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ON KRAMER-MESNER MATRIX PARTITIONING CONJECTURE

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ABSTRACT. In 1977, Ganter and Teirlinck proved that any $2t \times 2t$ matrix with $2t$ nonzero elements can be partitioned into four submatrices of order t of which at most two contain nonzero elements. In 1978, Kramer and Mesner conjectured that any $mt \times nt$ matrix with kt nonzero elements can be partitioned into mn submatrices of order t of which at most k contain nonzero elements. In 1995, Brualdi et al. showed that this conjecture is true if $m = 2$, $k \leq 3$ or $k \geq mn - 2$. They also found a counterexample of this conjecture when $m = 4$, $n = 4$, $k = 6$ and $t = 2$. When $t = 2$, we show that this conjecture is true if $k \leq 5$.

1. Introduction

The following theorem is proved by Ganter and Teirlinck[3].

THEOREM 1. *Every $2t \times 2t$ matrix with $2t$ nonzero elements can be partitioned into four submatrices of order t of which at most two contain nonzero elements.*

In 1978, Kramer and Mesner conjectured the following.

CONJECTURE 2. *Let m, n, t and k be positive integers. Then every $mt \times nt$ matrix with kt nonzero elements can be partitioned into mn submatrices of order t of which at most k contain nonzero elements.*

Brualdi et al.[1] denoted the assertion of this conjecture by $KM(m, n, k, t)$. They mentioned its relation with the Zarankiewicz problem which

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This conjecture was stated in the talk "On the Distribution of nonzero elements in certain sparse matrices" given by D. M. Mesner at 9th Southeastern Conference held at Florida Atlantic University in 1978.

is stated as follows ([5]): Determine $Z(a,b;c,d)$, the smallest number M such that each $a \times b$ matrix with M zero elements contains a $c \times d$ zero matrix. For that they used $f(m,n,k,t)$ which they defined as the largest N such that each $mt \times nt$ matrix with N nonzero elements can be partitioned into mn submatrices of order t of which at most k contain nonzero elements. Thus the assertion $KM(m,n,k,t)$ is equivalent to $f(m,n,k,t) \geq kt$. Also $f(m,n,mn-rs,t) \geq (mt)(nt) - Z(mt,nt;rt,st)$ with equality if $r = s = 1$. They proved that $KM(m,n,k,t)$ is true if $m = 2$, $k \leq 3$ or $k \geq mn - 2$. They also showed that $KM(4,4,6,2)$ is false by finding a counter example which is shown in Fig 1. In this paper we extend the results in [1] by showing that $KM(m,n,k,2)$ is true if $k \leq 5$.

1 0	0 0	0 0	0 0
1 1	0 0	0 0	0 0
0 1	0 0	0 0	0 0
0 1	1 1	1 0	0 0
0 0	0 0	1 1	0 0
0 0	1 0	0 0	0 0
0 0	0 0	0 0	1 0
0 0	0 0	0 0	0 0

FIGURE 1

2. Preliminaries and basic results

We introduce some notations and definitions. Let G be a graph. Then $|G|$ denotes the order of G. For a vertex u of G, a neighbor of u is a vertex which is connected to u. $N(u)$ denotes the set of all neighbors of u. For a set U of vertices, $N(U)$ denotes the union of $N(u)$ for all elements u of U. G is called *bipartite* if its vertex set can be partitioned into two partite sets where no edge connects two vertices of one partite set. A bipartite graph G is called a *complete bipartite graph* if any two vertices in different partite sets are adjacent. A *complete bipartite graph* with two partite sets of m, n vertices, respectively is denoted by $K_{m,n}$. $K_{1,3}$ is called a *claw*. A path with n vertices is denoted by P_n and a cycle with n vertices is denoted by C_n. We call G a $P_3 - claw$ if it is a claw where each edge is replaced by P_3. For a P_3-claw, we call the vertex of degree 3 the *center*. Throughout this paper we view a matrix $A = [a_{ij}]$ as an adjacency matrix of a bipartite graph $G = G(U,V;E)$ where U is the set of vertices corresponding to the rows of A, V is the set of vertices corresponding to the columns of A and E is the set of edges determined by the nonzero elements in A. We say that A has a *matrix − crossing* if both $a_{i_1j_2}$ and $a_{i_2j_1}$ are nonzero for some i_1, i_2, j_1, j_2 such that $i_1 < i_2$ and $j_1 < j_2$. We also say that A is decomposed into two matrices A_1 and A_2 if there exist permutation matrices P and Q such that $PAQ = A_1 \oplus A_2$, the direct sum of A_1 and A_2.

LEMMA 3. [1] *Let c, m, n and t be positive integers with $c \leq m$. Assume that $q_1, \ldots, q_m$ is a nonincreasing sequence of nonnegative integers with $\sum_{i=1}^{m} q_i \leq mn + c$. Then $\sum_{i=c+1}^{m} q_i \leq n(m - c)$.*

Proof. This is a restatement of Lemma 2.1. of Brualdi et al.[1]. □

LEMMA 4. *If an $m \times n$ matrix A has no matrix-crossing, then A contains at most $m + n - 1$ nonzero elements.*

Proof. We prove by induction on (m, n). If the element in k-th row and l-th column of A, $a_{kl} = 0$ for all $(k, l) \neq (1, 1)$, (m, n), then the lemma is true clearly. Suppose $a_{kl} \neq 0$ for some $(k, l) \neq (1, 1)$, (m, n). Consider a $k \times l$ matrix which we obtain by omitting the last $m - k$ rows and the last $n - l$ columns of A and an $(m-k+1) \times (n-l+1)$ matrix which we obtain by omitting the first $k - 1$ rows and the first $l - 1$ columns of A. Note that these two matrices contain all the nonzero elements of A. By applying the induction hypothesis on these two matrices, the lemma follows. □

LEMMA 5. *If an $mt \times nt$ matrix A has no matrix-crossing, then A can be partitioned into mn submatrices of order t of which at most $m+n-1$ contain nonzero elements.*

Proof. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let A_{ij} be the submatrix of A of order t we obtain from A by omitting all the rows and columns except $(i - 1)t + 1, \ldots, it$-th rows and $(j - 1)t + 1, \ldots, jt$-th columns. For each $1 \leq i_1, i_2 \leq m$, $1 \leq j_1, j_2 \leq n$ such that $i_1 < i_2$ and $j_1 < j_2$, not both submatrices $A_{i_1j_2}$ and $A_{i_2j_1}$ contain nonzero elements as A has no matrix-crossings. Thus the lemma follows from the previous Lemma 4. □

LEMMA 6. *A tree contains a P_3-claw if and only if all of its adjacency matrix has a matrix-crossing.*

Proof. If a tree contains a P_3-claw, then all of its adjacency matrix has a matrix-crossing clearly. Suppose a tree contains no P_3-claw. Then each vertex of it has at most two neighbors of degree more than 1. Thus it is a path with some vertices of degree 1 added, which obviously has an adjacency matrix without matrix-crossing.

THEOREM 7. [4] *If $\alpha_1, \ldots, \alpha_{2t-1}$ is a sequence of elements in the elementary Abelian group $Z_t \times Z_t$, then some subsequence has sum $(0,0)$.*

The following lemma is in the proof of Theorem 3.2. of Brualdi et al.[1]. It uses the above theorem which was conjectured by Erdös[2] and proved by Olson[4](see [1]).

LEMMA 8. [1] *Let A be an $mt \times nt$ matrix. If there are at least $2t$ connected components of the bipartite graph $G(U, V, E)$ corresponding to the matrix A, then A is decomposed into two matrices A_1 and A_2 where A_1 is an $et \times ft$ matrix and A_2 is an $(m-e)t \times (n-f)t$ matrix for some $(e, f) \neq (0,0)$, (m, n).*

The following lemma, which uses the pigeon-hole principle, is in the proof of Lemma 3.3. of Brualdi et al.[1].

LEMMA 9. [1] *Let A be a matrix with kt nonzero elements. If the bipartite graph $G(U, V, E)$ corresponding to A contains at least $t+1$ nontrivial components, then A is decomposed into two matrices A_1 and A_2 where A_1 contains et nonzero elements and A_2 contains $(k-e)t$ nonzero elements for some $0 < e < k$.*

LEMMA 10. *For all $k \geq 3$, $KM(3, k-1, k, 2)$ is true if $KM(3, k-2, k, 2)$ is true.*

Proof. Let A be a $6 \times 2(k-1)$ matrix with $2k$ nonzero elements. By Lemma 8, we may assume that A is decomposed into A_1 and A_2 where A_1 is a $2e \times 2f$ matrix and A_2 is a $2(3-e) \times 2(k-f-1)$ matrix for a pair of integers $(e, f) \neq (0,0), (3, k-1)$. After exchanging A_1 and A_2 if needed, we may assume that $2 \leq e \leq 3$. If $e = 3$, then the lemma is proved as $KM(3, k-2, k, 2)$ is true. Let $e = 2$. If $f = 0$ or $f = k-1$, then the lemma is true as $KM(2, k-1, k, 2)$ is true. Let $0 < f < k-1$. If A_1 contains at most $2f+2$ nonzero elements, then the lemma is proved by applying $KM(2, f, f+1, 2)$ to A_1. Also if A_1 contains at least $2f+4$ nonzero elements, then A_2 contains at least two zero columns and hence the lemma is proved as $KM(3, k-2, k, 2)$ is true. Assume that A_1 and A_2 contain $2f+3$ nonzero elements and $2(k-f-1)-1$ nonzero elements, respectively where all the nonzero elements of A_2 are in different columns. Note that A_2 is decomposed into two matrices. For each $1 \leq j \leq 2f$, let q_j be the number of nonzero elements in column j of A_1. After rearranging columns we may assume that $q_1 \geq \ldots \geq q_{2f}$. By Lemma 3, $q_3 + \ldots + q_{2f} \leq 2(f-1)+1$. If $q_3 + \ldots + q_{2(k-1-f)} \leq 2(f-1)$, then the matrix we obtain from A_1 by omitting its first two columns has a partition into submatrices of order 2 of which at most $f-1$ contain nonzero elements and hence the lemma is proved. Assume that $q_3 + \ldots + q_{2f} = 2(f-1)+1$. Then $q_1 = q_2 = q_3 = 2$ and $q_i = 1$ for all $i \geq 4$. Let $G_1 = G_1(U_1, V_1; E_1)$ be the bipartite graph corresponding to A_1 where $U_1 = \{u_1, u_2, u_4\}$. Let $v_1, v_2, v_3 \in V_1$ be the the vertices corresponding to the first three columns of A_1. Firstly assume that G_1 contains a cycle. Then G_1 is disconnected and hence A_1 is decomposed into two matrices. Note that each of A_1 and A_2 has a direct summand with odd number of nonzero elements. Applying $KM(2,m,n,2)$ for some appropriate m and n to the direct sum of those summands, the lemma is proved. Secondly assume that G_1 doesn't contain a cycle. Then G_1 is a tree. If A_1 has no matrix-crossing, then the lemma is true by Lemma 5. Assume that A_1 has a matrix-crossing. Then G_1 contains a P_3-claw by Lemma 6. Let $u_1 \in U_1$ be the center of a P_3-claw. Then u_1 is connected to v_1, v_2 and v_3. As A_1 contains odd number of nonzero elements we may assume that $d(u_1) + d(u_2) \equiv 1 \pmod 2$ and hence A_1 is partitioned as shown in Fig. 2. Thus the lemma follows.

1 1	1 1 ... 1 1 1 ... 1	

1
  1
		1 ... 1
		     1 ... 1

FIGURE 2

3. A partial solution on Kramer-Mesner matrix partitioning conjecture for $KM(m,n,k,2)$ for $k \leq 5$

LEMMA 11. *$KM(3,3,4,2)$ is true.*

Proof. Considering that $KM(3,2,4,2)$ is true, the statement of the lemma is true by Lemma 10.

LEMMA 12. *$KM(3,3,5,2)$ is true.*

Proof. Let A be a 6×6 matrix with 10 nonzero elements. Let $G = G(U,V;E)$ be the bipartite graph corresponding to A and s be the number of connected components of G. Then by Lemma 8, $s \geq 2$. Firstly assume that $s \geq 4$. Then by Lemma 8 again, A is decomposed into two matrices A_1 and A_2 where A_1 is an $a \times b$ matrix for some $(a, b) \equiv (0, 0) \pmod 2$ and $(a, b) \neq (0, 0)$, $(6, 6)$. We may assume that $a \geq 3$. If $a = 6$, then the lemma is proved as $KM(3, 2, 5, 2)$ is true. If $a = 4$, then the lemma is true considering sizes of A_1 and A_2. Secondly assume that $s = 3$ and all the components of G are nontrivial. Then by Lemma 9, A is decomposed into two matrices A_1 and A_2 where A_1 contains exactly $2e$ nonzero elements for some $0 < e < 5$. By applying $KM(2, 3, e, 2)$ and $KM(2, 3, 5 - e, 2)$, A_1 and A_2 have partitions into submatrices of order 2 of which at most e and $5 - e$ contain nonzero elements, respectively and hence the lemma is proved. Finally assume that $s \leq 3$ and at most two components of G are nontrivial. We consider the cases where one or two components of G are nontrivial separately in the following two cases.

Case 1: G has two nontrivial components.

Subcase 1a: G has no trivial component. Then A is decomposed into A_1 and A_2 where A_1 is an $a \times b$ matrix and A_2 is a $(6 - a) \times (6 - b)$ matrix for some $0 < a, b < 6$. By the above argument when $s \geq 4$ or G has three nontrivial components, it is enough to consider the case where $a \equiv 1 \pmod 2$ and A_1 contains $2e - 1$ nonzero elements for some $1 \leq e \leq 5$. Also after exchanging A_1 and A_2 if needed, we may assume that $e \geq 3$. Note that the bipartite graphs corresponding to A_1 and A_2 are trees and hence $a + b = 2e$. Let $e = 3$. Then we may assume that $(a, b) = (3, 3)$ or $(5, 1)$. In the latter case, the lemma is clearly true. Let both A_1 and A_2 be 3×3 matrices. For each $1 \leq j \leq 3$, let q_j be the number of nonzero elements in column j of A_1. Rearrange columns so that $q_1 \geq q_2 \geq q_3$. Then $q_3 = 1$ by Lemma 3. Similarly A_2 has a column with only one nonzero element. Thus A has a partition into submatrices of order 2 where those two nonzero elements are in the same submatrix as shown in Fig 3. Therefore the lemma is proved. Let $e = 4$. Then we may assume that A_1 is a 3×5 matrix and A_2 is a 3×1 matrix. For each $1 \leq j \leq 5$, let q_j be the number of nonzero elements in column j of A_1. Rearrange columns so that $q_1 \geq \ldots \geq q_5$. Then $q_3 + q_4 + q_5 \leq 3$ by Lemma 3. The last three columns of A_1 and the first row of A_2 together contain 4 nonzero elements and hence their direct sum is partitioned into submatrices of order 2 of which at most two contain nonzero elements. Now the first two columns of A_1 are in two submatrices and the last two rows of A_2 are in one submatrix as shown in Fig 4. Therefore the lemma follows. Let $e = 5$. Then A_1 is a 5×5 matrix and hence A_2 is an 1×1 matrix. For each $1 \leq i, j \leq 5$ let p_i and q_j be the number of nonzero elements in row i and column j of A_1, respectively. After rearranging columns of A_1, we may assume that $q_1 \geq \ldots \geq q_5$. By Lemma 3, $q_3 + q_4 + q_5 \leq 5$. If $q_3 + q_4 + q_5 \leq 3$, then the direct sum of A_2 and the matrix we obtain from A_1 by omitting first two columns has a partition into submatrices of order 2 of which at most two contain nonzero elements and hence the lemma is proved. Thus we assume that $q_3 + q_4 + q_5 \geq 4$ and hence $(q_1, q_2, q_3, q_4, q_5) = (3, 2, 2, 1, 1)$ or $(2, 2, 2, 2, 1)$. Similarly we assume that $\{p_1, p_2, p_3, p_4, p_5\} = \{3, 2, 2, 1, 1\}$ or $\{2, 2, 2, 2, 1\}$. As the bipartite graph corresponding to A_1 is a tree, it is enough to consider the case where it contains a P_3-claw by Lemma 5 and Lemma 6. We may assume that $\{p_1, p_2, p_3, p_4, p_5\} = \{3, 2, 2, 1, 1\}$ and some $u \in U$ is the center of a P_3-claw. Firstly assume that $(q_1, q_2, q_3, q_4, q_5) = (3, 2, 2, 1, 1)$. Then the neighbors of u have degrees 3, 2, 2 and hence we may assume that A_1 contains a submatrix which is shown in Fig 5. Note that two nonzero elements in the last two columns are in the right top part and the right bottom part. If they are in one part, then the direct sum of A_2 and the left middle part is a submatrix of order 2 and hence the lemma follows as shown in Fig $5'$. Otherwise we may assume that they are in the second row and the fourth row and hence A_2 is partitioned as shown in Fig $5''$ after permuting the second row and the third row. Thus the lemma follows. Secondly assume that $(q_1, q_2, q_3, q_4, q_5) = (2, 2, 2, 2, 1)$. Then all the neighbor of u has degree 2 and hence we may assume that A_1 contains a submatrix which is shown in Fig 6. Thus A_1 is partitioned as shown in Fig $6'$ or $6''$ and the lemma follows.

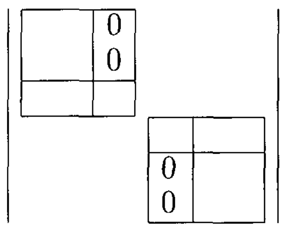

FIGURE 3

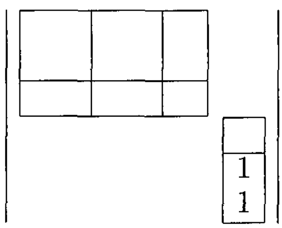

FIGURE 4

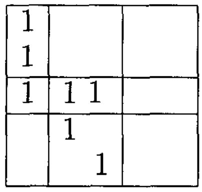

FIGURE 5

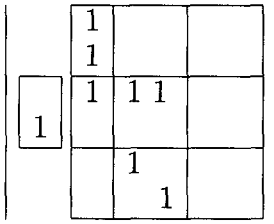

FIGURE 5′

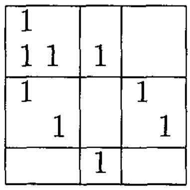

FIGURE 5″

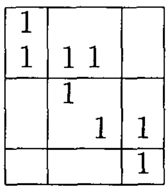

FIGURE 6

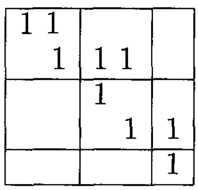

FIGURE 6′

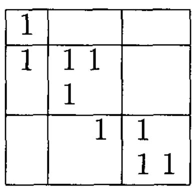

FIGURE 6″

Subcase 1b: G has one trivial component. Then we may assume that after omitting a zero column, A is decomposed into A_1 and A_2 where A_1 is an $a \times b$ matrix and A_2 is a $(6-a) \times (5-b)$ matrix for some $0 < a < 6$ and $0 < b < 5$. It is enough to consider the case where $a \equiv 1 \pmod 2$ and A_1 contains $2e-1$ nonzero elements for some $1 \leq e \leq 5$. We may assume that the bipartite graphs corresponding to A_1 and A_2 are an unicyclic graph and a tree, respectively and hence $e \geq 3$. When $e = 3$ or 4, A_1 and A_2 are same with those in Subcase 1a except that A_1 has one less column and hence by the same argument as in Subcase 1a, the lemma is proved. Assume that $e = 5$. Then A_1 is a 5×4 matrix and A_2 is an 1×1 matrix. For each $1 \leq i \leq 5$, let p_i be the number of nonzero elements in row i of A_1 and for each $1 \leq j \leq 4$, let q_j be the number of nonzero elements in column j of A_1. After rearranging columns we may assume that $q_1 \geq \ldots \geq q_4$. By Lemma 3, $q_3 + q_4 \leq 4$. If $q_3 + q_4 \leq 3$, then together with a zero column, the direct sum of A_2 and the matrix we obtain from A_1 by omitting the first two columns is partitioned into submatrices of order 2 of which at most two contain nonzero elements and hence the lemma is proved. Thus we may assume that $q_3 + q_4 = 4$ and hence $(q_1, q_2, q_3, q_4) = (3, 2, 2, 2)$. Similarly we may assume that $\{p_1, p_2, p_3, p_4, p_5\} = \{3, 2, 2, 1, 1\}$ or $\{2, 2, 2, 2, 1\}$. Let $G_1 = G_1(U_1, V_1; E_1)$ be the bipartite graph corresponding to A_1. In G_1, let C be the cycle and H be the graph induced by the vertices which are not in C. Consider the following three subcases.

Subcase 1ba: C has size 4. Then the adjacency matrix of C is a submatrix of order 2. As at most two vertices of C have degree 3, there are at most two edges connecting C and H. After subtracting them, A_1 is decomposed into an adjacency matrix of C and an adjacency matrix of H. Note that A_2 and the adjacency matrix of H together have 4 rows. Assume that there is one edge connecting C and H. Then the direct sum of A_2 and the adjacency matrix of H contains 5 nonzero elements and hence together with a zero column, it has a partition into submatrices of order 2 of which at most three contain nonzero elements by applying $KM(2,2,3,2)$. Thus the lemma is proved. When there are two edges connecting C and H, the lemma is proved similarly.

Subcase 1bb: C has size 6. Firstly if some vertex in V_1 which is in C has degree 3, then A_1 is partitioned as shown in Fig 7 where only one of the two right top parts contains a nonzero element, which proves the lemma. Secondly if all the vertex in V_1 which is in C has degree 2, then the other vertex in V_1 has degree 3 and is connected to both of the vertices in U_1

which are not in C. Thus A_1 is partitioned as shown in Fig 8 and the lemma is proved.

Subcase 1bc: C has size 8. Then A_1 is partitioned as shown in Fig 9 and the lemma is proved.

$$\begin{array}{|c|cc|c|}\hline 1 & 1 & & \\ 1 & & 1 & \\ \hline & 1 & 1 & \\ & & 1 & \\ \hline & & & 1 \\ \hline \end{array}$$

FIGURE 7

$$\begin{array}{|c|cc|c|}\hline 1 & 1 & & \\ 1 & & 1 & \\ \hline & 1 & 1 & 1 \\ & & & 1 \\ \hline & & & 1 \\ \hline \end{array}$$

FIGURE 8

$$\begin{array}{|c|cc|c|}\hline 1 & & 1 & \\ 1 & 1 & & \\ \hline & 1 & & 1 \\ & & 1 & 1 \\ \hline & & & 1 \\ \hline \end{array}$$

FIGURE 9

Case 2: G has one nontrivial component. Then we need to consider the subcases where G has one or two trivial components.

Subcase 2a: G has one trivial component. Then we may assume that an adjacency matrix of the nontrivial component is a 5×6 matrix. By a similar argument as in Subcase 1a with $e = 5$, the lemma is proved.

Subcase 2b: G has two trivial components. Then as $KM(2,3,5,2)$ is true, it is enough to consider the case where an adjacency matrix of the nontrivial component is a 5×5 matrix. By a similar argument as in Subcase 1b with $e = 5$, the lemma is proved.

LEMMA 13. *$KM(3,4,5,2)$ is true.*

Proof. The statement of lemma is true by Lemma 10 and Lemma 12.

The following two lemmas are extended from and proved similarly to Lemma 3.3 in Brualdi et al.[1].

LEMMA 14. [1] *For $k \geq 4$, let A be a $kt \times kt$ matrix with kt nonzero elements. If A has at most $3t-1$ zero rows and columns, then A can be partitioned into submatrices of order t of which at most k contain nonzero elements.*

LEMMA 15. [1] *For $k \geq 5$, let A be a $kt \times kt$ matrix with kt nonzero elements. If A has at most $4t-1$ zero rows and columns, then A can be partitioned into submatrices of order t of which at most k contain nonzero elements.*

THEOREM 16. *$KM(m, n, k, 2)$ is true if $k \leq 5$.*

Proof. As the assertion is true if $k \leq 3$ by Brualdi et al.[1], we only need to prove that it is true for $k = 4$ or $k = 5$. In each case, as in the proof of Theorem 3.4. of Brualdi et al.[1], $KM(k, k, k, 2)$ will suffice. Assume that $k = 4$. Let A be an 8×8 matrix with 8 nonzero elements. If A has at most 5 zero rows and columns, then the theorem follows by Lemma 14. Otherwise A has at least 6 zero rows and columns and hence the theorem follows by Lemma 11 considering that $KM(2, 4, 4, 2)$ is true. When $k = 5$, the theorem follows similarly by Lemma 13 and Lemma 15 considering that $KM(2, 5, 5, 2)$ is true. □

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