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SOME COMPANIONS OF GRÜSS INEQUALITY IN 2-INNER PRODUCT SPACES AND APPLICATIONS FOR DETERMINANTAL INTEGRAL INEQUALITIES

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Abstract. Some companions of Grüss inequality in 2-inner product spaces and applications for determinantal integral inequalities are given.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book ([1]). Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,

($2\mathrm{I}_2$) $(x, x|z) = (z, z|x)$,

($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,

($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a *2-inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows ([2]):

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(1) If $\mathbb{K} = \mathbb{R}$, then $(2I_3)$ reduces to

$$(y, x|z) = (x, y|z).$$

(2) From $(2I_3)$ and $(2I_4)$, we have

$$(0, y|z) = 0, \ (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{1.1}$$

(3) Using $(2I_2)$–$(2I_5)$, we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{1.2}$$

In the real case, (1.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{1.3}$$

and, using this formula, it is easy to see, for any $\alpha \in \mathbb{R}$, that

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{1.4}$$

In the complex case, using (1.1) and (1.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (1.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x+y) - (z, z|x-y)] + \frac{i}{4}[(z, z|x+iy) - (z, z|x-iy)]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha \in \mathbb{C}$, that

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{1.6}$$

However, for $\alpha \in \mathbb{R}$, (1.6) reduces to (1.4). Also, from (1.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows that

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent.

In any given 2-inner product space $(X, (\cdot, \cdot \,|\, \cdot))$, we can define a function $\|\cdot \,|\, \cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$.

It is easy to see that, this function satisfies the following conditions:

($2N_1$) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

($2N_2$) $\|z|x\| = \|x|z\|$,

($2N_3$) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,

($2N_4$) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.

Any function $\|\cdot \,|\, \cdot\|$ defined on $X \times X$ and satisfying the conditions ($2N_1$)–($2N_4$) is called a *2-norm* on X and $(X, \|\cdot \,|\, \cdot\|)$ a *linear 2-normed space* [6]. Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot \,|\, \cdot\|)$ with the 2-norm defined by (1.10).

In the recent paper ([5]), the authors have established the following Grüss' type inequality holding in 2-inner product spaces.

THEOREM 1. *Let $(X, (.,.|.))$ be a 2-inner product space over $\mathbb{K}(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e, z \in X$, with $\|e|z\| = 1$. If $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ and $x, y \in X$ such that the conditions*

$$\operatorname{Re}(\Phi e - x, x - \varphi e|z) \geq 0, \quad \operatorname{Re}(\Gamma e - y, y - \gamma e|z) \geq 0 \tag{1.11}$$

hold or, equivalently, the following assumptions

$$(1.12)\quad \left\| x - \frac{\varphi + \Phi}{2} \cdot e|z \right\| \le \frac{1}{2} |\Phi - \varphi|, \left\| y - \frac{\gamma + \Gamma}{2} \cdot e|z \right\| \le \frac{1}{2} |\Gamma - \gamma|$$

are valid, then one has the inequality

$$(1.13)\quad |(x, y|z) - (x, e|z)(e, y|z)| \le \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible.

The following result improving (1.13) holds (cf. [5, Theorem 1]).

THEOREM 2. *Let $(X, (.,.|.))$ be a 2-inner product space over $\mathbb{K}(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e, z \in X$, with $\|e|z\| = 1$. If $\varphi, \gamma, \Phi, \Gamma \in \mathbb{K}$ and $x, y \in X$ such that the conditions* (1.11) *or, equivalently,* (1.12) *hold, then we have the inequality*

$$\begin{aligned} & |(x, y|z) - (x, e|z)(e, y|z)| \\ \le & \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \\ & - [\mathrm{Re}(\Phi e - x, x - \varphi e|z)]^{\frac{1}{2}} [\mathrm{Re}(\Gamma e - y, y - \gamma e|z)]^{\frac{1}{2}} \\ & \left(\le \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma|\right). \end{aligned}$$

The constant $\frac{1}{4}$ is best possible.

The following companion of Grüss inequality in 2-inner product spaces also holds (see [5, Theorem 4]).

THEOREM 3. *Let $(X, (.,.|.))$ be a 2-inner product space over $\mathbb{K}(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and let $e, z \in X$ with $\|e|z\| = 1$. If $\gamma, \Gamma \in \mathbb{K}$ and $x, y \in X$ are so that either with*

$$(1.14)\quad \mathrm{Re}\left(\Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e|z\right) \ge 0$$

or, equivalently,

$$(1.15)\quad \left\| \frac{x+y}{2} - \frac{\gamma + \Gamma}{2} \cdot e|z \right\| \le \frac{1}{2} |\Gamma - \gamma|$$

holds, then we have the inequality

$$(1.16)\quad \mathrm{Re}\left[(x, y|z) - (x, e|z)(e, y|z)\right] \le \frac{1}{4} |\Gamma - \gamma|^2.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The following corollary can be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}\left[(x,y|z)-(x,e|z)\,(e,y|z)\right]$$

(see, for details, [5]).

COROLLARY 1. *Let $(X,(.,.|.))$ be a 2-inner product space over $\mathbb{K}(\mathbb{K}=\mathbb{R},\mathbb{C})$ and let $e,z\in X$ with $\|e|z\|=1$. If $\gamma,\Gamma\in\mathbb{K}$ and $x,y\in X$ are so that*

$$\operatorname{Re}\left(\Gamma e-\frac{x\pm y}{2},\frac{x\pm y}{2}-\gamma e|z\right)\geq 0 \tag{1.17}$$

or, equivalently,

$$\left\|\frac{x\pm y}{2}-\frac{\gamma+\Gamma}{2}\cdot e|z\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.18}$$

then we have the inequality

$$\left|\operatorname{Re}\left[(x,y|z)-(x,e|z)\,(e,y|z)\right]\right|\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}. \tag{1.19}$$

If the inner product space X is real, then, for $m,M\in\mathbb{R}$ with $M>m$,

$$\left(Me-\frac{x\pm y}{2},\frac{x\pm y}{2}-me|z\right)\geq 0 \tag{1.20}$$

or, equivalently,

$$\left\|\frac{x\pm y}{2}-\frac{m+M}{2}\cdot e|z\right\|\leq\frac{1}{2}\left(M-m\right) \tag{1.21}$$

implies

$$\left|(x,y|z)-(x,e|z)\,(e,y|z)\right|\leq\frac{1}{4}\left(M-m\right)^{2}. \tag{1.22}$$

In both the inequalities (1.19) and (1.22), the constant $\frac{1}{4}$ is best possible.

It is the main aim of this paper to provide other inequalities of Grüss type in the general setting of 2-inner product spaces over the real or complex number field $\mathbb{K}$. Applications for determinantal integral inequalities are pointed out as well.

2. A Grüss type inequality for 2-inner products

The following Grüss type inequality in 2-inner product spaces holds.

THEOREM 4. *Let* $(X, (.,.|.))$ *be a 2-inner product space,* $x, y, z, e \in X$ *with* $\|e|z\| = 1$ *and* $a, A, b, B \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *such that* $\operatorname{Re}(\bar{a}A) > 0$ *and* $\operatorname{Re}(\bar{b}B) > 0$. *If*

$$\operatorname{Re}(Ae - x, x - ae|z) \geq 0, \quad \operatorname{Re}(Be - y, y - be|z) \geq 0 \tag{2.1}$$

or, equivalently,

$$\left\|x - \frac{a + A}{2} e|z\right\| \leq \frac{1}{2}|A - a|, \quad \left\|y - \frac{b + B}{2} e|z\right\| \leq \frac{1}{2}|B - b|, \tag{2.2}$$

then we have the inequality

$$\begin{aligned} &|(x, y|z) - (x, e|z)(e, y|z)| \\ &\leq \frac{1}{4} \frac{|A - a||B - a|}{\sqrt{\operatorname{Re}(\bar{a}A)\operatorname{Re}(\bar{b}B)}} |(x, e|z)(e, y|z)|. \end{aligned} \tag{2.3}$$

The constant $\frac{1}{4}$ *in* (2.3) *is best possible in the sense that it cannot be replaced by a smaller constant.*

PROOF. Let $u, x, z, U \in X$ with $u \neq U$. We claim (see also [5]) that

$$\operatorname{Re}(U - x, x - u|z) \geq 0$$

if and only if

$$\left\|x - \frac{u + U}{2}|z\right\| \leq \frac{1}{2}\|U - u|z\|.$$

Define

$$I_1 := \operatorname{Re}(U - x, x - u|z), \quad I_2 := \frac{1}{4}\|U - u|z\|^2 - \left\|x - \frac{u + U}{2}|z\right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \operatorname{Re}[(x, u|z) + (U, x|z)] - \operatorname{Re}(U, u|z) - \|x|z\|^2$$

and thus, obviously, $I_1 \geq 0$ if and only if $I_2 \geq 0$ showing, for $U := Ae, u := ae$ $(U := Be, u := be)$, so that the required equivalence holds true.

Apply Schwarz's inequality in $(X, (\cdot, \cdot|\cdot))$ for $x - (x, e|z)e$ and $y - (y, e|z)e$ to get (see also [3]) that

$$\begin{aligned} &|(x, y|z) - (x, e|z)(e, y|z)|^2 \\ &\leq \left(\|x|z\|^2 - |(x, e|z)|^2\right)\left(\|y|z\|^2 - |(y, e|z)|^2\right). \end{aligned} \tag{2.4}$$

Now, assume that $u, v \in X$ and $c, C \in \mathbb{K}$ with $\operatorname{Re}(\bar{c}C) > 0$ and $\operatorname{Re}(Cv - u, u - cv|z) \geq 0$. This last inequality is equivalent to

$$\|u|z\|^2 + \operatorname{Re}(\bar{c}C)\|v|z\|^2 \leq \operatorname{Re}\left[C\overline{(u,v|z)} + \bar{c}(u,v|z)\right] = \operatorname{Re}\left[\left(\bar{C}+\bar{c}\right)(u,v|z)\right] \tag{2.5}$$

since it is obvious that

$$\operatorname{Re}\left[C\overline{(u,v|z)}\right] = \operatorname{Re}\left[\bar{C}(u,v|z)\right].$$

Dividing this inequality by $[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} > 0$, from (2.5), we deduce

$$\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u|z\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}\|v|z\|^2 \leq \frac{\operatorname{Re}\left[\left(\bar{C}+\bar{c}\right)(u,v|z)\right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}. \tag{2.6}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha}q^2 \geq 2pq, \quad \alpha > 0, \quad p, q \geq 0,$$

we deduce

$$2\|u|z\|\,\|v|z\| \leq \frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u|z\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}\|v|z\|^2. \tag{2.7}$$

Making use of (2.6) and (2.7) and the fact that, for any $z \in \mathbb{C}$ with $\operatorname{Re}(z) \leq |z|$, we get

$$\|u|z\|\,\|v|z\| \leq \frac{\operatorname{Re}\left[\left(\bar{C}+\bar{c}\right)(u,v|z)\right]}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \leq \frac{|c+C|}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}|(u,v|z)|.$$

Consequently, we have

$$\|u|z\|^2\|v|z\|^2 - |(u,v|z)|^2 \leq \left[\frac{|c+C|^2}{4[\operatorname{Re}(\bar{c}C)]} - 1\right]|(u,v|z)|^2 = \frac{1}{4}\frac{|C-c|^2}{\operatorname{Re}(\bar{c}C)}|(u,v|z)|^2. \tag{2.8}$$

Now, if we write (2.8) for the choices $u = x$, $v = e$ and $u = y$, $v = e$ respectively and use (2.4), we deduce the desired result (2.2). The sharpness of the constant will be proved in the case where X is a real 2-inner product space. □

The following corollary which provides a simpler Grüss type inequality for real constants (and, in particular, for real 2-inner product spaces) holds.

COROLLARY 2. *With the assumptions of Theorem 4 and if $m, M, n, N \in \mathbb{R}$ are such that $M > m > 0$, $N > n > 0$ and either*

$$(Me - x, x - me|z) \geq 0, \quad (Ne - y, y - ne|z) \geq 0$$

or, equivalently,

$$\left\| x - \frac{m+M}{2} e|z \right\| \leq \frac{1}{2}(M - m), \quad \left\| y - \frac{n+N}{2} e|z \right\| \leq \frac{1}{2}(N - n) \tag{2.9}$$

holds, then we have the inequality

$$\begin{aligned} &|(x, y|z) - (x, e|z)(e, y|z)| \\ &\quad \leq \frac{1}{4} \cdot \frac{(M - m)(N - n)}{\sqrt{mnMN}} |(x, e|z)(e, y|z)|. \end{aligned} \tag{2.10}$$

The constant $\frac{1}{4}$ is best possible.

PROOF. We will prove the best constant in (2.10) for $x = y$. To do that, let assume there is a constant $k > 0$ so that

$$\|x|z\|^2 - (x, e|z)^2 \leq k \cdot \frac{(M - m)^2}{mM} (x, e|z)^2 \tag{2.11}$$

provided

$$(Me - x, x - me|z) \geq 0, \tag{2.12}$$

where $M > m > 0$, $e, z \in X$ with $\|e|z\| = 1$ and $x \in X$.

For $\varepsilon > 0$, consider $M := 2 + \varepsilon, m := \varepsilon > 0, y \in X$ with $\|y|z\| = 1$ and $(e, y|z) = 0$. Define $x := (1 + \varepsilon)e + y$. With these choices, we have

$$\begin{aligned} (Me - x, x - me|z) &= ((2 + \varepsilon)e - (1 + \varepsilon)e - y, (1 + \varepsilon)e + y - \varepsilon e|z) \\ &= (e - y, e + y|z) = \|e|z\|^2 - \|y|z\|^2 = 0 \end{aligned}$$

and thus the condition (2.12) is obviously satisfied.

On the other hand,

$$\|x|z\|^2 = \|(1 + \varepsilon)e + y|z\|^2 = (1 + \varepsilon)^2 + 1$$

and

$$(x, e|z)^2 = ((1 + \varepsilon)e + y, e|z)^2 = (1 + \varepsilon)^2.$$

Replacing the above values in the inequality (2.11), we get

$$1 \leq \frac{4k(\varepsilon + 1)^2}{\varepsilon(\varepsilon + 2)} \tag{2.13}$$

for any $\varepsilon > 0$. Letting $\varepsilon \to +\infty$, we deduce $k \geq \frac{1}{4}$, which proves the fact that $\frac{1}{4}$ is the best constant in (2.10). □

REMARK 1. If $(x,e|z),(y,e|z)$ are assumed not to be zero, then the inequality (2.3) is equivalent to

$$\left|\frac{(x,y|z)}{(x,e|z)(e,y|z)}-1\right|\leq\frac{1}{4}\frac{|A-a|\,|B-a|}{\sqrt{\operatorname{Re}(\bar{a}A)\operatorname{Re}(\bar{b}B)}}, \tag{2.14}$$

while the inequality (2.10) is equivalent to

$$\left|\frac{(x,y|z)}{(x,e|z)(e,y|z)}-1\right|\leq\frac{1}{4}\cdot\frac{(M-m)(N-n)}{\sqrt{mnMN}}. \tag{2.15}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

3. Some related results

The following result which provides a generalization of Theorem 3 holds.

THEOREM 5. *Let $(X,(.,.|.))$ be a 2-inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{C},\mathbb{R}$). If $\gamma,\Gamma\in\mathbb{K}$, $e,x,y,z\in X$ with $\|e|z\|=1$ and $\lambda\in(0,1)$ are such that*

$$\operatorname{Re}(\Gamma e-(\lambda x+(1-\lambda)y),(\lambda x+(1-\lambda)y)-\gamma e|z)\geq 0 \tag{3.1}$$

or, equivalently,

$$\left\|\lambda x+(1-\lambda)y-\frac{\gamma+\Gamma}{2}e|z\right\|\leq\frac{1}{2}|\Gamma-\gamma|, \tag{3.2}$$

then we have the inequality

$$\operatorname{Re}[(x,y|z)-(x,e|z)(e,y|z)]\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2. \tag{3.3}$$

The constant $\frac{1}{16}$ is the best possible constant in (3.3) in the sense that it cannot be replaced by a smaller one.

PROOF. We know that, for any $z,u\in X$, one has

$$\operatorname{Re}(z,u|z)\leq\frac{1}{4}\|z+u|z\|^2.$$

Then, for any $a,b\in X$ and $\lambda\in(0,1)$, one has

$$\operatorname{Re}(a,b|z)\leq\frac{1}{4\lambda(1-\lambda)}\|\lambda a+(1-\lambda)b|z\|^2. \tag{3.4}$$

Since

$$\begin{aligned}&(x,y|z)-(x,e|z)(e,y|z)\\&\quad=(x-(x,e|z)e,y-(y,e|z)e|z)\ \text{ (as }\|e|z\|=1),\end{aligned}$$

using (3.4), we have

$$\begin{aligned}
&\operatorname{Re}\left[(x,y|z)-(x,e|z)\,(e,y|z)\right]\\
&=\operatorname{Re}\left[(x-(x,e|z)\,e,y-(y,e|z)\,e|z)\right]\\
&\le\frac{1}{4\lambda\,(1-\lambda)}\left\|\lambda\,(x-(x,e|z)\,e)+(1-\lambda)\,(y-(y,e|z)\,e)\,|z\right\|^2 \qquad (3.5)\\
&=\frac{1}{4\lambda\,(1-\lambda)}\left\|\lambda x+(1-\lambda)\,y-(\lambda x+(1-\lambda)\,y,e|z)\,e|z\right\|^2.
\end{aligned}$$

Since, for $m,e,z\in X$ with $\|e|z\|=1$, one has the equality

$$\|m-(m,e|z)\,e|z\|^2=\|m|z\|^2-|(m,e|z)|^2, \tag{3.6}$$

then, by (3.5), we deduce the inequality

$$\begin{aligned}
&\operatorname{Re}\left[(x,y|z)-(x,e|z)\,(e,y|z)\right]\\
&\le\frac{1}{4\lambda\,(1-\lambda)}\left[\|\lambda x+(1-\lambda)\,y|z\|^2-|(\lambda x+(1-\lambda)\,y,e|z)|^2\right].
\end{aligned} \tag{3.7}$$

Now, if we apply Grüss' inequality

$$0\le\|a|z\|^2-|(a,e|z)|^2\le\frac{1}{4}\,|\Gamma-\gamma|^2$$

provided

$$\operatorname{Re}\,(\Gamma e-a,a-\gamma e|z)\ge 0$$

for $a=\lambda x+(1-\lambda)\,y$, then we have

$$\|\lambda x+(1-\lambda)\,y|z\|^2-|(\lambda x+(1-\lambda)\,y,e|z)|^2\le\frac{1}{4}\,|\Gamma-\gamma|^2. \tag{3.8}$$

Utilising (3.7) and (3.8) we deduce the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.3) holds with a constant $C>0$ provided (3.1) is valid, i.e.,

$$\operatorname{Re}\left[(x,y|z)-(x,e|z)\,(e,y|z)\right]\le C\cdot\frac{1}{\lambda\,(1-\lambda)}\,|\Gamma-\gamma|^2. \tag{3.9}$$

If in (3.9) we choose $x=y$ provided (3.1) holds with $x=y$ and $\lambda\in(0,1)$, then

$$\|x|z\|^2-|(x,e|z)|^2\le C\cdot\frac{1}{\lambda\,(1-\lambda)}\,|\Gamma-\gamma|^2 \tag{3.10}$$

provided

$$\operatorname{Re}\,(\Gamma e-x,x-\gamma e|z)\ge 0.$$

Since we know, in Grüss' inequality, that the constant $\frac{1}{4}$ is best possible, then, by (3.10), one has

$$\frac{1}{4} \leq \frac{C}{\lambda(1-\lambda)}$$

for $\lambda \in (0,1)$, which gives, for $\lambda = \frac{1}{2}$, that $C \geq \frac{1}{16}$. The theorem is completely proved. □

The following corollary is a natural consequence of the above result:

COROLLARY 3. *Assume that $\gamma, \Gamma, e, x, y, z$ and λ are as in Theorem 5. If*

$$\operatorname{Re}\left(\Gamma e - (\lambda x \pm (1-\lambda) y), (\lambda x \pm (1-\lambda) y) - \gamma e | z\right) \geq 0 \tag{3.11}$$

or, equivalently,

$$\left\| \lambda x \pm (1-\lambda) y - \frac{\gamma + \Gamma}{2} e | z \right\| \leq \frac{1}{2} |\Gamma - \gamma|^2, \tag{3.12}$$

then we have the inequality

$$\left| \operatorname{Re}\left[(x, y|z) - (x, e|z)(e, y|z)\right] \right| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \tag{3.13}$$

The constant $\frac{1}{16}$ is best possible in (3.13).

PROOF. Using Theorem 5 for $(-y)$ instead of y, we have that

$$\operatorname{Re}\left(\Gamma e - (\lambda x - (1-\lambda) y), (\lambda x - (1-\lambda) y) - \gamma e | z\right) \geq 0,$$

which implies that

$$\operatorname{Re}\left[-(x, y|z) + (x, e|z)(e, y|z)\right] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2.$$

This last inequality shows that

$$\operatorname{Re}\left[(x, y|z) - (x, e|z)(e, y|z)\right] \geq -\frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \tag{3.14}$$

Consequently, by (3.3) and (3.14), we deduce the desired inequality (3.13). □

REMARK 2. If $M, m \in \mathbb{R}$ with $M > m$ and, for $\lambda \in (0,1)$,

$$\left\| \lambda x + (1-\lambda) y - \frac{M+m}{2} e | z \right\| \leq \frac{1}{2}(M - m), \tag{3.15}$$

then

$$(x, y|z) - (x, e|z)(e, y|z) \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} (M - m)^2.$$

If (3.15) holds with "±" instead of "+" , then

$$|(x,y|z)-(x,e|z)(e,y|z)| \le \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}(M-m)^2. \tag{3.16}$$

REMARK 3. If $\lambda=\frac{1}{2}$ in (3.1) or (3.2), then we obtain the result from [5] mentioned in Theorem 3.

For $\lambda=\frac{1}{2}$, Corollary 3 and Remark 2 will produce the corresponding results obtained in [5]. We omit the details.

The following similar result with the one incorporated in Theorem 5 may be stated as well:

THEOREM 6. *Assume that* $\gamma,\Gamma,e,x,y,z,$ *and* λ *are as in Theorem 5. If* $\operatorname{Re}(\Gamma\overline{\gamma})>0$, *then we have the inequality:*

$$\begin{aligned}&\operatorname{Re}\left[(x,y|z)-(x,e|z)(e,y|z)\right]\\ &\le \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|(\lambda x+(1-\lambda)y,e|z)|^2.\end{aligned} \tag{3.17}$$

The constant $\frac{1}{16}$ *is best possible in* (3.17).

PROOF. If we apply the companion of Grüss inequality incorporated in Theorem 4, then we may state that

$$\begin{aligned}&\|\lambda x+(1-\lambda)y|z\|^2-|(\lambda x+(1-\lambda)y,e|z)|^2\\ &\le \frac{|\Gamma-\gamma|^2}{4\operatorname{Re}(\Gamma\overline{\gamma})}|(\lambda x+(1-\lambda)y,e|z)|^2.\end{aligned} \tag{3.18}$$

Utilising (3.7) and (3.18), we deduce the desired inequality (3.17). The sharpness of the constant may be shown in a similar way to the one used in Theorem 5 and we omit the details. □

Finally, we may state the following:

COROLLARY 4. *Assume that* γ,Γ,e,x,y,z *and* λ *are as in Corollary 3. If* $\operatorname{Re}(\Gamma\overline{\gamma})>0$, *then we have the inequality:*

$$\begin{aligned}&|\operatorname{Re}\left[(x,y|z)-(x,e|z)(e,y|z)\right]|\\ &\le \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\cdot\frac{|\Gamma-\gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|(\lambda x+(1-\lambda)y,e|z)|^2.\end{aligned} \tag{3.19}$$

The constant $\frac{1}{16}$ *is best possible in* (3.19).

REMARK 4. The particular case $\lambda=\frac{1}{2}$ may produce some particular inequalities of interest, but we omit the details.

4. Determinantal integral inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L_{\rho}^{2}(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_{\Omega} \rho(s)|f(s)|^{2} d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L_{\rho}^{2}(\Omega)$ by formula

$$
(4.1)\qquad (f,g|h)_{\rho} := \frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix} \times \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t)\end{vmatrix} d\mu(s)d\mu(t),
$$

where by

$$
\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}
$$

we denote the determinant of the matrix

$$
\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t)\end{bmatrix}
$$

generating the 2-norm on $L_{\rho}^{2}(\Omega)$ expressed by

$$
(4.2)\qquad \|f|h\|_{\rho} := \left(\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t)\end{vmatrix}^{2} d\mu(s)d\mu(t)\right)^{1/2}
$$

A simple calculation with integrals reveals that

$$
(4.3)\qquad (f,g|h)_{\rho} = \begin{vmatrix} \int_{\Omega}\rho f g d\mu & \int_{\Omega}\rho f h d\mu \\ \int_{\Omega}\rho g h d\mu & \int_{\Omega}\rho h^{2} d\mu \end{vmatrix}
$$

and

$$
(4.4)\qquad \|f|h\|_{\rho} = \begin{vmatrix} \int_{\Omega}\rho f^{2} d\mu & \int_{\Omega}\rho f h d\mu \\ \int_{\Omega}\rho f h d\mu & \int_{\Omega}\rho h^{2} d\mu \end{vmatrix}^{1/2},
$$

where, for simplicity, instead of $\int_{\Omega} \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_{\Omega} \rho f g d\mu$.

We recall that the pair of functions $(q,p) \in L_\rho^2(\Omega) \times L_\rho^2(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for *a.e.* $x, y \in \Omega$.

We note that, if $\Omega = [a,b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L_\rho^2(\Omega)$ is such that $h(x) \neq 0$ for *a.e.* $x \in \Omega$. Then, by the definition of 2-inner product $(f,g|h)_\rho$, we have

$$\begin{aligned}(f,g|h)_\rho = \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t)h^2(s)h^2(t)\left(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)}\right) \\ \times \left(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)}\right) d\mu(s)\,d\mu(t)\end{aligned} \tag{4.5}$$

and thus a *sufficient condition* for the inequality

$$(f,g|h)_\rho \geq 0 \tag{4.6}$$

to hold, that is, the functions $\left(\frac{f}{h}, \frac{g}{h}\right)$ are synchronous. It is obvious that this condition is not necessary.

Using the representations (4.3), (4.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities as follows:

Proposition 1. *Let* $f,g,h,u \in L_\rho^2(\Omega)$ *with* $h \neq 0$ a.e. *and*

$$\int_\Omega \rho u^2 d\mu \int_\Omega \rho h^2 d\mu - \left(\int_\Omega \rho u h\, d\mu\right)^2 = 1. \tag{4.7}$$

If $M > m$ *and* $N > n$ *are real numbers with the property that the functions*

$$\left(M \cdot \frac{u}{h} - \frac{f}{h}, \frac{f}{h} - m \cdot \frac{u}{h}\right), \quad \left(N \cdot \frac{u}{h} - \frac{g}{h}, \frac{g}{h} - n \cdot \frac{u}{h}\right) \tag{4.8}$$

are synchronous on Ω, then we have the following determinantal integral Grüss type inequality

$$\left|\det\begin{bmatrix}\int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right.$$

$$\left. - \det\begin{bmatrix}\int_\Omega \rho fu d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho uh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix} \cdot \det\begin{bmatrix}\int_\Omega \rho gu d\mu & \int_\Omega \rho gh d\mu \\ \int_\Omega \rho uh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|$$

$$\leq \frac{1}{4}\frac{(M-m)(N-n)}{\sqrt{mMnM}}$$

$$\times \left|\det\begin{bmatrix}\int_\Omega \rho fu d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho uh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix} \cdot \det\begin{bmatrix}\int_\Omega \rho gu d\mu & \int_\Omega \rho gh d\mu \\ \int_\Omega \rho uh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}\right|.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 4 applied for the 2-inner product $(.,.|.)_\rho$ defined in (4.1).

PROPOSITION 2. *Let $f, g, h, u \in L^2_\rho(\Omega)$ with $h \neq 0$ a.e. and*

$$\int_\Omega \rho u^2 d\mu \int_\Omega \rho h^2 d\mu - \left(\int_\Omega \rho uh d\mu\right)^2 = 1.$$

If $M > m$ and $N > n$ and $\lambda \in (0,1)$ are real numbers with the property that the functions

$$\left(M \cdot \frac{u}{h} - \left(\lambda\frac{f}{h} + (1-\lambda)\frac{g}{h}\right), \lambda\frac{f}{h} + (1-\lambda)\frac{g}{h} - m \cdot \frac{u}{h}\right) \tag{4.9}$$

are synchronous on Ω, then we have the following determinantal integral Grüss type inequality

$$J := \det\begin{bmatrix}\int_\Omega \rho fg d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho gh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix} - \det\begin{bmatrix}\int_\Omega \rho fu d\mu & \int_\Omega \rho fh d\mu \\ \int_\Omega \rho uh d\mu & \int_\Omega \rho h^2 d\mu\end{bmatrix}$$

$$\cdot \det \begin{bmatrix} \int_\Omega \rho g u d\mu & \int_\Omega \rho g h d\mu \\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}$$
$$\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \cdot (M-m)^2 .$$

If (4.9) *holds with* " $\pm$ " *instead of* " $+$ ", *then*

$$|J| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \cdot (M-m)^2 .$$

The proof is obvious by the inequality (3.15) and we omit the details.

REMARK 5. It is obvious that, if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.

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