45-ii), and (45-v); repeating this construction using only $\{e_n\}_{n>n_1}^\infty$, it is easy to get the rest of the r_n^1 as required. To construct $\{s_n^1\}_{n=1}^\infty$, repeat the procedure just given with the $\{e_n\}_{n=1}^\infty$ replaced by a thin subsequence of $\{r_n^1\}_{n=1}^\infty$, z_1 replaced by w_1, and of course with reference to the condition desired in (45-iv), i.e., with w_1 on the right instead of the left. (Recall that $\|[a \otimes b]\|_{\mathcal{A}_T} = \|[b \otimes a]\|_{\mathcal{A}_{T^*}}$ and use $\mathcal{A}_{T^*}$ has property $\mathbf{S}_{o.n.}$.) Continuing this process, we may indeed construct the required sequences $\{r_n^i\}_{n=1}^\infty$ and $\{s_n^i\}_{n=1}^\infty$, and the proof is finally complete. □

REMARKS. We noted above that property $\mathbf{S}_{o.n.}$ is *a priori* weaker than property $\mathbf{S}$; we do not know whether it is in fact weaker, nor, in fact, any but the obvious relationships between the properties. Also, the referee of an earlier version of this paper asks the natural question whether the unilateral shift U of multiplicity one has property $\mathbf{S}$. While we are unable to resolve this, we conjecture that it does not, based on the following observations. First, if U has property $\mathbf{S}$, observe that the unilateral shift $U^{(n)}$ of any multiplicity (even $\aleph_0$) has property $\mathbf{S}$, since for any particular vector w, verification of the definition comes down to what happens in the cyclic subspace generated by w; either this is trivially (0), or the restriction of $U^{(n)}$ to this space is simply the shift of multiplicity one. Since via Proposition 2.6 any operator dilated by a pure shift will then inherit property $\mathbf{S}$, the class of operators with this property would be very large. In particular, any contraction in the class $C_{\cdot 0}$ will have this property, since these operators have minimal isometric dilation a pure shift. We believe this is unlikely, especially in light of the observation that property $\mathbf{S}$ is associated in some sense with the class $C_{0\cdot}$, not $C_{\cdot 0}$.

Observe that an effort to prove U does not have property $\mathbf{S}$ along the lines of Proposition 3.2 founders on the difference between the norm in $\mathcal{H}$ of vectors $\{e_n\}_{n=1}^{\infty}$ and the norm in $\mathcal{Q}_T$ of $\{[w \otimes e_n]\}_{n=1}^{\infty}$. This exposes clearly the difference between the conditions $\mathcal{A}$ has property $\mathbf{S}$ (or the variants) and $\mathcal{Q}_T$ has property $\mathbf{S}_B$. Indeed, since it is well known that $\|[w \otimes x]\|_{\mathcal{A}_T} \leq \|w\| \cdot \|x\|$, clearly the image set $\{[w \otimes e_n]\}_{n=1}^{\infty}$ of a bounded set $\{e_n\}_{n=1}^{\infty}$ would be bounded; therefore, if L^1/H_0^1 had property $\mathbf{S}_B$ then every operator in $\mathbb{A}^M$ would have property $\mathbf{S}_b$.

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