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A CENTRAL LIMIT THEOREM FOR GENERAL WEIGHTED SUM OF LNQD RANDOM VARIABLES AND ITS APPLICATION

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Abstract. In this paper we derive the central limit theorem for $\sum_{i=1}^{n} a_{ni}\xi_i$, where $\{a_{ni}, 1 \leq i \leq n\}$ is a triangular array of nonnegative numbers such that $\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty$, $\max_{1\leq i\leq n} a_{ni} \to 0$ *as* $n \to \infty$ and $\xi_i^{'}s$ are a linearly negative quadrant dependent sequence. We also apply this result to consider a central limit theorem for a partial sum of a generalized linear process $X_n = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j$.

1. Introduction and results

Lehmann[6] introduced a simple and natural definition of positive (negative) dependence: A sequence $\{\xi_i,\ i \in Z\}$ of random variables is said to be pairwise positive (negative) quadrant dependent (pairwise PQD(NQD)) if for any real α_i, α_j and $i \neq j$, $P(\xi_i > \alpha_i, \xi_j > \alpha_j) \geq (\leq)$ $P(\xi_i > \alpha_i)P(\xi_j > \alpha_j)$. A concept stronger than PQD(NQD) was introduced by Newman[7]: A sequence $\{\xi_i,\ i \in Z\}$ of random variables is said to be linearly positive(negative) quadrant dependent(LPQD(LNQD)) if for every pair of disjoint subsets A, $B \subset Z$ and positive $r_j^{'}s$

$$\sum_{i\in A} r_i\xi_i \text{ and } \sum_{j\in B} r_j\xi_j \text{ are } PQD\ (NQD). \tag{1.1}$$

Newman[7] established the central limit theorem for a strictly stationary LPQD(or LNQD) process and Birkel[2] also obtained a functional central limit theorem for LPQD process which is used to obtain the functional central limit theorem for LNQD process. Kim and

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Baek[5] extended this result to a stationary linear process of the form $X_k = \sum_{j=0}^{\infty} a_j \xi_{k-j}$, where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\{\xi_k\}$ is a strictly stationary LPQD process with $E\xi_i = 0,\ 0 < E\xi_i^2 < \infty$, which can be extended to the LNQD case by similar method.

In this paper, we derive a central limit theorem for a linearly negative quadrant dependent sequence in a double array, weakening the strictly stationarity assumption with uniform integrability (see Theorem 1.1 below) and apply this result to obtain a central limit theorem for a partial sum of a linear process $X_n = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j$ generated by linearly negative quadrant dependent sequence $\{\xi_j\}$ (see Theorem 1.2 below).

THEOREM 1.1. *Let $\{a_{ni},\ 1 \le i \le n\}$ be a triangular array of nonnegative numbers such that*

$$\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty \tag{1.2}$$

and

$$\max_{1 \le i \le n} a_{ni} \to 0 \text{ as } n \to \infty\ . \tag{1.3}$$

Let $\{\xi_i\}$ be a centered sequence of linearly negative quadrant dependent random variables such that

$$\{\xi_i^2\} \text{ is an uniformly integrable family,} \tag{1.4}$$

$$\mathrm{Var}(\sum_{i=1}^{n} a_{ni}\xi_i) = 1, \tag{1.5}$$

and

$$\sum_{j:|i-j| \ge u} \mathrm{Cov}(\xi_i, \xi_j)^- \to 0 \text{ as } u \to \infty \text{ uniformly in } i \ge 1 \tag{1.6}$$

(see Cox and Grimmet[3]). *Then*

$$\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \text{ as } n \to \infty.$$

REMARK. Theorem 1.1 extends the Newman's[7] central limit theorem for strictly stationary LNQD sequence from equal weights to general weights, weakening at the same time the assumption of stationarity.

COROLLARY 1.1. *Let $\{\xi_i\}$ be a centered sequence of linearly negative quadrant dependent random variables such that $\{\xi_i^2\}$ is a uniformly integrable family and let $\{a_{ni},\ 1 \le i \le n\}$ be a triangular array of nonnegative numbers such that*

$$\sup_n \sum_{i=1}^{n} \frac{a_{ni}^2}{\sigma_n^2} < \infty, \tag{1.7}$$

$$\max_{1\le i\le n} \frac{a_{ni}}{\sigma_n} \to 0 \text{ as } n \to \infty, \tag{1.8}$$

where $\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^n a_{ni}\xi_i)$. If (1.6) holds then, as $n \to \infty$

$$\frac{1}{\sigma_n}\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{\mathcal{D}} \mathrm{N}(0,1). \tag{1.9}$$

THEOREM 1.2. *Let $\{a_j, j \in Z\}$ be a sequence of nonnegative numbers such that $\sum_j a_j < \infty$ and let $\{\xi_j, j \in Z\}$ be a sequence of linearly negative quadrant dependent random variables which is uniformly integrable in L_2 and satisfies (1.6). Let*

$$X_k = \sum_{j=-\infty}^{\infty} a_{k+j}\xi_j \text{ and } S_n = \sum_{i=1}^{n} X_i.$$

Assume

$$\inf_{n\ge 1} n^{-1}\sigma_n^2 > 0, \tag{1.10}$$

where $\sigma_n^2 = \mathrm{Var}(S_n)$. Then

$$\frac{S_n}{\sigma_n} \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \text{ as } n \to \infty. \tag{1.11}$$

This result is an extension of Theorem 18.6.5 in Ibragimov and Linnik[4] from i.i.d. to the linearly negative quadrant dependence case by adding the condition (1.6) and improves on Kim and Baek's[5] result about central limit theorem for alinear processes generated by LNQD sequences.

2. Proofs

We starts with the following lemma.

LEMMA 2.1. (Newman[8]) *Let* $\{Z_i, 1 \le i \le n\}$ *be a sequence of linearly negative quadrant dependent random variables with finite second moments. Then*

$$|E\exp(it\sum_{j=1}^{n} Z_j) - \prod_{j=1}^{n} E\exp(itZ_j)| \le Ct^2|\mathrm{Var}(\sum_{j=1}^{n} Z_j) - \sum_{j=1}^{n}\mathrm{Var}(Z_j)|$$

for all $t \in \mathbb{R}$, *where* $C > 0$ *is an arbitrary constant, not depending on* n.

PROOF OF THEOREM 1.1. Without loss of generality we assume that $a_{ni} = 0$ for all $i > n$ and $\sup E\xi_n^2 = M < \infty$. For every $1 \le a < b \le n$ and $1 \le u \le b-a$ we have, after a simple manipulations,

$$\begin{aligned} 0 &\le \sum_{i=a}^{b-u} a_{ni} \sum_{j=i+u}^{b} a_{nj}\mathrm{Cov}(\xi_i, \xi_j)^- \\ &\le \sup_k \left(\sum_{j:|k-j|\ge u} \mathrm{Cov}(\xi_k, \xi_j)^- \right) \left(\sum_{i=a}^{b} a_{ni}^2 \right). \end{aligned} \tag{2.1}$$

In particular by definition of LNQD, we also have

$$\mathrm{Var}\left(\sum_{i=a}^{b} a_{ni}\xi_i\right) \le M \sum_{i=a}^{b} a_{ni}^2.$$

We shall construct now a triangular array of random variables $\{Z_{ni}, 1 \le i \le n\}$ for which we shall make use of Lemma 2.1. Fix a small positive ϵ and find a positive integer $u = u_\epsilon$ such that, for every $n \ge u+1$

$$\begin{aligned} 0 &\le (\sum_{i=1}^{n-u} a_{ni} \sum_{j=i+u}^{b} a_{nj}\mathrm{Cov}(\xi_i, \xi_j)^-) \\ &\le \epsilon. \end{aligned}$$

This is possible because of (2.1) and (1.6). Denote by $[x]$ the integer part of x and define

$$K = [\frac{1}{\epsilon}]$$

$$Y_{nj} = \sum_{i=uj+1}^{u(j+1)} a_{ni}\xi_i, \ j = 0, 1, \cdots,$$

$$A_j = \left\{ i : 2Kj \le i < 2Kj + K, \mathrm{Cov}(Y_{ni}, Y_{n,i+1})^- \le \frac{2}{K} \sum_{i=2Kj}^{2Kj+K} \mathrm{Var}(Y_{ni}) \right\}.$$

Since $2\text{Cov}(Y_{ni}, Y_{n,i+1})^- \leq \text{Var}(Y_{ni}) + \text{Var}(Y_{n,i+1})$, we get that for every j the set A_j is not empty. Now we define the integers $m_1, m_2, \cdots, m_n$ recurrently by $m_0 = 0$:

$$m_{j+1} = \text{mim}\{m; m > m_j,\ m \in A_j\}$$

and define

$$Z_{nj} = \sum_{i=m_j+1}^{m_{j+1}} Y_{ni},\ j = 0, 1, \cdots,$$
$$\Delta_j = \{u(m_j+1)+1, \cdots, u(m_{j+1}+1)\}.$$

We observe that

$$Z_{nj} = \sum_{k \in \Delta_j} a_{nk}\xi_k,\ j = 0, 1, \cdots.$$

By definition of LNQD $Z_{nj}'s$ are linearly negative quadrant dependent, and from the fact that $m_j \geq 2K(j-1)$ and $m_{j+1} \leq K(2j+1)$ every set Δ_j contains no more than 3 Ku elements and $m_{j+1}/m_j \to 1$ as $j \to \infty$. Hence, for every fixed positive ϵ by (1.2)–(1.5) the array $\{Z_{nj} : i = 1, \cdots, n;\ n \geq 1\}$ satisfies the Lindeberg's condition(see Stout [9]). It remains to observe that by Lemma 2.1 and construction.

$$\begin{aligned}
&|E\exp(it\sum_{j=1}^{n} Z_{nj}) - \prod_{j=1}^{n} E\exp(itZ_{nj})| \\
&\leq ct^2|\{\text{Var}(\sum_{j=1}^{n} Z_{nj}) - \sum_{j=1}^{n}\text{Var}(Z_{nj})|\} \\
&\leq ct^2\{2(\sum_{i=1}^{n}\text{Cov}(Z_{ni}, Z_{n,i+1})^-) + 2(\sum_{i=1}^{n-2}\sum_{j=i+2}^{n}\text{Cov}(Z_{ni}, Z_{nj})^-)\} \\
&\leq ct^2\{4\sum_{i=1}^{n-u} a_{ni}\sum_{j=i+u}^{n} a_{nj}\text{Cov}(\xi_i, \xi_j)^- + 2\sum_{j=1}^{n}\text{Cov}(Y_{n,m_j}, Y_{n,m_j+1})^-\} \\
&\leq ct^2\{4\epsilon + \frac{8}{K}\sum_{i=1}^{n}\text{Var}(Y_{ni})\} \\
&= ct^2\{4\epsilon + \frac{8}{K}\sum_{j=1}^{n}\text{Var}(\sum_{i=uj+1}^{u(j+1)} a_{n_i}\xi_i)\}
\end{aligned}$$

$$\le ct^2\{4\epsilon + \frac{8M}{K}\sum_{j=1}^{n}\sum_{i=uj+1}^{u(j+1)} a_{n_i}^2\}$$

$$\le c_1 t^2\epsilon\{1+\sup_n \sum_{i=1}^{n} a_{ni}^2\}$$

$$\le c_2 t^2\epsilon.$$

Now the proof is complete by Theorem 4.2 in Billingslely[1]. □

PROOF OF COROLLARY 1.1. Let $A_{ni} = \frac{a_{ni}}{\sigma_n}$. Then we have

$$\max_{1\le i\le n} A_{ni} \to 0 \text{ as } n \to \infty,$$

$$\sup_n \sum_{i=1}^{n} A_{ni}^2 < \infty,$$

$$\mathrm{Var}(\sum_{i=1}^{n} A_{ni}\xi_i) = 1.$$

Hence, by Theorem 1.1 the desired result (1.11) follows.

PROOF OF THEOREM 1.2. First note that $\sum_j a_j^2 < \infty$ and without restricting the generality, we can assume $\sup E\xi_k^2 = 1$. Let

$$S_n = \sum_{k=1}^{n} X_k = \sum_{j=-\infty}^{\infty} (\sum_{k=1}^{n} a_{k+j})\xi_j.$$

In order to apply Theorem 1.1, we fix W_n such that $\sum_{|j|>W_n} a_j^2 < n^{-3}$ and take $k_n = W_n + n$. Then

$$\frac{S_n}{\sigma_n} = \sum_{|j|\le k_n} (\sum_{k=1}^{n} a_{k+j})\frac{\xi_j}{\sigma_n} + \sum_{|j|>k_n} (\sum_{k=1}^{n} a_{k+j})\frac{\xi_j}{\sigma_n}$$
$$= T_n + U_n(\text{say}).$$

By Cauchy Schwarz inequality and assumptions we have the following estimate

$$\mathrm{Var}(U_n) \le \sum_{|j|>k_n} \mathrm{Var}\left(\sum_{k=1}^{n} a_{k+j}\frac{\xi_j}{\sigma_n}\right)$$

$$\leq \sum_{|j|>k_n} (\sum_{k=1}^{n} a_{k+j}/\sigma_n)^2 E\xi_j^2$$
$$\leq n\sigma_n^{-2} \sum_{|j|>k_n} (\sum_{k=1}^{n} a_{k+j}^2)$$
$$\leq n^2\sigma_n^{-2} \sum_{|j|>k_n-n} a_j^2$$
$$\leq n^2\sigma_n^{-2} \sum_{|j|>W_n} a_j^2$$
$$\leq n^{-1}\sigma_n^{-2} \to 0 \text{ as } n \to \infty,$$

which yields

$$U_n \to 0 \text{ in probability as n} \to \infty. \tag{2.2}$$

It remains only to prove that $T_n \xrightarrow{\mathcal{D}} \mathrm{N}(0,1)$ by Theorem 4.1 of Billingsley[1]. Put

$$a_{nk} = \frac{\sum_{j=1}^{n} a_{k+j}}{\sigma_n} . \tag{2.3}$$

From assumption $\sum_j a_j < \infty$ $(a_j > 0)$, (1.10) and (2.3) we obtain

$$\frac{\sup_{-\infty<k<\infty} \sum_{j=1}^{n} a_{k+j}}{\sigma_n} \to 0 \text{ as } n \to \infty,$$

$$\max_{1\leq k\leq n} a_{nk} \to 0 \text{ as } n \to \infty,$$

$$\sup_n \sum_{k=1}^{n} a_{nk}^2 < \infty.$$

Hence, by Theorem 1.1

$$T_n \xrightarrow{\mathcal{D}} \mathrm{N}(0,1) \tag{2.4}$$

and from (2.2) and (2.4) the desired result (1.10) follows. □

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