J. Korean Math. Soc. **42** (2005), No. 4, pp. 847–856

ON TWO GRAPH PARTITIONING QUESTIONS

YOOMI RHO

ABSTRACT. M. Jünger, G. Reinelt, and W. R. Pulleyblank asked the following questions ([2]). (1) Is it true that every simple planar 2-edge connected bipartite graph has a 3-partition in which each component consists of the edge set of a simple path? (2) Does every simple planar 2-edge connected graph have a 3-partition in which every component consists of the edge set of simple paths and triangles? The purpose of this paper is to provide a positive answer to the second question for simple outerplanar 2-vertex connected graphs and a positive answer to the first question for simple planar 2-edge connected bipartite graphs one set of whose bipartition has at most 4 vertices.

1. Introduction

Let G be a simple graph, where each edge connects two different vertices and at most one edge connects two given vertices. Let u, v be vertices of G. Then $\overline{uv}$ denotes an *edge* connecting u and v, and $u - v$ denotes a *path* connecting u and v. In each case, u and v are called *endvertices.* G is called *planar* if it can be drawn in the plane without edge crossings. Especially G is called *outerplanar* if it has an embedding in the plane such that every vertex lies on the unbounded face. G is called *2-edge connected* if we need to delete at least two edges to disconnect it. G is called *2-vertex connected* if we need to delete at least two vertices to disconnect it. The *degree* of a vertex is the number of edges containing it. G is called *bipartite* if its vertex set can be partitioned into two sets where no edge connects two vertices of one set. When those two sets have orders m and n, respectively, we denote G by $B_{m,n}$. An *ear decomposition* of G is a partition of the edge set of G

Received October 12, 2004.

2000 Mathematics Subject Classification: Primary 05C70; Secondary 05C38, 05C40.

Key words and phrases: planar graphs, outerplanar graphs, 2-edge connected graphs, 2-vertex connected graphs, bipartite graphs, 3-partitions.

This work was supported by Incheon University in the program, 2003.

into a cycle C and paths $P_1, \ldots, P_k$ such that P_i has only its endvertices on $C \cup P_1 \cup \ldots \cup P_{i-1}$ for each $i = 1, \ldots, k$.

For each positive integer s, M. Jünger, G. Reinelt, and W. R. Pulleyblank defined an s-partition of G as a partition of its edge set E into $k = \lceil \frac{|E|}{s} \rceil$ sets $E_1, \ldots, E_k$, where $| E_i |= s$ for $i = 1, \ldots, k-1$ and $| E_k | \leq s$ such that each subgraph induced by E_i is connected for $i = 1, \ldots, k$. They called E_k a small part if $| E_k |< s$. They proved that

(i) if G is connected, then there exists a 2-partition, but not necessarily a 3-partition;
(ii) if G is 2-edge connected, then there exists a 3-partition, but not necessarily a 4-partition;
(iii) if G is 3-edge connected, then there exists a 4-partition;
(iv) if G is 4-edge connected, then there exists an s-partition for all s.

They also asked four questions (see [2]). The purpose of this paper is to prove two theorems, which answer two of those questions in certain cases.

2. Outerplanar 2-vertex connected graphs

LEMMA 1. (see [3]) *A graph is* 2*-vertex connected if and only if it has an ear decomposition.*

This lemma leads to the following lemma considering the definition of an outerplanar graph.

LEMMA 2. *Let G be an outerplanar* 2*-vertex connected graph. Then G is a cycle with some chords of order* 1*, where no two of them are crossing.*

Let G be an outerplanar 2-vertex connected graph. A path $u - v$ on the cycle which connects endvertices of a chord is called an *end-path* if it doesn't intersect with other chords. In this case, the chord $\overline{uv}$ is called an *end-chord.* G has at most two end-chords and at most two end-paths.

Consider the following example in Fig 1. The end-chords are $\overline{15}$ and $\overline{34}$. The end-paths are A and B. We call A, the *left end-chord* and B, the *right end-chord.*

The following theorem answers the second question of M. Jünger, G. Reinelt and W. R. Pulleyblank in the case that G is a simple outerplanar 2-vertex connected graph.

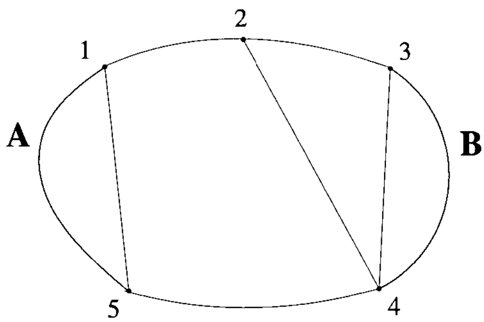

FIGURE 1

THEOREM 3. *Let G be a outerplanar 2-vertex connected graph. Then for each end-path of G, G has a 3-partition whose small part, if it exists, is on the end-path.*

Proof. We prove that G has a 3-partition whose small part, if it exists, is on the right end-path by an induction on the number of chords of G. Then by similar arguments, G has a 3-partition whose small part, if it exists, is on the left end-path. Thus the theorem is true.

If G has no chords, then the result clearly holds. Assume that there is a chord of G (see Fig 2). Choose an end-chord $\overline{xy}$ of G. Let D be the end-path which connects x and y. We may assume that $\overline{xy}$ is the only chord which is incident to x. Let G' be the graph obtained from G by excluding D and then adding $\{x, y\}$. Among the endvertices of a chord of G', let u be the vertex which is closest to x as one proceeds from x to y avoiding the chord $\overline{xy}$. Then by the induction hypothesis, G' has a 3-partition where the small part, if it exists, is on $u - x - y$.

If there is no small part of G', then clearly G has a 3-partition whose small part, if it exists, is on the right end-path.

Assume that there is a small part of G'. Label the endvertices of the small part as z and w, where w is closer to y than z is. We may assume that z is on $u - x$.

Consider the following two cases.

Case 1: The small part $\overline{zw}$ is an edge. Then there are two subcases to consider.

Subcase 1a: $z = x$. Then $\overline{zw} = \overline{xy}$. Now $\overline{zw}$ and the 2-path of D from x together compose a 3-path or a triangle of G.

Subcase 1*b*: $z \neq x$. Then w is on $u - x$ too. Consider the following three subcases.

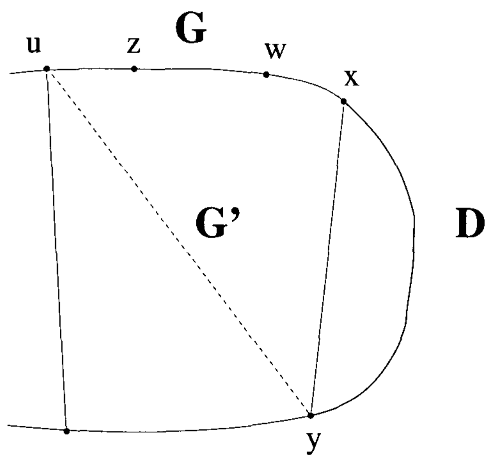

FIGURE 2

Subcase $1ba$: The distance between w and x on $u-x$ is 0 (mod 3). Then $w-x$ on $u-x$ consists of zero or more 3-paths of G'. Since $z-x$ and the 2-path of D from x can be partitioned into 3-paths of G, we are done.

Subcase $1bb$: The distance between w and x on $u-x$ is 1 (mod 3). Then $w-x$ on $u-x$ consists of zero or more 3-paths of G' and one edge. This edge should compose a 3-path of G' together with $\overline{xy}$ and an edge incident to y. Now the last edge and the 2-path of D from y together compose a 3-path of G. Since $z-y$ can be partitioned into 3-paths of G, we are done.

Subcase $1bc$: The distance between w and x of $u-x$ is 2 (mod 3). Then $w-x$ on $u-x$ consists of zero or more 3-paths of G' and a 2-path. This 2-path should compose a 3-path of G' together with $\overline{xy}$. Now $\overline{xy}$ and the 2-path of D from y together compose a 3-path of G. Since $z-x$ can be partitioned into 3-paths of G, we are done.

Case 2: The small part $z-w$ is a 2-path. Then there are three subcases to consider.

Subcase $2a$: $z=x$. Then $z-w$ and the edge of D which is incident to x together compose a 3-path of G.

Subcase $2b$: The distance between z and x on $u-x$ is 1, i.e., $w=y$. Then it is similar to Subcase $2a$.

Subcase $2c$: In the other cases, w is on $u-x$ too. Consider the following three subcases.

Subcase $2ca$: The distance between w and x on $u-x$ is 0 (mod 3). Then $w-x$ on $u-x$ consists of zero or more 3-paths of G'. Since $z-x$ and the edge of D incident to x can be partitioned into 3-paths of G, we are done.

Subcase $2cb$: The distance between w and x on $u-x$ is 1 (mod 3). Then $w-x$ on $u-x$ consists of zero or more 3-paths of G' and an edge. The last edge should compose a 3-path of G' together with $\overline{xy}$ and an edge incident to y. Now the last two edges and the edge of D which is incident to x together compose a 3-path of G. Since $z-x$ can be partitioned into 3-paths of G, we are done.

Subcase $2cc$: The distance between w and x on $u-x$ is 2 (mod 3). Then $w-x$ on $u-x$ consists of zero or more 3-paths of G' and a 2-path. This 2-path should compose a 3-path of G' together with $\overline{xy}$. Since $z-y$ and the edge of D which is incident to y can be partitioned into 3-paths of G, we are done.

In each of the above cases, G has a 3-partition whose small part, if it exists, is on the right end-path and hence the theorem is true. □

3. Bipartite graphs one set of whose bipartition has at most four vertices

LEMMA 4. *Let $G = B_{m,n}$ $(m \leq n)$ be a 2-edge connected planar graph. Then G has at most $2(m+n-2)$ edges.*

Proof. A 2-edge connected planar graph with girth $g \geq 3$ has at most $\frac{g}{g-2}$ $(\mid V(G) \mid -2)$ edges (see pp.18 of [1]). As G is a bipartite graph, $g \geq 4$ and hence $\frac{g}{g-2}(\mid V(G) \mid -2) \leq 2(m+n-2)$. □

LEMMA 5. *Let G be an edge maximal planar 2-edge connected bipartite graph. Then G has a 3-partition.*

Proof. Every edge maximal planar 2-edge connected bipartite graph has a 3-partition in which every component consists of the edge set of paths and triangles ([2]). As G is a bipartite graph, no component is a triangle. □

LEMMA 6. [3] *Let G be a k-edge connected graph. Then each vertex of G has degree at least k.*

The following theorem answers the first question of M. Jünger, G. Reinelt and W. R. Pulleyblank in the case that G is a simple planar 2-edge connected bipartite graph.

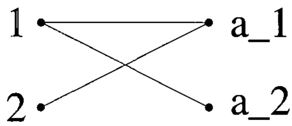

FIGURE 3

THEOREM 7. *Let $G = B_{m,n}$ $(m \leq n)$ be a 2-edge connected planar graph. If $m \leq 4$, then G has a 3-partition.*

Proof. Let $G = B_{m,n}$ $(m \leq n)$ be a 2-edge connected planar graph. By Lemma 6, $m \geq 2$. When $m = 2$, it is clear that G has a 3-partition. Thus the theorem is a consequence of the following two lemmas. □

LEMMA 8. *Let $G = B_{3,n}$ $(n \geq 3)$ be a 2-edge connected planar graph. Then G has a 3-partition.*

Proof. By Lemma 4, G has at most $2n + 2$ edges. By Lemma 5, it is enough to consider the cases where G has $2n$ or $2n + 1$ edges. Label the vertices of one set of bipartition as 1, 2, 3 and the vertices of the other set as $a_1, \ldots, a_n$. Among vertices 1, 2, and 3, line up the ones which are adjacent to a_1 on the first line, and the ones which are adjacent to a_2 on the second line, etc. By Lemma 6, each line contains at least two vertices. Suppose that the first two lines are as follows:

12
1.

Then there is a 3-path $2 - a_1 - 1 - a_2$ (see Fig 3). Actually, two vertices on one line and one of them on another line together compose a 3-path. Similarly, two vertices on a line together or the same vertex on two lines together compose a 2-path.

Case 1: $2n$ edges. Then each line contains two vertices. Suppose there are three lines

12
13
23.

Then there are 3-paths $1 - a_1 - 2 - a_3$ and $1 - a_2 - 3 - a_3$, in other terms, each vertex 2 or 3 on the third line 23 composes a 3-path together with the first line 12 or the second line 13, respectively (see Fig 4).

As one set of bipartition has only three vertices 1, 2 and 3, any two lines have a common vertex and any three lines together compose 3-paths. If the number of lines is 0 (mod 3), then G has a 3-partition with no small part. If the number of lines is 1 (mod 3), then G has a

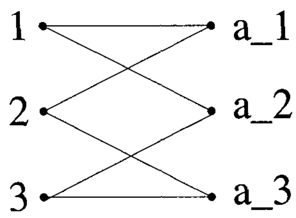

FIGURE 4

3-partition where the small part is a 2-path. If the number of lines is 2 (mod 3), then G has a 3-partition where the small part is an edge. Hence in any case, G has a 3-partition.

Case 2: $2n+1$ edges. Then we may assume that the first line has three vertices 123 and so each of the other lines has two vertices. By Lemma 6, there is another line containing the vertex 1, say 12. Then it composes a 3-path together with the vertex 1 on the first line 123 and the case reduces to Case 1. □

LEMMA 9. *Let $G = B_{4,n}$ ($n \geq 4$) be a 2-edge connected planar graph. Then G has a 3-partition.*

Proof. By Lemma 4 and Lemma 5, it is enough to consider the cases where G has $2n, \ldots, 2n+3$ edges. Label the vertices of one set of bipartition as $1, \ldots, 4$ and the vertices of the other set as $a_1, \ldots, a_n$. Line up the vertices among 1, 2, 3 and 4 as in the proof of Lemma 8.

Case 1: $2n$ edges. Then each line has two vertices. We may assume that two of the lines are 12 and 13. Three lines containing the vertex 4 where two of them are the same compose two 3-paths. Thus after repeatedly deleting those three lines together, we need to consider only the cases where the vertex 4 is on zero, one, two lines or on three different lines 14, 24 and 34.

Subcase 1a: No line contains the vertex 4. Then the result follows by Lemma 8.

In each of the following subcases, after partitioning some part of G into 3-paths as shown below, it reduces to Subcase 1a.

Subcase 1b: Only one line contains the vertex 4. Then there are two subcases to consider.

Subcase 1*ba:* The line containing the vertex 4 is 24 or 34. Then it composes two 3-paths together with a line 12 and a line 13.

Subcase 1*bb*: The line containing the vertex 4 is 14. Then by Lemma 6, there is another line containing the vertex 2. Consider the following two subcases.

Subcase 1*bba*: There is another line 12 or a line 23. Then it composes two 3-paths together with a line 12 and the line 14.

Subcase 1*bbb*: A line 24 is deleted. Then it is deleted together with two other lines containing the vertex 4. Either those two lines are the same or one of them is 24. In the first case, those two lines compose 3-paths together with the line 14, and the case reduces to Subcase 1*bba*. In the latter case, those two lines compose 3-paths together with a line 12, and the other line 24 composes 3-paths together with a line 13 and the line 14.

Subcase 1*c*: Only two lines contain the vertex 4. If one of them is a line 14 or 24, then they compose 3-paths together with a line 12. In the other case, we can proceed similarly.

Subcase 1*d*: Only three lines 14, 24 and 34 contain the vertex 4. Then the line 24 composes 3-paths together with a line 12 and a line 13. If there is a line 23, then the line 14 and the line 34 compose 3-paths together with a line 23. Otherwise, the vertex 4 on the line 14 composes a 3-path together with the line 34.

In each of the following cases, we can proceed as in Case 1.

Case 2: $2n+1$ edges. Then we may assume that the first line has three vertices 123 and so each of the other lines has two vertices.

Subcase 2*a*: No line contains the vertex 4. If a line 12, 13 or 23 exists, then it composes a 3-path together with an appropriate vertex on the first line 123 and the case reduces to Subcase 1*a*. If not, then by Lemma 6, there should be another line containing the vertex 4 and hence WLOG, we may say that two lines 14's and another line containing the vertex 4 are deleted together. Then two subcases to consider follow.

Subcase 2*aa*: The last line is 24 or 34. Then it composes 3-paths together with a line 14 and the vertices 23 on the first line 123, and the other line 14 composes a 3-path together with the remaining vertex 1 of the first line 123.

Subcase 2*ab*: The last line is 14. Then by Lemma 6, a line 24 is deleted together with two other lines containing the vertex 4. Either those two lines are the same or one of them is 14. In the first case, they compose 3-paths together with a line 14. In the latter case, if the other line is 14, then they compose 3-paths together with a line 14. Otherwise the other line is 34 and then two lines 24's compose 3-paths together with a line 14. In any case, the case reduces to Subcase $2aa$.

Subcase $2b$: Only one line contains the vertex 4. Then it composes a 3-path together with an appropriate vertex on the first line 123.

Subcase $2c$: Only two lines contain the vertex 4. WLOG, we may assume that one of them is a line 14. Suppose there is a line 12. Then it composes a 3-path together with the vertex 2 on the first line 123 and those two lines containing the vertex 4 compose 3-paths together with the vertices 13 on the first line 123. In the other case we can proceed similarly.

Subcase $2d$: Only three lines 14, 24 and 34 contain the vertex 4. Then each of them composes a 3-path together with each vertex 1, 2 and 3 on the first line 123, respectively.

Case 3: $2n + 2$ edges. Then there are two subcases to consider.

Subcase $3a$: Each of the first two lines has three vertices and so each of the other lines has two vertices. Then we may assume that the first two lines are 123 and 123, or 123 and 124. In each case, two subcases to consider follow.

Subcase $3aa$: No other line contains the vertex 4. Then the first two lines compose 3-paths together.

Subcase $3ab$: There are exactly one, two, or three other lines 14, 24 and 34 containing the vertex 4. In each case, one of those lines composes a 3-path together with the vertex 3 or 4 on the second line 123 or 124, respectively. Then the case reduces to Subcase $2a$, Subcase $2b$, or Subcase $2c$, respectively.

Subcase $3b$: The first line has four vertices 1234 and so each of the other lines has two vertices.

Subcase $3ba$: No other line contains the vertex 4. By Lemma 6, three lines containing the vertex 4 where two of them are the same are deleted together. Then one of them composes a 3-path together with the vertex 4 on the first line 1234 and the case reduces to Subcase $2c$.

Subcase $3bb$: Only one other line contains the vertex 4. Then it composes 3-paths together with the first line 1234.

Subcase $3bc$: Only two other lines contain the vertex 4. Then one of them composes a 3-path together with the vertex 4 on the first line 1234 and the case reduces to Subcase $2b$.

Subcase $3bd$: Only three other lines 14, 24 and 34 contain the vertex 4. Then the line 34 composes a 3-path together with the vertex 4 on the first line 1234 and the case reduces to Subcase $2c$.

Case 4: $2n+3$ edges. Then there are two subcases to consider.

Subcase $4a$: Each of the first three lines has three vertices and so each of the other lines has two vertices. Then as G is planar, we may assume that the first three lines to be either 123, 123, and 124 or 123, 124, and 134. In the latter case, we consider the following two subcases. In the first case, we can proceed similarly.

Subcase $4aa$: No other line contains the vertex 4. Then the first three lines compose 3-paths together.

Subcase $4ab$: There are exactly one, two, or three other lines 14, 24 and 34 containing the vertex 4. Then one of those lines composes a 3-path together with the vertex 4 on the line 134 and the case reduces to Subcase $3a$.

Subcase $4b$: The first two lines has four vertices and three vertices, respectively and so each of the other lines has two vertices. Then we may assume the first two lines are 1234 and 123 respectively.

Subcase $4ba$: No other line contains the vertex 4. Then the vertices 34 on the first line and the vertex 3 on the second line compose a 3-path together.

Subcase $4bb$: There are exactly one, two, or three other lines 14, 24 and 34 containing the vertex 4. Then one of those lines composes a 3-path together with the vertex 4 on the first line 1234 and the case reduces to Subcase $3a$. □

References

[1] B. Bollobás, *Graph theory*, Springer-Verlag, New York, 1979.
[2] M. Jünger, G. Reinelt, and W. R. Pulleyblank, *On partitioning the edges of graphs into connected subgraphs*, J. Graph Theory **9** (1985), 539–549.
[3] Douglas B. West, *Introduction to graph theory*, Prentice-Hall, New Jersey, 1996.

Department of Mathematics
Incheon University
Incheon 402-749, Korea
E-mail: rho@incheon.ac.kr