ON THE CONCIRCULAR CURVATURE TENSOR
OF A CONTACT METRIC MANIFOLD

D. E. BLAIR, JEONG-SIK KIM, AND MUKUT MANI TRIPATHI

ABSTRACT. We classify $N(\kappa)$-contact metric manifolds which satisfy $Z(\xi, X) \cdot Z = 0$, $Z(\xi, X) \cdot R = 0$ or $R(\xi, X) \cdot Z = 0$.

1. Introduction

As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies

$$R(X, Y) \cdot R = 0, \quad X, Y \in TM,$$

where $R(X, Y)$ acts on R as a derivation. In contact geometry, S. Tanno[12] showed that a semi-symmetric K-contact manifold M^{2n+1} is locally isometric to the unit sphere $S^{2n+1}(1)$.

We remark that a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat (see [3] or [9] pp.98–99). A contact metric manifold M^{2n+1} satisfying $R(X, Y)\xi = 0$, where ξ is the characteristic vector field of the contact structure, is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat in dimension 3 ([3] or see [4] p.101).

In [11], D. Perrone studied contact metric manifolds satisfying $R(\xi, X) \cdot R = 0$ and that under additional assumptions the manifold is either Sasakian (and of constant curvature $+1$) or $R(X, \xi)\xi = 0$. Ch. Baikoussis and Th. Koufogiorgos[2] showed that if ξ belongs to the κ-nullity distribution and if $R(\xi, X) \cdot C = 0$, C being the Weyl conformal curvature tensor, the contact metric manifold M^{2n+1} is locally isometric

to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. This generalizes a result of Chaki and Tarafdar[7] that a Sasakian manifold M^{2n+1} such that $R(\xi, X) \cdot C = 0$ is locally isometric to $S^{2n+1}(1)$. B. J. Papantoniou[10] showed that a semi-symmetric contact metric manifold M^{2n+1} with ξ belonging to the (κ, μ)-nullity distribution is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$. Both Perrone and Papantoniou also studied $R(\xi, X) \cdot S = 0$ where S denotes the Ricci tensor. Perrone showed that if ξ belongs to the κ-nullity distribution and if $R(\xi, X) \cdot S = 0$, then the contact metric manifold is locally isometric to $E^{n+1} \times S^n(4)$ or is Sasakian-Einstein.

After the curvature tensor and the Weyl conformal curvature tensor, the concircular curvature tensor is the next most important $(1, 3)$-type curvature tensor from the Riemannian point of view. We find it interesting that studying the concircular curvature tensor on contact metric manifolds leads to new, noteworthy examples, as we will see in this paper. The paper is organized as follows. Section 2 contains necessary details about contact metric manifolds and the concircular curvature tensor. In section 3, we give a brief account of (κ, μ)-manifolds and $\mathcal{D}$-homothetic deformation. Then we construct a key example for later use. Section 4 contains our main results. In the last section, as an application we classify concircularly symmetric $N(\kappa)$-contact metric manifolds. Then we show the non-existence of $N(\kappa)$-contact metric manifolds with non-vanishing recurrent concircular curvature tensor.

2. Preliminaries

An odd-dimensional manifold M^{2n+1} is said to admit an *almost contact structure*, sometimes called a (φ, ξ, η)-*structure*, if it admits a tensor field φ of type $(1, 1)$, a vector field ξ, and a 1-form η satisfying

$$(2.1) \qquad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

The first and one of the remaining three relations in (2.1) imply the other two relations in (2.1). An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t the coordinate of $\mathbb{R}$ and λ a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric

with (φ, ξ, η), that is,

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y)$$

or equivalently,

$$g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in TM$. Then, M becomes an *almost contact metric manifold* equipped with an almost contact metric structure (φ, ξ, η, g).

An almost contact metric structure becomes a *contact metric structure* if

$$g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in TM.$$

The 1-form η is then a contact form and ξ is its characteristic vector field. A normal contact metric manifold is a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if

$$\nabla_X \varphi = R_0(\xi, X), \quad X \in TM,$$

where ∇ is Levi-Civita connection and

$$R_0(X, Y)W = g(Y, W)X - g(X, W)Y, \quad X, Y, W \in TM.$$

A contact metric manifold M is Sasakian if and only if the curvature tensor R satisfies

$$(2.2) \qquad R(X, Y)\xi = R_0(X, Y)\xi, \quad X, Y \in TM.$$

A contact metric manifold is called a *K-contact manifold* if the characteristic vector field ξ is a Killing vector field. An almost contact metric manifold is K-contact if and only if $\nabla \xi = -\varphi$. A K-contact manifold is a contact metric manifold, while the converse is true if $h = 0$, where $2h$ is the Lie derivative of φ in the characteristic direction ξ. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold. Thus a 3-dimensional contact metric manifold is a Sasakian manifold if and only if $h = 0$. For more details we refer to [3].

The *concircular curvature tensor* Z in a Riemannian manifold (M^n, g) is defined by ([14, 15])

$$Z = R - \frac{r}{n(n-1)} R_0,$$

where r is the scalar curvature of M^n. We observe immediately from the form of the concircular curvature tensor that Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular curvature tensor as a measure of the failure of a Riemannian manifold to be of constant curvature.

Also a necessary and sufficient condition that a Riemannian manifold be reducible to a Euclidean space by a suitable concircular transformation is that its concircular curvature tensor vanishes.

3. (κ, μ)-manifolds

It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$. ([4]) On the other hand, as we have noted (equation (2.2)), on a Sasakian manifold

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case; D. Blair, Th. Koufogiorgos and B. J. Papantoniou[5] considered the (κ, μ)-*nullity condition* on a contact metric manifold and gave several reasons for studying it. The (κ, μ)-*nullity distribution* $N(\kappa, \mu)$ ([5, 10]) of a contact metric manifold M is defined by

$$N(\kappa, \mu) : p \to N_p(\kappa, \mu)$$
$$= \{W \in T_p M \mid R(X, Y)W = (\kappa I + \mu h)R_0(X, Y)W\}$$

for all $X, Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(\kappa, \mu)$ is called a (κ, μ)-*manifold*. In particular on a (κ, μ)-manifold, we have

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

On a (κ, μ)-manifold $\kappa \leq 1$. If $\kappa = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $\kappa < 1$, the (κ, μ)-nullity condition determines the curvature of M^{2n+1} completely [5]. In fact, for a (κ, μ)-manifold, the conditions of being a Sasakian manifold, a K-contact manifold, $\kappa = 1$ and $h = 0$ are all equivalent. In a (κ, μ)-manifold the covariant derivative of φ satisfies

$$\nabla_X \varphi = R_0(\xi, (I + h)X).$$

Moreover, we have

$$Q\xi = 2n\kappa\xi, \quad h^2 = (\kappa - 1)\varphi^2,$$

where Q is Ricci operator. If $\mu = 0$, the (κ, μ)-nullity distribution $N(\kappa, \mu)$ is reduced to the κ-*nullity distribution* $N(\kappa)$ [13], where the κ-*nullity distribution* $N(\kappa)$ of a Riemannian manifold M is defined by

$$N(\kappa) : p \to N_p(\kappa) = \{W \in T_p M \mid R(X, Y)W = \kappa R_0(X, Y)W\};$$

κ being a constant. If $\xi \in N(\kappa)$, then we call a contact metric manifold M an $N(\kappa)$-*contact metric manifold*. If $\kappa = 1$, an $N(\kappa)$-contact metric manifold is Sasakian and if $\kappa = 0$, an $N(\kappa)$-contact metric manifold is locally isometric to $E^{n+1} \times S^n(4)$. In [1], $N(\kappa)$-contact metric manifolds were studied in some detail. In particular, if $\kappa < 1$, the scalar curvature is $r = 2n(2n - 2 + \kappa)$. For more details we refer to [1] and [5].

The standard contact metric structure on the tangent sphere bundle T_1M satisfies the (κ, μ)-nullity condition if and only if the base manifold M is of constant curvature. In particular if M has constant curvature c, then $\kappa = c(2 - c)$ and $\mu = -2c$.

We also recall the notion of a $\mathcal{D}$-*homothetic deformation*. For a given contact metric structure (φ, ξ, η, g), this is the structure defined by

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\varphi} = \varphi, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,$$

where a is a positive constant. While such a change preserves the state of being contact metric, K-contact, Sasakian or strongly pseudo-convex CR, it destroys a condition like $R(X, Y)\xi = 0$ or $R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y)$. However the form of the (κ, μ)-nullity condition is preserved under a $\mathcal{D}$-homothetic deformation with

$$\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}, \quad \bar{\mu} = \frac{\mu + 2a - 2}{a}.$$

Given a non-Sasakian (κ, μ)-manifold M, E. Boeckx[6] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - \kappa}}$$

and showed that for two non-Sasakian (κ, μ)-manifolds $(M_i, \varphi_i, \xi_i, \eta_i, g_i)$, $i = 1, 2$, we have $I_{M_1} = I_{M_2}$ if and only if up to a $\mathcal{D}$-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus we know all non-Sasakian (κ, μ)-manifolds locally as soon as we have for every odd dimension $2n + 1$ and for every possible value of the invariant I, one (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ with $I_M = I$. For $I > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where we have $I = \frac{1+c}{|1-c|}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I \leq -1$.

Using this invariant, we now construct an example of a $(2n + 1)$-dimensional $N\left(1 - \frac{1}{n}\right)$-contact metric manifold, $n > 1$.

EXAMPLE 3.1. Since the Boeckx invariant for a $\left(1 - \frac{1}{n}, 0\right)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting $\mathcal{D}$-homothetic deformation will be a $\left(1 - \frac{1}{n}, 0\right)$-manifold. That is, for $\kappa = c(2 - c)$ and $\mu = -2c$ we solve

$$1 - \frac{1}{n} = \frac{\kappa + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a}$$

for a and c. The result is

$$c = \frac{\left(\sqrt{n} \pm 1\right)^2}{n - 1}, \quad a = 1 + c$$

and taking c and a to be these values we obtain an $N\left(1 - \frac{1}{n}\right)$-contact metric manifold.

The above example will be used in Theorem 4.1.

4. $N(\kappa)$-contact metric manifolds satisfying $Z(\xi, X) \cdot Z = 0$

First, we recall from the Introduction that a contact metric manifold of constant curvature is necessarily a Sasakian manifold of constant curvature $+1$ or is 3-dimensional and flat, and a contact metric manifold M^{2n+1} satisfying $R(X, Y)\xi = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat in dimension 3.

We now prove the following theorem in which Example 3.1 arises naturally in contrast to $E^{n+1} \times S^n(4)$ (cf. Theorem 4.3 below).

THEOREM 4.1. *A* $(2n+1)$-*dimensional* $N(\kappa)$-*contact metric manifold* M *satisfies*

$$Z(\xi, X) \cdot Z = 0$$

if and only if M *is locally isometric to the sphere* $S^{2n+1}(1)$, M *is locally isometric to the* Example 3.1 *or* M *is 3-dimensional and flat.*

Proof. We first note that on an $N(\kappa)$-contact metric manifold M^{2n+1}

(4.1) $$Z(X, Y)\xi = \left(\kappa - \frac{r}{2n(2n+1)}\right) R_0(X, Y)\xi,$$

(4.2) $$Z(\xi, X) = \left(\kappa - \frac{r}{2n(2n+1)}\right) R_0(\xi, X).$$

From the condition $Z(\xi, U) \cdot Z = 0$, we get

$$0 = [Z(\xi, U), Z(X, Y)]\xi - Z(Z(\xi, U)X, Y)\xi - Z(X, Z(\xi, U)Y)\xi,$$

which in view of (4.2) gives

$$0 = \left(\kappa - \frac{r}{2n(2n+1)}\right)\{g(U, Z(X,Y)\xi)\xi - \eta(Z(X,Y)\xi)U$$
$$- g(U,X)Z(\xi,Y)\xi + \eta(X)Z(U,Y)\xi - g(U,Y)Z(X,\xi)\xi$$
$$+ \eta(Y)Z(X,U)\xi - \eta(U)Z(X,Y)\xi + Z(X,Y)U\}.$$

Equation (4.1) then gives

$$\left(\kappa - \frac{r}{2n(2n+1)}\right)\left(Z - \left(\kappa - \frac{r}{2n(2n+1)}\right)R_0\right) = 0.$$

Therefore either $r = 2n(2n+1)\kappa$ or M is of constant curvature. In the second case M is either 3-dimensional and flat, or locally isometric to the sphere $S^{2n+1}(1)$.

If $r = 2n(2n+1)\kappa$, recall that the scalar curvature of an $N(\kappa)$-contact metric manifold is $r = 2n(2n-2+\kappa)$. Comparing gives $\kappa = 1 - \frac{1}{n}$ and hence M is locally isometric to the Example 3.1 for $n > 1$ and to the flat case if $n = 1$.

Conversely, the first and third cases are of constant curvature and therefore $Z = 0$; in the second case $\kappa = \frac{r}{2n(2n+1)}$ giving $Z(\xi, X) = 0$. □

Using the fact that $Z(\xi, X) \cdot R$ denotes $Z(\xi, X)$ acting on R as a derivation, we have the following theorem as a corollary of Theorem 4.1.

THEOREM 4.2. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies*

$$Z(\xi, X) \cdot R = 0$$

if and only if M is locally isometric to the sphere $S^{2n+1}(1)$, M is locally isometric to Example 3.1 or M is 3-dimensional and flat.

On the other hand, reversing the order of Z and R gives the following result.

THEOREM 4.3. *A $(2n+1)$-dimensional $N(\kappa)$-contact metric manifold M satisfies*

$$R(\xi, X) \cdot Z = 0$$

if and only if M is locally isometric to the sphere $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$.

Proof. The condition $R(\xi, U) \cdot Z = 0$ implies that

$$0 = [R(\xi, U), Z(X, Y)]\xi - Z(R(\xi, U)X, Y)\xi - Z(X, R(\xi, U)Y)\xi,$$

which in view of $R(\xi, X) = \kappa R_0(\xi, X)$ gives

$$
\begin{aligned}
0 = \kappa \{ & g(U, Z(X,Y)\xi)\xi - \eta(Z(X,Y)\xi)U \\
& - g(U,X)Z(\xi,Y)\xi + \eta(X)Z(U,Y)\xi - g(U,Y)Z(X,\xi)\xi \\
& + \eta(Y)Z(X,U)\xi - \eta(U)Z(X,Y)\xi + Z(X,Y)U \}.
\end{aligned}
$$

In view of (4.1) the previous equation yields

$$
\kappa \left[Z - \left(\kappa - \frac{r}{2n(2n+1)} \right) R_0 \right] = 0.
$$

Therefore, either we have $\kappa = 0$ or M is of constant curvature and the result follows.

Conversely, $Z = 0$ in the first case and $R(\xi, X) = 0$ in the second case. $\square$

5. Some applications

A Riemannian manifold is said to be *concircularly symmetric* if the concircular curvature tensor Z is parallel, that is

$$(5.1) \qquad\qquad\qquad \nabla Z = 0.$$

EXAMPLE 5.1. $E^{n+1} \times S^n(4)$ *is concircularly symmetric.*

Now, we prove the following theorem.

THEOREM 5.2. *Let M^{2n+1} be a concircularly symmetric $N(\kappa)$-contact metric manifold. Then M is locally isometric to either $E^{n+1} \times S^n(4)$ or the sphere $S^{2n+1}(1)$.*

Proof. Suppose the $N(\kappa)$-contact metric manifold M^{2n+1} is concircularly symmetric. Then, from (5.1) it follows that $R(\xi, X) \cdot Z = 0$. Consequently, the result follows at once from Theorem 4.3. $\square$

REMARK 5.3. We note that while Z is a concircular invariant, the connection ∇ is not and hence $\nabla Z = 0$ is not a concircular invariant. It would also be natural to study spaces which are concircularly equivalent to a locally symmetric space.

The concircular curvature tensor Z in a Riemannian manifold M is said to be *recurrent* if

$$(5.2) \qquad\qquad\qquad \nabla Z = \alpha \otimes Z,$$

where α is an everywhere non-vanishing 1-form. Then, we prove the following theorem.

THEOREM 5.4. *An $N(\kappa)$-contact metric manifold with non-vanishing recurrent concircular curvature tensor does not exist.*

Proof. If possible, let M^{2n+1} be an $N(\kappa)$-contact metric manifold with non-vanishing recurrent concircular curvature tensor. Then from (5.2), we get

$$\nabla_X \nabla_Y Z = (X\alpha(Y) + \alpha(X)\alpha(Y)) Z,$$

which implies that

$$(5.3) \qquad\qquad R(X,Y) \cdot Z = 2d\alpha(X,Y) Z.$$

We define a function f on M^{2n+1} by

$$(5.4) \qquad\qquad f = (g(Z,Z))^{1/2},$$

where g is the usual extension to the inner product between the tensor fields ([8], pp.156–157). Since, $\nabla_X g = 0$, from (5.2) and (5.4) it follows that

$$f(Xf) = f^2 \alpha(X),$$

or,

$$(5.5) \qquad\qquad Xf = f\alpha(X).$$

Using (5.5), a straightforward calculation yields

$$(5.6) \qquad 2d\alpha(X,Y) f = (X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])) f = 0.$$

Since f is non-vanishing by assumption, the 1-form α is closed. Thus, from (5.3) and (5.6) we get $R(X,Y) \cdot Z = 0$, which in view of Theorem 4.3 and the assumption of non-vanishing Z, shows that M^{2n+1} is locally isometric to $E^{n+1} \times S^n(4)$. But $E^{n+1} \times S^n(4)$ satisfies $\nabla Z = 0$, hence our assumption is not possible. □

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D. E. Blair
Department of Mathematics
Michigan State University
East Lansing, MI 48824-1027, U.S.A.
E-mail: blair@math.msu.edu

Jeong-Sik Kim
Department of Mathematic and Mathematical Information
Yosu National University
Yosu 550-749, Korea
E-mail: sunchon@yosu.ac.kr

Mukut Mani Tripathi
Department of Mathematics and Astronomy
Lucknow University
Lucknow-226 007, India
E-mail: mmtripathi66@yahoo.com