

RELATIVE SEQUENCE ENTROPY PAIRS FOR A MEASURE AND RELATIVE TOPOLOGICAL KRONECKER FACTOR

YOUNG-HO AHN*, JUNGSEOB LEE** AND KYEWON KOH PARK***

ABSTRACT. Let (X, \mathcal{B}, μ, T) be a dynamical system and (Y, \mathcal{A}, ν, S) be a factor. We investigate the relative sequence entropy of a partition of X via the maximal compact extension of (Y, \mathcal{A}, ν, S) . We define relative sequence entropy pairs and using them, we find the relative topological μ -Kronecker factor over (Y, ν) which is the maximal topological factor having relative discrete spectrum over (Y, ν) . We also describe the topological Kronecker factor which is the maximal factor having discrete spectrum for any invariant measure.

1. Introduction

A topological flow is a pair (X, T) where X is a compact metric space and T is a homeomorphism of X to itself. A topological factor of the flow is a pair (Y, S) where there exists a map $\pi : (X, T) \rightarrow (Y, S)$ which is continuous, onto, and satisfies the intertwining relation $\pi \circ T = S \circ \pi$. We have the corresponding definitions in measure theoretic dynamical systems. Let (X, \mathcal{B}, μ, T) be a measure preserving dynamical system. If there exists a measure preserving map $\phi : (X, \mathcal{B}, \mu) \rightarrow (Y, \mathcal{A}, \nu)$ which satisfies the intertwining relation $\phi \circ T = S \circ \phi$, then (Y, \mathcal{A}, ν, S) is called a factor of (X, \mathcal{B}, μ, T) .

In the metric theory of dynamical system, i.e., measure theoretical dynamical system, it is well known that there exists the largest measure theoretical dynamical system $(Z, \mathcal{C}, \eta, F)$ between (X, \mathcal{B}, μ, T) and

Received November 18, 2004.

2000 Mathematics Subject Classification: 28D05, 47A35.

Key words and phrases: relative sequence entropy, relative sequence entropy pairs, relative weakly mixing, compact extension, relative Kronecker factor, equicontinuous factor, null factor.

*Partially supported by Korea Research Foundation Grant(KRF-2002-037-C00002). **Partially supported by Ajou University faculty grant. ***Partially supported by KOSEF(R05-2004-000-10864-0).

(Y, \mathcal{A}, ν, S) which satisfies $h(Z, F) = h(Y, S)$. That is, any factor of (X, \mathcal{B}, μ, T) which is an extension of (Y, \mathcal{A}, ν, S) having the same entropy with (Y, S) is a factor of $(Z, \mathcal{C}, \eta, F)$. We call $(Z, \mathcal{C}, \eta, F)$ the *relative Pinsker factor* of (X, \mathcal{B}, μ, T) with respect to (Y, \mathcal{A}, ν, S) . In general, the relative Pinsker factor does not have good topological structures. There have been attempts to extend the measure theoretic dynamical properties to topological dynamics. For example, the notion of entropy pairs was introduced to find topological property corresponding to K-mixing which implies mixing of all orders ([2, 3, 9, 10]). Recall that a pair of points $(x, x') \in X^2$ is called an entropy pair if any standard cover distinguishing x and x' has positive entropy ([2]). The topological Pinsker factor is defined via the entropy pairs.

Furstenberg and Zimmer's structure theorem says that if (Y, \mathcal{A}, ν, S) is a measure theoretic factor of (X, \mathcal{B}, μ, T) , then there exists the maximal relative Kronecker factor $(Z, \mathcal{C}, \eta, F)$ of (X, \mathcal{B}, μ, T) which is an extension of (Y, \mathcal{A}, ν, S) . Namely, if $(Z', \mathcal{C}', \eta', F')$ has discrete spectrum relative to (Y, \mathcal{A}, ν, S) , then $(Z, \mathcal{C}, \eta, F)$ is an extension of $(Z', \mathcal{C}', \eta', F')$. We also call this maximal relative Kronecker factor as maximal compact extension of (Y, \mathcal{A}, ν, S) . The maximal relative Kronecker factor does not have good topological structures as in the case of the relative Pinsker factor. For a given measure μ on X , we find the maximal topological factor having relative discrete spectrum over (Y, ν) . We call it the *relative topological μ -Kronecker factor* over (Y, ν) . We will also find the *topological Kronecker factor* which is independent of its invariant measures and different from the *maximal null factor*.

2. Sequence entropy and weakly mixing

Let (X, \mathcal{B}, μ, T) be an invertible measure theoretic dynamical system on a standard Borel space and let (Y, \mathcal{A}, ν, S) be a factor. We denote the fiber measures by $\{\mu_y\}_{y \in Y}$. We recall the definition of conditional entropy. Given a partition ξ , let

$$H_{\mathcal{A}}(\xi) = \int H_y(\xi) d\nu$$

where $H_y(\cdot)$ denotes the entropy with respect to the fiber measure μ_y . Note that $H_{\mathcal{A}}(\xi) = H(\xi|\mathcal{A})$.

We also define $H_{\mathcal{A}}(\xi|\beta)$ the conditional entropy relative to a finite partition β as

$$H_{\mathcal{A}}(\xi|\beta) = \int H_y(\xi|\beta) d\nu$$

where $H_y(\cdot|\beta)$ denotes the conditional entropy with respect to μ_y . When \mathcal{D} is a sub σ -algebra, then $H_{\mathcal{A}}(\xi|\mathcal{D})$ is similarly defined.

Let $\Gamma = \{t_n\}_{n=1}^\infty$ be an increasing sequence of natural numbers. Recall that Γ -entropy of T relative to \mathcal{A} is defined as follows. Let ξ be a finite partition of X . then

$$h_{\mathcal{A}}^\Gamma(T, \xi) = \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\mathcal{A}}\left(\bigvee_{i=1}^n T^{-t_i} \xi\right)$$

and

$$h_{\mathcal{A}}^\Gamma(T) = \sup_{\xi} \{h_{\mathcal{A}}^\Gamma(T, \xi)\},$$

where the supremum is taken over all finite partitions.

If \mathcal{A} is trivial then $h_{\mathcal{A}}^\Gamma(T)$ is just the usual sequence entropy. If $\Gamma = \{n\}$ and T is a skew product with base (Y, S) , then $h_{\mathcal{A}}^\Gamma(T)$ corresponds to Abramov's and Rokhlin's mixed entropy ([8]).

Recall that T is ergodic relative to a sub σ -algebra \mathcal{C} if all T -invariant functions are \mathcal{C} -measurable. We say that T has discrete spectrum relative to \mathcal{A} if (X, T) is a compact extension of (Y, S) and T is weakly mixing relative to \mathcal{A} if one of the followings holds.

- (i) $T \times_Y T$ is ergodic relative to (Y, S) .
- (ii) $\frac{1}{n} \sum_{i=0}^{n-1} |E(fT^i \cdot g|\mathcal{A})| \rightarrow 0$ a.e. μ and in $L^1(X, \mu)$ for all $f \in L^2(\mathcal{A})^\perp, g \in L^2(X, \mu)$ where $E(\cdot|\mathcal{A})$ is the conditional expectation operator with respect to \mathcal{A} .

We will denote the maximal relative Kronecker factor of (X, \mathcal{B}, μ, T) with respect to a factor (Y, \mathcal{A}, ν, S) by $(X, \mathcal{K}_{(Y, \mu)}, \mu, T)$. If there is no confusion, we simply denote it by (X, \mathcal{K}, μ, T) .

The following theorems are well known ([8]).

THEOREM 1. *Let (X, \mathcal{B}, μ, T) be a dynamical system and (Y, \mathcal{A}, ν, S) be a factor of it. Then T has discrete spectrum relative to \mathcal{A} if and only if $h_{\mathcal{A}}^\Gamma(T) = 0$ for every increasing sequence Γ of natural numbers.*

THEOREM 2. *Let (X, \mathcal{B}, μ, T) be a dynamical system and (Y, \mathcal{A}, ν, S) be a factor. Then $L^2(X, \mathcal{B}, \mu)$ can be decomposed as*

$$L^2(X, \mathcal{B}, \mu) = H_{Y, \mathcal{K}} \oplus H_{Y, w\mu},$$

where

$$H_{Y,K} = L^2(X, \mathcal{K}_{(Y,\mu)}, \mu)$$

and

$$H_{Y,wm} = \{f \in L^2(X, \mathcal{B}, \mu) : \frac{1}{n} \sum_{i=0}^{n-1} |E(fT^i \cdot g|\mathcal{A})| \rightarrow 0$$

a.e., μ and in $L^1(X, \mu)$ for all $g \in L^2(X, \mathcal{B}, \mu)\}$.

The following two lemmas are the relativized versions of Lemma 2.2 and Theorem 2.3 in [7].

LEMMA 1. Let (X, \mathcal{B}, μ, T) be a measure theoretic dynamical system and \mathcal{A} be a T -invariant sub σ -algebra of \mathcal{B} . For any finite measurable partition α of X and any increasing sequence of natural numbers Γ , $h_{\mathcal{A}}^{\Gamma}(T, \alpha) \leq H_{\mathcal{A}}(\alpha|\mathcal{K}_{(Y,\mu)})$

Proof. Let $\{\beta_k : k \in \mathbb{N}\}$ be a sequence of $\mathcal{K}_{(Y,\mu)}$ -measurable partitions such that $\lim_{n \rightarrow \infty} h_{\mathcal{A}}(\alpha|\beta_k) = h_{\mathcal{A}}(\alpha|\mathcal{K}_{(Y,\mu)})$. Thus for a fixed $k \in \mathbb{N}$ and $\Gamma = \{t_n\}_{n=1}^{\infty}$,

$$\begin{aligned} h_{\mathcal{A}}^{\Gamma}(T, \alpha) &= \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\mathcal{A}}\left(\bigvee_{i=1}^n T^{-t_i} \alpha\right) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \int H_y\left(\bigvee_{i=1}^n T^{-t_i} \alpha\right) d\nu(y) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \int \left(H_y\left(\bigvee_{i=1}^n T^{-t_i} (\alpha \vee \beta_k)\right) - H_y\left(\bigvee_{i=1}^n T^{-t_i} \beta_k\right)\right) d\nu(y) \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \int H_y\left(\bigvee_{i=1}^n T^{-t_i} \alpha \middle| \bigvee_{i=1}^n T^{-t_i} \beta_k\right) d\nu(y) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \int \sum_{j=1}^n H_y(T^{-t_j} \alpha \middle| \bigvee_{i=1}^n T^{-t_i} \beta_k) d\nu(y) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \int \sum_{i=1}^n H_y(T^{-t_i} \alpha \middle| T^{-t_i} \beta_k) d\nu(y) \\ &= \int H_y(\alpha|\beta_k) d\nu(y) \\ &= H_{\mathcal{A}}(\alpha|\beta_k), \end{aligned}$$

where in the third equality we use $h_{\mathcal{A}}^{\Gamma}(T, \beta_k) = 0$ for $\beta_k \subseteq \mathcal{K}_{(Y,\mu)}$ from Theorem 1. Since k is arbitrary, the proof is complete. \square

LEMMA 2. Let (X, \mathcal{B}, μ, T) be a measure theoretic dynamical system and \mathcal{A} be a T -invariant sub σ -algebra of \mathcal{B} . Let α be a finite measurable partition of X . Then there exists an increasing sequence of natural numbers $\Gamma = \{t_n\}_{n=1}^\infty$ such that $h_{\mathcal{A}}^{\Gamma}(T, \alpha) = H_{\mathcal{A}}(\alpha|\mathcal{K}_{(Y, \mu)})$

Proof. In this proof, we briefly denote $\mathcal{K}_{(Y, \mu)}$ by \mathcal{K} . Let $\alpha = \{A_1, \dots, A_k\}$ and $\beta = \{B_1, \dots, B_l\}$ be two finite partitions of X . We will first show that for an $\epsilon > 0$, there exists an integer m such that

$$H_{\mathcal{A}}(T^{-m}\alpha|\beta) \geq H_{\mathcal{A}}(\alpha|\mathcal{K}) - \epsilon.$$

Since we have $\mathbf{1}_A - E(\mathbf{1}_{A_i}|\mathcal{K}) \in H_{Y,wm}$ for any $A \in \mathcal{B}$, we may assume that there exists a subset S of \mathbb{N} with density 1, such that

$$\lim_{n \in S} \int_Y | \langle U_T^n \mathbf{1}_{A_i} - E(\mathbf{1}_{A_i}|\mathcal{K}), \mathbf{1}_{B_j} \rangle_y | d\nu(y) = 0$$

for all $1 \leq i \leq k, 1 \leq j \leq l$.

By Jensen's inequality, we deduce that for each y ,

$$\begin{aligned} & - \langle U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}), \mathbf{1}_{B_j} \rangle_y \log \left(\frac{\langle U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}), \mathbf{1}_{B_j} \rangle_y}{\mu_y(B_j)} \right) \\ & \geq \int_{B_j} -U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}) \log(U_T^n E(\mathbf{1}_{A_i}|\mathcal{K})) d\mu_y. \end{aligned}$$

Hence

$$\begin{aligned} & \liminf_{n \in S} H_{\mathcal{A}}(T^{-n}\alpha|\beta) \\ & = \liminf_{n \in S} \int \sum_{i,j} - \langle U_T^n \mathbf{1}_{A_i}, \mathbf{1}_{B_j} \rangle_y \log \left(\frac{\langle U_T^n \mathbf{1}_{A_i}, \mathbf{1}_{B_j} \rangle_y}{\mu_y(B_j)} \right) d\nu(y) \\ & = \liminf_{n \in S} \int \sum_{i,j} - \langle U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}), \mathbf{1}_{B_j} \rangle_y \\ & \quad \times \log \left(\frac{\langle U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}), \mathbf{1}_{B_j} \rangle_y}{\mu_y(B_j)} \right) d\nu(y) \\ & \geq \int \left(\sum_{i,j} \int_{B_j} -U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}) \log(U_T^n E(\mathbf{1}_{A_i}|\mathcal{K})) d\mu_y \right) d\nu(y) \\ & = \int \left(\sum_i \int_X -U_T^n E(\mathbf{1}_{A_i}|\mathcal{K}) \log(U_T^n E(\mathbf{1}_{A_i}|\mathcal{K})) d\mu_y \right) d\nu(y) \\ & = H_{\mathcal{A}}(\alpha|\mathcal{K}). \end{aligned}$$

Thus we have

$$\liminf_{n \in S} H_{\mathcal{A}}(T^{-n}\alpha|\beta) \geq H_{\mathcal{A}}(\alpha|\mathcal{K}).$$

Now we can choose an increasing sequence of natural numbers $\Gamma = \{0 \leq t_1 < t_2 < \dots\}$ such that

$$H_{\mathcal{A}}(T^{-t_n}\alpha|\bigvee_{i=1}^{n-1} T^{-t_i}\alpha) \geq H_{\mathcal{A}}(\alpha|\mathcal{K}) - \frac{1}{2^n}.$$

Hence

$$\begin{aligned} h_{\mathcal{A}}^{\Gamma}(T, \alpha) &= \limsup_{n \rightarrow \infty} \frac{1}{n} H_{\mathcal{A}}\left(\bigvee_{i=1}^n T^{-t_i}\alpha\right), \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H_{\mathcal{A}}\left(T^{-t_k}\alpha|\bigvee_{i=1}^{k-1} T^{-t_i}\alpha\right), \\ &\geq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(H_{\mathcal{A}}(\alpha|\mathcal{K}) - \frac{1}{2^k}\right), \\ &= H_{\mathcal{A}}(\alpha|\mathcal{K}). \end{aligned} \quad \square$$

By the previous two lemmas, it is easy to obtain the following theorem.

THEOREM 3. *Let (X, \mathcal{B}, μ, T) be a measure theoretic dynamical system and \mathcal{A} be a T -invariant sub σ -algebra of \mathcal{B} . Let α be a finite measurable partition of X . Then we have*

$$\sup_{\Gamma} h_{\mathcal{A}}^{\Gamma}(T, \alpha) = H_{\mathcal{A}}(\alpha|\mathcal{K}_{(Y, \mu)}).$$

3. Relative sequence entropy pairs and topological relative μ -Kronecker factor

The topological Pinsker factor which is the maximal factor of zero topological entropy, was defined via topological entropy pairs ([2]). That is, it is the largest factor without topological entropy pairs. It is shown that there exists a measure μ so that the set of μ -entropy pairs is the set of topological entropy pairs ([1]). It is clear that the topological Pinsker factor is the maximal topological factor which has measure theoretic entropy zero for any invariant measures ([1, 9, 10]).

Denote by $\langle A \rangle$ the smallest closed $T \times T$ -invariant equivalence relation containing A . It is well known that there is a one to one correspondence

between topological factors of a topological dynamical system (X, T) and the closed $T \times T$ -invariant equivalence relations on $X \times X$ ([2]).

Let $SE(X, T)$ be the set of topological sequence entropy pairs. Recall that a *null factor* is a factor which has no topological sequence entropy pairs, and it is a factor of $X/\langle SE(X, T) \rangle$ which is the maximal null factor. We define the *topological Kronecker factor* as the largest topological factor which has discrete spectrum for any invariant measure ([6]). We will see in Theorem 6 that the topological Kronecker factor is different from the maximal null factor.

Let (X, T) be a topological flow and $\pi : (X, T) \rightarrow (Y, S)$ a factor map. Unlike the relative equicontinuous factor, we could not yet define the relative null factor, because there is no clear notion of relative topological entropy. For a given T -invariant measure μ , we define the *relative topological μ -Kronecker factor* which has relative discrete spectrum with respect to given factor (Y, ν) . Using these factors, we will describe the *relative topological Kronecker factor* which is the largest topological factor having relative discrete spectrum over (Y, S) for any invariant measure.

DEFINITION 1. Let (X, T) be a topological flow and $\pi : (X, T) \rightarrow (Y, S)$ a factor map. Given a T invariant probability measure μ on (X, \mathcal{B}) , a pair $(x, y) \in X \times X$, $x \neq y$ is called a *relative sequence entropy pair for μ* with respect to the factor (Y, \mathcal{A}) if for any Borel partition $\mathcal{F} = \{F, F^c\}$ of X with $x \in \text{Int}(F)$ and $y \in \text{Int}(F^c)$, there exist an increasing sequence Γ such that $h_{\mathcal{A}}^{\Gamma}(T, \mathcal{F}) > 0$. This is equivalent to saying that $H_{\mathcal{A}}(\mathcal{F} | \mathcal{K}_{(Y, \mu)}) > 0$ by Theorem 3, i.e., \mathcal{F} is not measurable in the relative Kronecker factor $\mathcal{K}_{(Y, \mu)}$. We denote the set of relative sequence entropy pairs by $SE_{(Y, \mu)}$.

When it is clear, we denote $(X, \mathcal{K}_{(Y, \mu)}, \mu)$ by (Z, m) .

Let $\mu = \int_Z \mu_z dm(z)$ be the disintegration of μ over (Z, m) . Set

$$\lambda_{\mu} = \int_Z (\mu_z \times \mu_z) dm(z),$$

the relatively independent joining of μ over m . Finally let Λ_{μ} be the topological support of λ_{μ} .

THEOREM 4. For a measure $\mu \in M_T(X)$, $SE_{(Y, \mu)} = \Lambda_{\mu} \setminus \Delta$ where Δ is the diagonal of $X \times X$.

Proof. We follow the idea of Glasner in describing entropy pairs ([5]). Suppose $(x, y) \notin SE_{(Y, \mu)} \cup \Delta$, then there exist a Borel partition $\mathcal{F} =$

$\{F, F^c\}$ of X with $x \in \text{Int}(F)$ and $y \in \text{Int}(F^c)$ and such that $H_{\mathcal{A}}(\mathcal{F}|\mathcal{K}) = 0$. This implies that F is \mathcal{K} -measurable. Hence

$$\lambda_{\mu}(F \times F^c) = \int \mu_z(F)\mu_z(F^c) dm(z) = 0.$$

Thus $(x, y) \notin \Lambda_{\mu}$.

Conversely, suppose $(x, y) \notin \Lambda_{\mu} \cup \Delta$. Then there exist disjoint open neighborhoods A and B of x and y respectively with

$$0 = \lambda_{\mu}(A \times B) = \int \mu_z(A)\mu_z(B) dm(z) = \int E(\mathbf{1}_A|\mathcal{K})E(\mathbf{1}_B|\mathcal{K}) d\mu(x).$$

Now by the property of conditional expectation operator, this implies the existence of a Borel subset F of X such that $A \subset F, B \subset F^c$ and F is in \mathcal{K} . Thus $(x, y) \notin SE_{(Y, \mu)}$. Hence the proof is complete. \square

Let R_{μ} be the $T \times T$ -invariant equivalence relation generated by topological support Λ_{μ} , i.e., $R_{\mu} = \langle \Lambda_{\mu} \rangle$.

THEOREM 5. *If W is a topological factor of X containing Y which has relative discrete spectrum over Y then it is a topological factor of X/R_{μ} . That is, X/R_{μ} is the largest topological factor having relative discrete spectrum over (Y, ν) .*

Proof. Let $\pi_{(Y, \mu)} : (X, \mathcal{B}, \mu) \rightarrow (Z, \mathcal{K}_{(Y, \mu)}, m)$ be the measure theoretical factor map. By $P_{\pi_{(Y, \mu)}}$ denote the measure theoretic $T \times T$ -invariant equivalence relation given by $(x, x') \in P_{\pi_{(Y, \mu)}}$ if $\pi_{(Y, \mu)}(x) = \pi_{(Y, \mu)}(x')$. It is easy to see that $\langle P_{\pi_{(Y, \mu)}} \rangle = R_{\mu}$. Take a topological factor $\pi' : X \rightarrow W$ with W having relative discrete spectrum over Y . Since $\mathcal{K}_{(Y, \mu)}$ is the maximal measure theoretical relative Kronecker factor, $P_{\pi_{(Y, \mu)}} \subset P_{\pi'} \pmod{\lambda}_{\mu}$. Hence $R_{\mu} \subset P_{\pi'}$, which means that W is a topological factor of X/R_{μ} . \square

DEFINITION 2. Let (X, \mathcal{B}, μ, T) be a measure theoretic dynamical system and (Y, \mathcal{A}, ν, S) be a factor. We will call X/R_{μ} the *relative topological μ -Kronecker factor* over (Y, ν) .

Let $M_T(X)$ be the set of T -invariant measures on X . By the previous theorems, it is immediate to get the following corollary.

COROLLARY 1. *Let (X, T) be a topological flow and $\pi : (X, T) \rightarrow (Y, S)$ a factor map.*

- (i) *If we fix an invariant measure ν on Y , then the maximal topological factor having relative discrete spectrum for any measure*

whose projection on Y is ν , i.e., the relative topological Kronecker factor over (Y, ν) has the form $X/R(\nu)$ where $R(\nu) = \langle \bigcup_{\mu \in M_T(X), \mu|_Y = \nu} R_\mu \rangle$.

Furthermore $R(\nu) = \langle \bigcup_{\mu \in M_T(X), \mu|_Y = \nu} SE_{(Y, \mu)} \rangle$.

- (ii) The relative topological Kronecker factor over (Y, S) has the form X/R where R is the closed invariant equivalence relation generated by $\{R(\nu) : \nu \in M_S(Y)\}$, that is, $R = \langle \bigcup_{\nu \in M_S(Y)} R(\nu) \rangle$.

Recall that for a dynamical system (X, T) , $(x_1, x_2) \in X \times X \setminus \Delta$ is a weakly mixing pair if for any open neighborhood U_i of x_i , $i = 1, 2$, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$, where $N(U, V) = \{n \in \mathbb{N} : U \cap T^{-n}V \neq \emptyset\}$. The set of weakly mixing pairs is denoted by $WM(X, T)$. It is well known that the maximal equicontinuous factor of a flow (X, T) has the form $X/\langle WM(X, T) \rangle$ ([6]).

Consider an Anzai skew product transformation T on $X = [0, 1) \times [0, 1)$ defined by

$$(x, y, z) \mapsto (x + \alpha, y + x) \pmod{1}$$

where α is irrational. Let (Y, S) be a factor of (X, T) which is restricted to the first coordinate. The topological Kronecker factor is (Y, S) and the relative topological Kronecker factor over (Y, S) is (X, T) . This example has the property that the topological Kronecker factor is the same with the maximal equicontinuous factor. However we have the following theorem.

THEOREM 6. *Let (X, T) be a topological dynamical system. Let (Y_1, T) , (Y_2, T) and (Y_3, T) be the topological Kronecker factor, the maximal null factor and the maximal equicontinuous factor respectively. Then we have the factor relations, $(Y_1, T) \rightarrow (Y_2, T) \rightarrow (Y_3, T)$. In general, these three factors are different from each other.*

Proof. Since measure theoretical sequence entropy pairs is a subset of topological sequence entropy pairs, the first factor relation follows by Theorems 4 and Corollary 1 ([7]). The second factor relation follows from the fact that the set of topological sequence entropy pairs is a subset of the set of weakly mixing pairs.

Since there exists a uniquely ergodic model which has discrete spectrum with non empty topological sequence entropy pairs ([5]), it is clear that the topological Kronecker factor is not the same with the maximal null factor. There are substitutions which make shift spaces without topological sequence entropy pairs ([5]). Since every shift space has weakly mixing pairs, Y_3 can be a proper factor of Y_2 . \square

REMARK 1. Although we do not have the complete generalization of the above theorem to the relative case, it is not hard to see that the relative equicontinuous factor is a factor of the relative topological Kronecker factor.

For a T -invariant measure μ , let $S(\mu)$ be the topological support of μ and $S^2(\mu) = \{(x, x) : x \in S(\mu)\}$. Let $(X_i, \mathcal{B}_i, \mu_i)$ be extensions of (Y, \mathcal{C}, ν) for $i = 1, 2$. Note that the measure $(\mu_1 \times_\nu \mu_1) \times_\nu (\mu_2 \times_\nu \mu_2)$ on $(X_1 \times X_1) \times (X_2 \times X_2)$ can be identified with the measure $(\mu_1 \times_\nu \mu_2) \times_\nu (\mu_1 \times_\nu \mu_2)$ on $(X_1 \times X_2) \times (X_1 \times X_2)$ via appropriate coordinate exchanges. In the following, when it is clear, we use this coordinate identification without mentioning it.

THEOREM 7. Let (X_1, T) and (X_2, T) be two topological systems with invariant measures μ_i , and let (Y, ν) be a common factor of two flows X_1 and X_2 via the homomorphisms π_i , $i = 1, 2$. Then

- (1) $\lambda_{\mu_1 \times_\nu \mu_2} = \lambda_{\mu_1} \times_\nu \lambda_{\mu_2}$, whence $\Lambda_{\mu_1 \times_\nu \mu_2} = \Lambda_{\mu_1} \times_\nu \Lambda_{\mu_2}$.
- (2) $SE_{(Y, \mu_1 \times_\nu \mu_2)} = (SE_{(Y, \mu_1)} \times_\nu SE_{(Y, \mu_2)}) \cup (SE_{(Y, \mu_1)} \times_\nu S^2(\mu_2)) \cup (S^2(\mu_1) \times_\nu SE_{(Y, \mu_2)})$.

Proof. (1) As in the case of Y consisting of a single point, it is known that the relative Kronecker factor of a relative product is the relative product of the relative Kronecker factor ([4]), that is,

$$\mathcal{K}_{(Y, \mu_1 \times_\nu \mu_2)} = \mathcal{K}_{(Y, \mu_1)} \times_\nu \mathcal{K}_{(Y, \mu_2)}.$$

By the above remark, the disintegration of $\mu_1 \times_\nu \mu_2$ over $\mathcal{K}_{(Y, \mu_1 \times_\nu \mu_2)}$ is given by

$$\mu_1 \times_\nu \mu_2 = \int_{Z_1 \times_\nu Z_2} (\mu_1)_s \times (\mu_2)_t d(m_1 \times_\nu m_2)(s, t)$$

where $(Z_i, m_i) = (X_i, \mathcal{K}_{(Y, \mu_i)}, \mu_i)$ for $i = 1, 2$. Therefore we have

$$\begin{aligned} & \lambda_{\mu_1 \times_\nu \mu_2} \\ &= \int_Y (\lambda_{\mu_1})_y \times (\lambda_{\mu_2})_y d\nu(y) \\ &= \int_Y \left(\int_{Z_1} (\mu_1)_s \times (\mu_1)_s d(m_1)(s) \right)_y \\ & \quad \times \left(\int_{Z_2} (\mu_2)_t \times (\mu_2)_t d(m_2)(t) \right)_y d\mu(y) \end{aligned}$$

$$\begin{aligned}
 &= \int_{Z_1 \times_\nu Z_2} ((\mu_1)_s \times (\mu_2)_t) \times ((\mu_1)_s \times (\mu_2)_t) d(m_1 \times_\nu m_2)(s, t) \\
 &= \lambda_{\mu_1 \times_\nu \mu_2}.
 \end{aligned}$$

This proves the first statement.

(2) Now (1) implies

$$\begin{aligned}
 &SE_{(Y, \mu_1 \times \mu_2)} \\
 &= \Lambda_{\mu_1 \times_\nu \mu_2} \setminus \Delta_{X_1 \times X_2} = \Lambda_{\mu_1} \times_\nu \Lambda_{\mu_2} \setminus \Delta_{X_1 \times X_2} \\
 &= ((\Lambda_{\mu_1} \setminus \Delta_{X_1}) \times_\nu (\Lambda_{\mu_2} \setminus \Delta_{X_2})) \cup ((\Lambda_{\mu_1} \setminus \Delta_{X_1}) \times_\nu (S^2(\mu_2))) \\
 &\quad \cup ((S^2(\mu_1)) \times_\nu (\Lambda_{\mu_1} \setminus \Delta_{X_2})) \\
 &= (SE_{(Y, \mu_1)} \times_\nu SE_{(Y, \mu_2)}) \cup (SE_{(Y, \mu_1)} \times_\nu S^2(\mu_2)) \\
 &\quad \cup (S^2(\mu_1) \times_\nu SE_{(Y, \mu_2)}).
 \end{aligned}$$

Hence the proof is completed. □

In the case of a product space, we have the following.

COROLLARY 2. *Let Y be the one-point flow then we have*

- (1) $\lambda_{\mu_1 \times \mu_2} = \lambda_{\mu_1} \times \lambda_{\mu_2}$, whence $\Lambda_{\mu_1 \times \mu_2} = \Lambda_{\mu_1} \times \Lambda_{\mu_2}$.
- (2) $SE_{(\mu_1 \times \mu_2)} = (SE_{(\mu_1)} \times SE_{(\mu_2)}) \cup (SE_{(\mu_1)} \times S^2(\mu_2)) \cup (S^2(\mu_1) \times SE_{(\mu_2)})$.

Now we investigate the relative topological $\mu_1 \times_\nu \mu_2$ -Kronecker factor of $X = X_1 \times_Y X_2$. Let $A \subset X_1 \times X_1$ and $B \subset X_2 \times X_2$ be nonempty subsets containing diagonals. It is easy to see that $\langle A \times B \rangle = \langle A \rangle \times \langle B \rangle$ ([10]).

THEOREM 8. *Let (X_1, T) and (X_2, T) be two topological systems with invariant probability measures $\mu_i, i = 1, 2$, and let (Y, ν) be a common factor of two flows X_1 and X_2 . The relative topological $\mu_1 \times_\nu \mu_2$ -Kronecker factor of $X = X_1 \times_Y X_2$ with respect to (Y, ν) has the form*

$$X/R_{\mu_1 \times_\nu \mu_2} = (Z_1 \times_Y Z_2) \cup (X \setminus S(\mu_1) \times_Y S(\mu_2))$$

where Z_i is the relative topological μ_i -Kronecker factor of X_i with respect to Y for $i = 1, 2$.

Proof. Since $\Lambda_{\mu_1 \times_\nu \mu_2} \subset (S(\mu_1) \times_Y S(\mu_2)) \times_Y (S(\mu_1) \times_Y S(\mu_2))$, all points in $X \setminus S(\mu_1) \times S(\mu_2)$ have trivial equivalence classes with respect to $X/R_{\mu_1 \times_\nu \mu_2}$. Note that Λ_{μ_i} projects onto $S(\mu_i)$ and $\Delta_{S(\mu_i)} \subset \Lambda_{\mu_i}$ for

$i = 1, 2$. Theorem 7 with the relative version of the fact that $\langle A \times B \rangle = \langle A \rangle \times \langle B \rangle$ gives

$$\begin{aligned} & X / \langle \Lambda_{\mu_1 \times_\nu \mu_2} \rangle \\ &= (S(\mu_1) \times_Y S(\mu_2)) / \langle \Lambda_{\mu_1 \times_\nu \mu_2} \rangle \cup (X \setminus S(\mu_1) \times_Y S(\mu_2)) \\ &= (S(\mu_1) \times_Y S(\mu_2)) / \langle \Lambda_{\mu_1} \rangle \times_Y \langle \Lambda_{\mu_2} \rangle \cup (X \setminus S(\mu_1) \times_Y S(\mu_2)) \\ &= (S(\mu_1) / \langle \Lambda_{\mu_1} \rangle \times_Y S(\mu_2) / \langle \Lambda_{\mu_2} \rangle) \cup (X \setminus S(\mu_1) \times_Y S(\mu_2)). \quad \square \end{aligned}$$

The following is an immediate consequence of the above theorem.

COROLLARY 3. *Let Z_i be the relative topological μ_i -Kronecker factor of (X_i, μ_i) with respect to a given factor (Y, ν) , $i = 1, 2$.*

- (1) *If both μ_1 and μ_2 have full supports, then the relative topological $\mu_1 \times_\nu \mu_2$ -Kronecker factor of $X = X_1 \times_Y X_2$ with respect to (Y, ν) has the form $(Z_1 \times_Y Z_2)$.*
- (2) *Let Y be the one-point flow. The topological $\mu_1 \times \mu_2$ -Kronecker factor of $X = X_1 \times X_2$ has the form $(Z_1 \times Z_2) \cup (X \setminus S(\mu_1) \times S(\mu_2))$*

References

- [1] F. Blanchard, E. Glasner, and E. Host, *A variation on the variational principle and applications to entropy pairs*, Ergodic Theory Dynam. Systems **17** (1997), 29–43.
- [2] F. Blanchard and Y. Lacroix, *Zero entropy factors of topological flows*, Proc. Amer. Math. Soc. **119** (1993), no. 2, 985–992.
- [3] E. Glasner, *A simple characterization of the set of μ -entropy pairs and application*, Israel J. Math. **102** (1997), 13–27.
- [4] ———, *Ergodic theory via joinings*, Math. Surveys Monogr. **101** (2003).
- [5] T. Goodman *Topological sequence entropy*, Proc. London Math. Soc. **29** (1974), no. 3, 331–350.
- [6] W. Huang, S. Li, S. Shao, and X. Ye, *Null systems and sequence entropy pairs*, Ergodic Theory Dynam. Systems **23** (2003), no. 3, 1505–1523.
- [7] W. Huang, A. Maass, and X. Ye, *Sequence entropy pairs for a measure*, Ann. Inst. Fourier, to appear.
- [8] P. Hulse, *Sequence entropy relative to an invariant σ -algebra*, J. London Math. Soc. **33** (1986), no. 2, 59–72.
- [9] M. Lemańczyk and A. Siemaszko, *A note on the existence of a largest topological factors with zero entropy*, Proc. Amer. Math. Soc. **129** (2001), 475–485.
- [10] K. K. Park and A. Siemaszko, *Relative topological Pinsker factors and entropy pairs*, Monatsh. Math. **134** (2001), 67–79.

Young-Ho Ahn
Department of Mathematics
Mokpo National University
Muan 534-729, Korea
E-mail: youngho.ahn@gmail.com

Jungseob Lee and Kyewon Koh Park
Department of Mathematics
Ajou University
Suwon 442-749, Korea
E-mail: jslee@madang.ajou.ac.kr
kkpark@madang.ajou.ac.kr