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On Teaching Materials by Using the Rotations about the Origin and the Reflections in Lines through the Origin[1]

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When notions of numbers are expanded from natural number to complex number, a similar mathematical phenomenon can be observed in each number. As a case study, to complex number, the phenomenon is investigated carefully and teaching materials are created. Then complex number is expressed with matrices and is geometrically treated, so a new number which is an extension of complex number is discovered. Thus, teaching material regarding to complex number and matrices is made for students of ordinary level. Moreover, for talented students, material about an extension of complex number can be added to the previous one.

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1. INTRODUCTION

When the notion of numbers is extended from natural numbers, integers, rational numbers, and real numbers to the complex numbers, there are analogous mathematical

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phenomena as follows. For example, let $\mathbb{R}^+$ be the set of positive real numbers and $\mathbb{R}^-$ the set of negative real numbers, then the following relation holds :

$$\mathbb{R}^+\mathbb{R}^+ \subset \mathbb{R}^+, \quad \mathbb{R}^+\mathbb{R}^- \subset \mathbb{R}^-, \quad \mathbb{R}^-\mathbb{R}^+ \subset \mathbb{R}^-, \quad \mathbb{R}^-\mathbb{R}^- \subset \mathbb{R}^+.$$

Similar relation holds for the set $\mathbb{R}^*$ of non-zero real numbers and the set $\mathbb{R}^*i$ of pure imaginary numbers. These facts with regard to the product resemble to the fact with regard to the plus operation in the field of 2 elements: $0+0=0, 0+1=1, 1+1=0$.

Now we choose the complex number filed C as one number-system. We want to choose another number-system D such that C and D altogether satisfy similar structure of the field of 2 elements.In order to find out such D, we start from thinking the regular representation of C. Complex number $a+bi$ is represented by the matrix $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. We denote this by $C(a,b)$ and also matrix $\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$ by $S(a,b)$.

As the purpose of this paper is to make creative teaching materials and to practice it for general or capable students, we use logical descriptions in order to express its contents hereafter.

2. COMPLEX NUMBERS, MATRICES WHICH REPRESENT THE SCALAR-MULTIPLE OF REFLECTIONS IN THE LINES THROUGH THE ORIGIN

2.1. Matrices which represent the scalar-multiples of reflections in the lines through the origin.

Let the point (a,b), or vector (a,b) in the orthogonal coordinate plane be corresponding with the complex number $a+bi$ $(i^2=-1)$. Then, for the sum $z+w$ of complex numbers $z=a+bi$ and $w=c+di$, the vector sum $(a,b)+(c,d)$ corresponds to $z+w$. For complex numbers z, w, they satisfy the properties:

$$|zw| = |z||w|, \quad \arg zw = \arg z + \arg w,$$

where $|z|$ is a length of z and $\arg z$ is the augment of z. From these properties, we see that to multiply w on the left by z amount to the $|z|$-multiple of rotation by $\arg z$ radian about the origin for the vector w. By using these properties, we can draw the vector zw.

Since the distance is preserved by rotations for example, we deal with the distance-preserving linear transformations on the orthogonal coordinate plane.

If A is a matrix corresponding to such linear transformation, this condition implies that $A^tA = E$ (E is the identity matrix). So we can exhibit A as

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \text{ or } \begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The former is the matrix which corresponds to the rotation by θ about the origin. The latter is the one which exhibit the reflection in the line $y = \tan(\frac{\theta}{2})x$. In fact, in the orthogonal coordinate plane, reflection in the line $y = mx$ corresponds to the matrix

$$\begin{pmatrix}\frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2}\\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2}\end{pmatrix}$$

If we put $m = \tan(\frac{\theta}{2})$, then $\frac{1-m^2}{1+m^2} = \cos\theta$, $\frac{2m}{1+m^2} = \sin\theta$. Thus the quoted fact follows.

By considering scalar-multiples for these matrices, we get the following matrices:

$$\begin{pmatrix}a & -b\\ b & a\end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}a & b\\ b & -a\end{pmatrix} = \begin{pmatrix}a & -b\\ b & a\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

where $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$.

The former matrix is $C(a, b)$ that represents $a + bi$ and the latter one is exhibited by $S(a, b)$ that corresponds $\sqrt{a^2+b^2}$ multiple of reflection in the line $y = (\tan\frac{\theta}{2})x$ through origin.

We denote M_0, M_1 as follows:

$$M_0 := \left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix};\ a, b \in \mathbb{R}\right\},$$

$$M_1 := \left\{\begin{pmatrix}a & b\\ b & -a\end{pmatrix};\ a, b \in \mathbb{R}\right\}.$$

Then M_0, M_1 are vector spaces and we see that

$$M_0M_0 \subset M_0,\ \ M_0M_1 \subset M_1,\ \ M_1M_0 \subset M_1,\ \ M_1M_1 \subset M_0.$$

These relations are similar to the one which was established between positive real numbers and negative real numbers.

By multiplying the imaginary unit i to the complex number $a + bi$, point (a, b) is rotated $\frac{\pi}{2}$ radian in positive sense about the origin. This rotation can be obtained by applying the reflection in the line $y = x$ and continuing the reflection in the line $x = 0$. Also, this can be achieved by applying the reflection in the line $y = 0$ and continuing reflection in the line

$y = x$ too. These facts are exhibited by the following matrix forms:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We may say that the imaginary unit i is represented by the composition of two reflections. From the above formulas, we see that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are very important matrices. So, from now on, we denote these matrices by J, H, K respectively.

Let M_0^* and M_1^* be the sets of regular matrices in the vector spaces M_0 and M_1 respectively. The union G of M_0^* and M_1^* becomes a non-commutative group with regard to the matrix product. Then M_0^* is a subgroup of G and M_1^* is a right coset in G. Clearly M_0^* is closed under product AB of two elements A, B in M_0^*. M_1^* is not closed under the matrix product AB of two elements, but is closed under the triple product ABC or $AB^{-1}C$ (A, B, C in M_1^*).

2.2. Properties of matrices which represent the scalar-multiples of reflections in lines through the origin.

The matrix $S(a, b)$ can be exhibited by $S(a, b) = aH + bK$, where $H = S(1, 0)$, $K = S(0, 1)$.

For H, J, K, we have the following table of multiplication:

	J	H	K
J	$-E$	K	$-H$
H	$-K$	E	$-J$
K	H	J	E

For $S(a, b)$, $S(a, b)^2 = (a^2 + b^2)E$, so that inverse matrix $S(a, b)^{-1} = \frac{1}{a^2+b^2}S(a, b)$ in case $(a, b) \neq (0, 0)$. Also we have $S(a, b) = HS(b, a)K$.

If we denote $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ by $A(\theta)$, then

$$A(\alpha)A(\beta)A(\gamma) = A(\alpha)A(\beta)^{-1}A(\gamma) = A(\alpha - \beta + \gamma) \qquad (*)$$

Therefore mapping $\theta \to A(\theta)$ is a homomorphism with respect to the operations for 3-terms between angles and matrices. We can state the formula $(*)$ by using a geometrical expression, as follows:

To operate the reflection in line $y = (\tan(\frac{\alpha-\beta+\gamma}{2}))x$ is the same as to do three reflections in lines $y = (\tan(\frac{\alpha}{2}))x$, $y = (\tan(\frac{\beta}{2}))x$, $y = (\tan(\frac{\gamma}{2}))x$ (or reverse order) successively (Ryan 1986, p. 96; Yaglom 1962, p. 53, Prop. 4).

M_1 is not closed under the matrix product of 2 elements in M_1. but M_1 is closed under the triple product ABC of A, B, C in M_1, and is commutative as $ABC = CBA$. Therefore we consider the new product $\cdot$ for A, B in M_1 by $A \cdot B := AHB$. Then $A(\alpha) \cdot A(\beta) = A(\alpha + \beta)$ is approved. This means that mapping $\theta \longrightarrow A(\theta)$ is a homomorphism with respect to the operation for 2-terms between sums $\alpha + \beta$ of angles and product $A \cdot B$.

2.3. Products of matrices which represent the scalar-multiple of reflections in lines through the origin.

We define the mapping $f : M_0 \to M_1$ by $f(A) = AH$. Then f is an isomorphism from M_0 onto M_1 as vector spaces. Clearly M_0 is algebra over $\mathbb{R}$, but M_1 is not. So, we define a new matrix product for the elements AH, BH in $M_1(A, B \in M_0)$ by $(AH) \cdot (BH) := (AH)H(BH)$. Then M_1 becomes an algebra over $\mathbb{R}$ with regard to the operations $+$ and $\cdot$. Easily we get the equation $f(A) \cdot f(B) = f(AB)$ so that f gives the algebra isomorphism between M_0 and M_1.

2.4. Characterization of matrices which exhibit the scalar-multiples of reflections in lines through origin.

We seek the condition that 2-by-2 real matrix A has the form $S(a, b)$.

Proposition. *The statement that 2-by-2 real matrix* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *can be expressed by the form* $S(a, b)$ *is equivalent to the following each one:*

(1) $A^t A = -(\det A)E$,

(2) $AJ + JA = O$ *or* $AHK = KHA$,

(3) A *satisfies*

$$(adA)^3 X = 2(a^2 + b^2 + c^2 + d^2)(adA)X \tag{$*$}$$

for arbitrary 2-by-2 real matrix X. *Where* $(adA)X = [A, X] = AX - XA$

Equivalence of (1) and (2) is proved easily. The following Lemma is effective for the proof of (3).

Lemma. *For arbitrary 2-by-2 real matrices* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, X, *it holds the following equation:*

$$(adA)^3 X = \left\{(a - d)^2 + 4bc\right\}(adA)X \tag{$**$}$$

For the proof of lemma, the following expression for 2-by-2 real matrix is effectively used:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha E + \beta H + \gamma K + \delta J,$$

where $\alpha = \frac{(a+d)}{2}$, $\beta = \frac{(a-d)}{2}$, $\gamma = \frac{(b+c)}{2}$, $\delta = \frac{-(b-c)}{2}$.

Remark. A formula of the form (3) is used for I-matirx or N-matrix by Hua (1951, Section 14).

2.5. Complex number plane and number plane that corresponds to the set of scalar-multiple of reflections in lines through origin.

Points, or vectors on the complex number plane π_0 are expressed by low vectors (a, b) or column vectors $\binom{a}{b}$. Then the following equations established for 2 points (a, b) and (c, d):

$$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a+c\\b+d\end{pmatrix},\quad \begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a&-b\\b&a\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}=a\begin{pmatrix}c\\d\end{pmatrix}+b\begin{pmatrix}-d\\c\end{pmatrix}.$$

Similarly, points, or vectors on the number plane π_1 that correspond to the set of scalar-multiple of reflections in lines through origin are expressed by low vectors $\{a, b\}$ or column vectors $\left\{\begin{smallmatrix}a\\b\end{smallmatrix}\right\}$. Then the following equations are established for 2 points $\{a, b\}$ and $\{c, d\}$:

$$\begin{Bmatrix}a\\b\end{Bmatrix}+\begin{Bmatrix}c\\d\end{Bmatrix}=\begin{Bmatrix}a+c\\b+d\end{Bmatrix}$$
$$\begin{Bmatrix}a\\b\end{Bmatrix}\begin{Bmatrix}c\\d\end{Bmatrix}=\begin{pmatrix}a&b\\b&-a\end{pmatrix}\begin{Bmatrix}c\\d\end{Bmatrix}=a\begin{pmatrix}c\\-d\end{pmatrix}+b\begin{pmatrix}d\\c\end{pmatrix}.$$

Points on π_0 and π_1 have the following product-relations:

$$\begin{pmatrix}a\\b\end{pmatrix}\begin{Bmatrix}c\\d\end{Bmatrix}=\begin{Bmatrix}ac-bd\\ad+bc\end{Bmatrix},\quad \begin{Bmatrix}a\\b\end{Bmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{Bmatrix}ac+bd\\-ad+bc\end{Bmatrix}$$

Let Q be a point which is given by the product of 2 points, (a, b) and (c, d) in the complex plane π_0 and let T be a point which is given by the product of 2 points $\{a, b\}$, $\{c, d\}$ in the number plane π_1. Then

$$Q := (a, b)(c, d) = (ac - bd, ad + bc) \text{ and } T := \{a, b\}\{c, d\} = (ac + bd, -ad + bc)$$

both belongs to the plane π_0, but $Q \neq T$ in general. Also, the angle of segments OQ and OT depends only to the point T.

We define the mapping f from π_0 to π_1 as follows:

$$f\left(\begin{pmatrix}a\\b\end{pmatrix}\right):=\begin{pmatrix}a\\b\end{pmatrix}\begin{Bmatrix}1\\0\end{Bmatrix}=\begin{Bmatrix}a\\b\end{Bmatrix}.$$

Then f is an isomorphism from π_0 onto π_1 with regard to triple sum and triple product. In fact,

$$f\left(\begin{pmatrix}a\\b\end{pmatrix}-\begin{pmatrix}p\\q\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}\right)=f\left(\begin{pmatrix}a\\b\end{pmatrix}\right)-f\left(\begin{pmatrix}p\\q\end{pmatrix}\right)+f\left(\begin{pmatrix}c\\d\end{pmatrix}\right),$$

$$f\left(\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}p\\q\end{pmatrix}^{-1}\begin{pmatrix}c\\d\end{pmatrix}\right)=f\left(\begin{pmatrix}a\\b\end{pmatrix}\right)f\left(\begin{pmatrix}p\\q\end{pmatrix}\right)^{-1}f\left(\begin{pmatrix}c\\d\end{pmatrix}\right),\ (p,q)\neq(0,0).$$

For $\left\{\begin{smallmatrix}a\\b\end{smallmatrix}\right\}$, $\left\{\begin{smallmatrix}c\\d\end{smallmatrix}\right\}$ in π_1, we define the product· by $\left\{\begin{smallmatrix}a\\b\end{smallmatrix}\right\}\cdot\left\{\begin{smallmatrix}c\\d\end{smallmatrix}\right\}:=\left\{\begin{smallmatrix}a\\b\end{smallmatrix}\right\}\left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}\left\{\begin{smallmatrix}c\\d\end{smallmatrix}\right\}$, and for $\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right)$ in π_0 , we set the product · by $\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right):=\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right)$ Then

$$\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}ac-bd\\ad+bc\end{pmatrix}\quad\begin{Bmatrix}a\\b\end{Bmatrix}\cdot\begin{Bmatrix}c\\d\end{Bmatrix}=\begin{Bmatrix}ac-bd\\ad+bc\end{Bmatrix}.$$

Thus the mapping $(a,b)\rightarrow\{a,b\}$ gives an isomorphism from π_0 onto π_1 with respect to the sum and the product "·".

Remark. we arrange the vectors (a,b), $\{c,d\}$ in this order, and think the 4-dimensional vector $a+bi+ch+dk$ with basis 1, i, h, k. 4-dimennsional algebra with the following multiplication table is called by pseudoquaternion (Rozenfel'd & Yaglom 1951, p. 205; Yaglom 1968):

	i	h	k
i	-1	k	$-h$
h	$-k$	1	$-i$
k	h	i	1

Matrix $S(a,b)$ is the representation of $ah+bk$ in the pseudoquaternion.

3. ON THE TEACHING GUIDELINE

We will consider a plan of lessons by using our teaching materials in this section.

3.1. Preliminaries of lessons.

About the matrices of M_1, we can think of them from two standpoints. One is that they are reflections in line through the origin (geometrical standpoint) and another is that they have a close connection to the complex numbers(algebraical standpoint). So, the following 5 objects are needed to the coming learning beforehand. Though, there exist something that overlaps with the study plan.

(1) definition of vectors and these operations, vector spaces, definition of matrices and these operations, definition of determinant.

(2) Coordinate plane, the correspondence between linear transformations and matrices, rotations, reflections,

(3) Definition of complex numbers and its properties, complex number plane, geometrical expression of complex numbers

(4) Trigonometrical function, addition theorem

(5) The others (group, subgroup, residue class, ring, homomorphism, isomorphism)

3.2. Plan of lessons.

(1) Definition of complex numbers: $a + bi$, pair of real numbers (a, b)

(2) Geometrical expression of orthogonal coordinate plane (complex number plane)

(3) Represented matrix $C(a, b)$ of complex number $a + bi$

(4) Relations between the left product of complex number and its represented matrix:

$$\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

(5) Distance preserving linear transformation in orthogonal coordinate plane, orthogonal matrix, classification of 2-by-2 real orthogonal matrices:

(6) Let M_0 be the set of $C(a, b)$, a, $b \in \mathbb{R}$, and let M_1 be the set of $S(a, b)$, a, $b \in \mathbb{R}$. Then the product of vector spaces M_0, M_1 satisfies following properties:

$$M_0 M_0 \subset M_0, \quad M_0 M_1 \subset M_1, \quad M_1 M_0 \subset M_1, \quad M_1 M_1 \subset M_0$$

Remark that the relation above is the same as the relation of product between positive numbers and negative numbers, or non-zero real numbers and pure imaginary numbers.

(7) Geometrical meanings of linear transformations afforded by the elements of M_0 and M_1

(8) M_1 is not closed under the product of two elements in M_1, but is closed under the product of three elements.

(9) Geometrical meanings of triple product ABC, where A, B, C are the matrices of the standard forms in M_1: Product of three reflections in lines through the origin is a reflection in a line through the origin.

(10) Some calculations of the elements in M_1: inverse matrix (method of formula or sweeping out method), eigenvalue, eigenvector, diagonalization.

(11) Extension from the elements of M_0 and M_1 to 4-by-4 matrix (pseudoquaternion)

(12) The explanation of Hamilton's quaternion and its matrix representation.

4. PROBLEMS IN THE TEACHING MATERIALS

(1) We consider the reflection in the line $y = (\tan\frac{\theta}{2})x$. When $\theta = 0°, 60°, 90°, 270°$, where do the point (x, y) moved by such reflections?

(2) Verify that to operate the reflection in line $y = (\tan 60°)x$ agree with to operate the reflections in lines $y = x$, $y = (\tan 30°)x$ and $y = x$ successively.

(3) Let A be a matrix $S(1, 2)$. Find the matrices X, Y that satisfy the following properties $H = AKX$, $H = YKA$.

(4) Let A be a matrix $C(a, b)$. Show that the following formula holds for arbitrary 2-by-2 real matrix X.

$$A^3X - XA^3 - 3A^2XA + 3AXA^2 = 4(AX - XA)$$

(5) Let $A = C(a, b), B = S(p, q), C = C(c, d), D = S(r, s)$.

(a) Show that $\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} C & D \\ D & C \end{pmatrix} = \begin{pmatrix} L & M \\ M & L \end{pmatrix}$, where

$$L = C(ac - bd + pr + qs, ad + bc - ps + qr),$$

$$M = S(ar - bs + pc + qd, as + br - pd + qc).$$

(b) Show that $\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = (\det A + \det B)^2$

(6) Let A be a 2-by-2 real matrix that satisfies the condition $A^2 = E$. Show that

$$A = \begin{pmatrix} a & \frac{1-a^2}{c} \\ c & -a \end{pmatrix}, \ c \neq 0, \quad \pm\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}, \quad \pm E.$$

(7) Let A be a 2-by-2 real matrix that satisfies the condition $A^2 = A$. Show that

$$A = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix}, \ c \neq 0, \quad \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \quad E.$$

(8) Find 2-by-2 real matrices that commute for each matrix H, K, $\begin{pmatrix} 1 & k \\ 0 & -1 \end{pmatrix}$.

(9) Show that 2-by-2 real matrix X satisfies the condition $XHX = -H$ if and only if X has a form $X = \begin{pmatrix} a & \frac{1+a^2}{c} \\ c & a \end{pmatrix}$, $c \neq 0$.

(10) Set $K(p, q) := \begin{pmatrix} p & \frac{1+p^2}{c} \\ q & p \end{pmatrix}$, $q \neq 0$, and put $S := \{K(p, q) : p, q \in \mathbb{R}\}$. Then

$$K(0, 1) = K.$$

(a) Show that $\det K(p, q) = -1$.

(b) Show that 2-by-2 real matrix X satisfies the condition $X = aH + bK(p, q)$, $q \neq 0$ if and only if X has a form $X = aH + bK(p, q)$.

(c) Show that S is closed under the following product:

$$K(p, q)K(r, s)^{-1}K(p, q), \quad p,\ q,\ r,\ s \in \mathbb{R}.$$

(11) $\sinh t = \frac{e^t - e^{-t}}{2}$, $\cosh t = \frac{e^t + e^{-t}}{2}$ are called by the hyperbolic sine and the hyperbolic cosine respectively. They satisfy the addition theorem:

$$\sinh(s+t) = \sinh s \cosh t + \cosh s \sinh t,$$
$$\cosh(s+t) = \cosh s \cosh t + \sinh s \sinh t.$$

Let

$$A = \begin{pmatrix} \sinh s & \cosh s \\ \cosh s & \sinh s \end{pmatrix},\ B = \begin{pmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{pmatrix},\ C = \begin{pmatrix} \sinh u & \cosh u \\ \cosh u & \sinh u \end{pmatrix}.$$

(a) Calculate AB and verify that

$$AB^{-1}C = \begin{pmatrix} \sinh(s-t+u) & \cosh(s-t+u) \\ \cosh(s-t+u) & \sinh(s-t+u) \end{pmatrix}.$$

(b) Let

$$M_0 = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},\ -\infty < t < \infty \right\},$$
$$M_1 = \left\{ \begin{pmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{pmatrix},\ -\infty < t < \infty \right\}.$$

Show the following properties:

$$M_0 M_0 \subset M_0, M_0 M_1 \subset M_1, M_1 M_0 \subset M_1, M_1 M_1 \subset M_0$$

(12) Find the inverse matrix for $\begin{pmatrix} p & \frac{1+p^2}{q} \\ q & p \end{pmatrix}$, $q \neq 0$, using the sweeping out method.

(13) Diagonalize the following 2-by-2 real matrices: $C(a,b)$, $S(a,b)$, $\begin{pmatrix} p & \frac{1+p^2}{q} \\ q & p \end{pmatrix}$, $q \neq 0$.

5. CLOSING REMARKS

In mathematics, it often appears that the research objects have similar properties. We paid attention to the properties of M_1 contrasting with the set of complex numbers. We can consider M_1 from algebraic aspect and from geometric aspect. By using such M_1, we have made teaching materials to deepen the understanding of the linear algebra. In the future, we will execute lessons according to the teaching plan in section 3, and will examine it to make appropriate teaching materials and better one. Furthermore, we want to refine it so that students will become to study of their own volition.

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