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ON THE ALMOST SURE CONVERGENCE OF WEIGHTED SUMS OF NEGATIVELY ASSOCIATED RANDOM VARIABLES

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Abstract. Let $\{X, X_n | n \geq 1\}$ be a sequence of identically negatively associated random variables under some conditions. We discuss strong laws of weighted sums for arrays of negatively associated random variables.

1. Introduction

Let $\{X,\ X_n \mid n \geq 1\}$ be a sequence of independent identically distribution (i.i.d.) random variables and let $\{a_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ be an array of constants. Up to now, many authors studied the almost sure (a.s.) limiting behavior of weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ of i.i.d. random variables. Bai and Cheng[3] proved the strong law of large numbers $\sum_{i=1}^{n} a_{ni}X_{ni}/b_n \longrightarrow 0$ *a.s.*, by Bernstein inequality when $\{X, X_n \mid n \geq 1\}$ is a sequence of i.i.d. random variables with $EX = 0$ and

$$E(e^{h|X|^{\gamma}}) < \infty \text{ for some } h > 0\ (\gamma > 0), \tag{1.1}$$

and $\{a_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants satisfying

$$A_{\alpha} = \lim_{n \to \infty} \sup A_{\alpha,n} < \infty,\ A_{\alpha,n}^{\alpha} = \sum_{i=1}^{n} |a_{ni}|^{\alpha}/n \tag{1.2}$$

for some $1 < \alpha < 2$, where $b_n = n^{1/\alpha}(\log n)^{1/\gamma + \gamma(\alpha-1)/\alpha(1+\gamma)}$ and $0 < \gamma \leq 1$.

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Sung[18] extended this result and obtained another a.s.limiting law when condition (1.2) is replaced by stronger condition

$$E(e^{h|X|^{\gamma}}) < \infty \text{ for any } h > 0 \ (\gamma > 0), \tag{1.3}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ if $0 < \gamma \leq 1$, $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\gamma(\alpha-1)/\alpha(1+\gamma)}$ if $\gamma > 1$.

In this paper, we obtain an almost sure limiting behavior on the weighted sums by using negatively associated random variables on the moment generating function $E(e^{h|X|^{\gamma}}) < \infty$ for any $h > 0$ $(\gamma > 0)$. Note that the results of Sung[18] extend from i.i.d. random variables to negatively associated setting. The definition of negatively associated random variables is as follows.

DEFINITION. A finite family of random variables $\{X_i \mid 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$,

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise increasing such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The notion of NA was introduced by Alam and Saxena[1] and were carefully studied by Joag-Dev and Proschan[10] and Block, Savits and Shaked[6]. Joag-Dev and Proschan[10] showed that a number of well known multivariate distributions possess the NA property, such as (a) multinomial, (b) convolution of unlike multinomials, (c) multivariate hypergeometric, (d) Dirichlet, (e) Dirichlet compound multinomial, (f) negatively correlated normal distribution, (g) permutation distribution, (h) random sampling without replacement, and (i) joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and reliability, the notion of NA has received considerable attention recently. We refer the reader to Joag-Dev and Proschan[10] for fundamental properties, Newman[14] for the central limit theorem, Matula[13] for the three series theorem, Su et al.[17] for a moment inequality, a weak invariance principle and an example to show that there exists an infinite family of non-degenerate non-independent strictly stationary NA random variables, Shao and Su[11] for the law of the iterated logarithm, Shao[16] for the Rosenthal type maximal inequality and the Kolmogorov exponential inequality, Liang and Su[11], Liang[12] and Baek, Kim and Liang[2] for complete convergence, and Roussas[15] for the central limit theorem of random fields, some examples and applications.

The organization of this paper is as follows. In Section 2, we give main results and some corollaries. Throughout this paper, $a_{ni} = a_{ni}^+ - a_{ni}^-$, where $a_{ni}^+ = \max(a_{ni}, 0)$, $a_{ni}^- = \max(-a_{ni}, 0)$, C denotes a positive constant whose values are unimportant.

2. Main results

We will deal with an almost sure convergence theorem for weighted sums of identically(but not necessary independent)distributed dependent case.

THEOREM 1. *Let $\{X_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$ for $1 \leq i \leq n$ and $n \geq 1$. Assume that $\{a_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ is an array of constants. Suppose that the following conditions hold;*

(a) *$|a_{ni}X_{ni}I(X_{ni} \geq 0)| \leq u_n |X_i|^\beta / \log n$ a.s., for some $0 < \beta \leq \gamma$ and some sequence $\{u_n\}$ of constants such that $u_n \to 0$,*
(b) *$X_{ni}^2 \sum_{i=1}^n a_{ni}^2 \leq v_n |X_i|^\delta / \log n$ a.s., for some $\delta > 0$ and some sequence $\{v_n\}$ of constants such that $v_n \to 0$.*

Let $\{X, X_n | n \geq 1\}$ be a sequence of identically distributed random variables satisfying (1.1), *then*

$$\sum_{n=1}^{\infty} n^r P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > \varepsilon) < \infty \text{ for any } \varepsilon > 0 \text{ and } r \geq 0.$$

PROOF. It suffices to show that

$$\sum_{n=1}^{\infty} n^r P(|\sum_{i=1}^{n} a_{ni}^+ X_{ni}| > \varepsilon) < \infty \text{ for any } \varepsilon > 0, \tag{2.1}$$

$$\sum_{n=1}^{\infty} n^r P(|\sum_{i=1}^{n} a_{ni}^- X_{ni}| > \varepsilon) < \infty \text{ for any } \varepsilon > 0. \tag{2.2}$$

We prove only (2.1), the proof of (2.2) is analogous. To prove (2.1), we need only to prove that

$$\sum_{n=1}^{\infty} n^r P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} > \varepsilon) < \infty \text{ for any } \varepsilon > 0, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^r P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} < -\varepsilon) < \infty \text{ for any } \varepsilon > 0. \tag{2.4}$$

We first prove (2.3). From the definition of NA variables, we know that $\{a_{ni}^{+}X_{ni} \mid 1 \le i \le n,\, n \ge 1\}$ is still an arrays of rowwise NA random variables. For all $x \in R$, by putting $t = M \log n/\varepsilon$, where M is a large constant and $|x|^{\delta} \le O(e^{(h/2)|x|^{\beta}})$ and inequality $e^{x} \le 1 + x + \frac{1}{2}x^{2}e^{|x|}$, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r} P\Big(\sum_{i=1}^{n} a_{ni}^{+}X_{ni} > \varepsilon\Big)\\
&\le \sum_{n=1}^{\infty} n^{r} e^{-\varepsilon t} E e^{t\sum_{i=1}^{n} a_{ni}^{+}X_{ni}}\\
&\le \sum_{n=1}^{\infty} n^{r-M} \prod_{i=1}^{n} E e^{t a_{ni}^{+}X_{ni}}\\
&\le \sum_{n=1}^{\infty} n^{r-M} \prod_{i=1}^{n} \Big(1 + (M^{2}/2\varepsilon^{2}) \log^{2} n (a_{ni}^{+})^{2}\\
&\quad \cdot (EX_{ni}^{2} + EX_{ni}^{2} I(X_{ni} \ge 0) e^{(M/\varepsilon)\log n |a_{ni}^{+}X_{ni}|})\Big)\\
&\le \sum_{n=1}^{\infty} n^{r-M} \prod_{i=1}^{n} \Big(1 + (M^{2}/2\varepsilon^{2}) v_{n} \log n (a_{ni}^{2}/\Sigma a_{ni}^{2})\\
&\quad \cdot (E(|X_{i}|^{\delta}) + E|X_{i}|^{\delta} e^{(M/\varepsilon)u_{n}|X_{ni}|^{\beta}})\Big)\\
&\le \sum_{n=1}^{\infty} n^{r-M} \prod_{i=1}^{n} \Big(1 + (M^{2}/2\varepsilon^{2}) v_{n} \log n (a_{ni}^{2}/\Sigma a_{ni}^{2})\\
&\quad \cdot E(|X_{i}|^{\delta} e^{(h/2)|X_{i}|^{\beta}})\Big)\\
&\le \sum_{n=1}^{\infty} n^{r-M} \prod_{i=1}^{n} \Big(1 + O(1) v_{n} \log n (a_{ni}^{2}/\Sigma a_{ni}^{2}) E(e^{h|X_{i}|^{\beta}})\Big)\\
&\le \sum_{n=1}^{\infty} n^{r-M} \prod_{i=1}^{n} \Big(1 + C \log n (a_{ni}^{2}/\Sigma a_{ni}^{2})\Big)\\
&\le \sum_{n=1}^{\infty} n^{r-M+C}\\
&< \infty,
\end{aligned}
$$

proved $M > r + C + 1$. Thus, (2.3) is proved. By replacing X_i by $-X_{ni}$ from the above statement and noticing $\{a_{ni}^+(-X_{ni}) \mid 1 \le i \le n,\ n \ge 1\}$ is still an arrays of rowwise NA random variables, we know that $\sum_{n=1}^{\infty} n^r P(\sum_{i=1}^{n} a_{ni}^+ X_{ni} < -\varepsilon) < \infty$ for any $\varepsilon > 0$. Hence the result follows by (2.3) and (2.4). □

The following corollary shows that if condition(1.1) of Theorem 1 is replaced by stronger condition(1.3), then condition(a) can be replaced by weaker condition (a′) $|a_{ni}X_{ni}I(X_{ni} \ge 0)| \le C|X_i|^{\beta}/\log n$ *a.s.*, for some $0 < \beta \le \gamma$ and some constant $C > 0$.

COROLLARY 1. *Let* $\{X, X_n \mid n \ge 1\}$, $\{X_{ni} \mid 1 \le i \le n,\ n \ge 1\}$, and $\{a_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ *be same as in Theorem 1 except that* (1.1) *and* (*a*) *are replaced by* (1.3) *and* (a′), *respectively. Then*

$$\sum_{n=1}^{\infty} n^r P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > \varepsilon) < \infty \text{ for any } \varepsilon > 0 \text{ and } r \ge 0.$$

PROOF. The proof is similar to that of Theorem 1 and is omitted. □

THEOREM 2. *Let* $\{X, X_n \mid n \ge 1\}$ *be a sequence of identically distributed NA random variables satisfying* $EX_n = 0$ *and* (1.3), *and let* $\{a_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ *be an array of constants satisfying* (1.2) *for* $1 < \alpha \le 2$. *Then for some* $0 < \gamma \le 1$ *and* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$,

$$\sum_{i=1}^{n} a_{ni}X_i/b_n \longrightarrow 0 \text{ a.s. as } n \longrightarrow \infty.$$

PROOF. Since $\sum_{i=1}^{n} a_{ni}X_i/b_n = \sum_{i=1}^{n} a_{ni}^+X_i/b_n - \sum_{i=1}^{n} a_{ni}^-X_i/b_n$, we may assume, without loss of generality, that $a_{ni} > 0$. Let $X'_{ni} = (\log n)^{1/\gamma}I(X_i > (\log n)^{1/\gamma}) + X_iI(|X_i| \le (\log n)^{1/\gamma})$ and $X''_{ni} = X_i - X'_{ni}$ for $1 \le i \le n$ and $n \ge 1$. Note that X''_{ni} and X'_{ni} are still NA random variables by property of NA (see, Joag and Proschan[10]) and so,

$$\left|\sum_{i=1}^{n} a_{ni}X_i\right|/b_n \le \left|\sum_{i=1}^{n} a_{ni}X'_{ni}\right|/b_n + \left|\sum_{i=1}^{n} a_{ni}X''_{ni}\right|/b_n := I_1 + I_2.$$

Note that $E(e^{|X|^{\gamma}}) < \infty$ is equivalent to $\sum_{n=1}^{\infty} P(|X_n| > (\log n)^{1/\gamma}) < \infty$. Hence, by the Borel-Cantelli Lemma, $\sum |X''_{ni}|$ is bounded *a.s.*, it follows that

$$\begin{aligned}
I_2 &= |\sum a_{ni}X''_{ni}| \ / \ b_n \\
&\le \max_{1\le i\le n} |a_{ni}| \sum_{i=1}^{n} |X''_{ni}| \ / \ b_n \\
&= \left(\frac{1}{n}\max_{1\le i\le n} |a_{ni}|^{\alpha}\right)^{1/\alpha} \sum_{i=1}^{n} |X''_{ni}| \ / \ \log^{1/\gamma} n \\
&\le \left(\frac{1}{n}\sum_{i=1}^{n} |a_{ni}|^{\alpha}\right)^{1/\alpha} \sum_{i=1}^{n} |X''_{ni}| \ / \ \log^{1/\gamma} n \\
&= A_{\alpha,n} \sum_{i=1}^{n} |X''_{ni}| \ / \ \log^{1/\gamma} n \longrightarrow 0 \ a.s. \ as \ n \longrightarrow \infty.
\end{aligned}$$

To prove $|a_{ni}X_{ni}|/b_n \to 0$ a.s. as $n \to \infty$, we will apply Corollary 1 to random variable X_{ni} and $b_n^{-1}a_{ni}$.

Note that $EX'_{ni} + EX''_{ni} = EX_i = 0$ and $EX''_{ni} \ge 0$, so $EX'_{ni} \le 0$. In addition, we observe that

$$\begin{aligned}
&|a_{ni}X'_{ni}I(X'_{ni} \ge 0)|/b_n \\
\le& \left|\frac{1}{n^{1/\alpha}}a_{ni}X_iI(0 \le X_i \le (\log n)^{1/\gamma})\right| /(\log n)^{1/\gamma} \\
&+ \left|\frac{1}{n^{1/\alpha}}a_{ni}I(X_i > (\log n)^{1/\gamma})\right| \\
\le& \left(\frac{1}{n}\sum_{i=1}^{n}|a_{ni}|^{\alpha}\right)^{1/\alpha} \{|X_i|I(|X_i| \le (\log n)^{1/\gamma})/(\log n)^{1/\gamma} \\
&+ I(X_i > (\log n)^{1/\gamma})\} \\
\le& A_{\alpha,n}\{(\log n)^{(1-\gamma)/\gamma}|X_i|^{\gamma}/(\log n)^{1/\gamma} + |X_i|^{\gamma}/\log n\} \\
\le& A_{\alpha,n}|X_i|^{\gamma}/\log n
\end{aligned}$$

and

$$\begin{aligned}
{X'_{ni}}^2 \sum_{i=1}^{n} a_{ni}^2/b_n^2 &\le X_i^2 \sum_{i=1}^{n} a_{ni}^2/b_n^2 \\
&\le A_{\alpha,n}^2 X_i^2/(\log n)^{2/\gamma}.
\end{aligned}$$

Hence, conditions (a') and (b) of Corollary 1 are satisfied the results above, and so Corollary 1 implies that

$$\sum_{i=1}^{n} a_{ni}X'_{ni}/b_n \longrightarrow 0 \; a.s. \; as \; n \longrightarrow \infty.$$

Hence the result of Theorem 2 follows by I_1 and I_2. □

The following theorem is a slight modification of Theorem 2. The following theorem shows if constant b_n is stronger than that of Theorem 2, then condition (1.3) of Theorem 2 can be replaced by weaker condition (1.1).

THEOREM 3. *Let $\{X, X_n \mid n \geq 1\}$ be a sequence of identically distributed NA random variables satisfying $EX_n = 0$ and* (1.1), *and let $\{a_{ni} \mid 1 \leq i \leq n, \; n \geq 1\}$ be an array of constants satisfying* (1.2) *for $1 < \alpha \leq 2$. Then for $0 < \gamma \leq 1$ and $b_n = n^{1/\alpha}(\log n)^{1/\gamma+\beta}$ $(\beta > 0)$,*

$$\sum_{i=1}^{n} a_{ni}X_i/b_n \longrightarrow 0 \; a.s. \; as \; n \longrightarrow \infty.$$

PROOF. Let $X'_{ni} = X_i I(|X_i| \leq (h^{-1}\log n)^{1/\gamma}) + (h^{-1}\log n)^{1/\gamma} I(X_i > (h^{-1}\log n)^{1/\gamma})$ and $X''_{ni} = X_i - X'_{ni}$ for $1 \leq i \leq n$ and $n \geq 1$. Since the proof is similar to that of Theorem 2, it is omitted. □

REMARK 2.1. Suppose that $\{X, X_n | n \geq 1\}$ are i.i.d. random variables with $EX = 0$. Sung[18] discussed Theorems 1, 2 and 3. Obviously, Theorems 1, 2 and 3 extend the result of Sung[18] from i.i.d. case to NA setting.

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