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FREE ACTIONS OF FINITE ABELIAN GROUPS ON 3-DIMENSIONAL NILMANIFOLDS

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ABSTRACT. We study free actions of finite abelian groups on 3-dimensional nilmanifolds. By the works of Bieberbach and Waldhausen, this classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy. All such actions are completely classified.

1. Introduction

The general question of classifying finite group actions on a closed 3-manifold is very hard. However, the actions on a 3-dimensional nilmanifold can be understood easily by the works of Bieberbach, L. Auslander and Waldhausen[4, 5, 13]. Free actions of finite, cyclic and abelian groups on the 3-torus were studied in [3], [6] and [9], respectively. It is interesting that if a finite group acts freely on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2$, then it is cyclic ([1]). In this paper we study free actions of finite abelian groups on 3-dimensional nilmanifolds which includes the main theorems 3.2 and 3.3 of [1] as corollaries.

Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e., $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1,$$

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where $\mathbb{R} = \mathcal{Z}(\mathcal{H})$, the center of $\mathcal{H}$. Hence $\mathcal{H}$ has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of $\mathcal{H}$. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes O(2)$ [11, 12]. All isometries of $\mathcal{H}$ preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a Nil-geometry if there is a subgroup π of $\mathrm{Isom}(\mathcal{H})$ so that π acts properly discontinuously and freely with quotient $M = \mathcal{H}/\pi$. The simplest such a manifold is the quotient of $\mathcal{H}$ by the lattice consisting of integral matrices. For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \middle| \; l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these Γ_p's produce infinitely many distinct nilmanifolds $\mathcal{N}_p = \mathcal{H}/\Gamma_p$ covered by $\mathcal{N}_1$. In this paper, we shall find all possible finite abelian groups acting freely on each $\mathcal{N}_p$.

Let G be a finite group acting freely on the nilmanifold $\mathcal{N}_p$. Then clearly, $M = \mathcal{N}_p/G$ is a topological manifold, and $\pi = \pi_1(M) \subset \mathrm{TOP}(\mathcal{H})$ is isomorphic to an almost Bieberbach group. Let π' be an embedding of π into $\mathrm{Aff}(\mathcal{H})$. Such an embedding always exists. Since any isomorphism between lattices extends uniquely to an automorphism of $\mathcal{H}$, we may assume the subgroup Γ_p goes to itself by the embedding $\pi \to \pi' \subset \mathrm{Aff}(\mathcal{H})$. Then the quotient group $G' = \pi'/\Gamma_p$ acts freely on the nilmanifold $\mathcal{N}_p = \mathcal{H}/\Gamma_p$. Moreover, $M' = \mathcal{N}_p/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N}_p)$ gives rise to an isometric action $(G', \mathcal{N}_p)$ on the nilmanifold $\mathcal{N}_p$. Clearly, $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ are sufficiently large, see [5, Proposition 2]. By works of Waldhausen[13] and Heil[4, Theorem A], M is homeomorphic to M'.

DEFINITION 1.1. Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is *topologically conjugate* to (G_2, M_2) if there exists an isomorphism $\theta\colon G_1 \to G_2$ and a homeomorphism $h\colon M_1 \to M_2$ such that $h(g \cdot x) = \theta(g) \cdot h(x)$ for all $x \in M_1$ and all $g \in G_1$. When $G_1 = G_2$ and $M_1 = M_2$, topologically conjugate is the same as *weakly equivariant.*

2. Criteria for conjugacy

In this section, we develop a technique for finding and classifying all possible finite abelian group actions on 3-dimensional nilmanifolds. The problem will be reduced to a purely group-theoretic one. We quote most of the Introduction and Section 2 of [1] in our Introduction and Section 2 for the reader's conveniences.

Let Γ be any lattice of $\mathcal{H}$ and $\mathcal{Z}(\mathcal{H})$ be the center of $\mathcal{H}$. Then $\mathbb{Z} = \Gamma \cap \mathcal{Z}(\mathcal{H})$ and $\Gamma/\Gamma \cap \mathcal{Z}(\mathcal{H})$ are lattices of $\mathcal{Z}(\mathcal{H})$ and $\mathcal{H}/\mathcal{Z}(\mathcal{H})$, respectively. Therefore, the lattice Γ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, that is, there is an exact sequence:

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 1.$$

Let a, b and c be elements of Γ such that the images of a and b in $\mathbb{Z}^2$ generate $\mathbb{Z}^2$ and c generates the center $\mathbb{Z}$. Then it is known that such Γ is isomorphic to one of the following groups, for some $k \neq 0$:

$$\Gamma_k = \langle a, b, c \mid [b, a] = c^k,\ [c, a] = [c, b] = 1 \rangle,$$

where $[b, a] = b^{-1}a^{-1}ba$. This group is realized as a uniform lattice of $\mathcal{H}$ if one takes

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ c = \begin{bmatrix} 1 & 0 & \frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark that Γ_k is equal to Γ_{-k}.

The following proposition gives a characterization of an almost Bieberbach group (see [7]).

PROPOSITION 2.1. *An abstract group π is the fundamental group of a 3-dimensional infra-nilmanifold if and only if π is torsion-free and contains Γ_k for some $k > 0$ as a maximal normal nilpotent subgroup of finite index.*

It is well known that all 3-dimensional infra-nilmanifolds are Seifert manifolds (see [8, 10]). Assume M is a 3-dimensional infra-nilmanifold.

Then M has a Seifert bundle structure; namely, M is a circle bundle over a 2-dimensional orbifold with singularities. It is known [2, Proposition 6.1.] that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant.

Note that if $M = \mathcal{H}/\pi$ is a 3-dimensional infra-nilmanifold, then there is a diffeomorphism f between $\mathcal{H}$ and $\mathbb{R}^3$, and an isomorphism φ between π and π', where π' is a subgroup of $\mathrm{Aff}(\mathbb{R}^3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ such that $(\pi, \mathcal{H})$ and $(\pi', \mathbb{R}^3)$ are weakly equivariant. Therefore, an infra-nilmanifold $M = \mathcal{H}/\pi$ is diffeomorphic to an affine manifold $M' = \mathbb{R}^3/\pi'$. The following proposition [1, Proposition 2.2.] presents an imbedding of $\mathrm{Aff}(\mathcal{H})$ into $\mathrm{Aff}(\mathbb{R}^3) \subset \mathrm{GL}(4,\mathbb{R})$, and will be used to realize an action of $G = \pi/N$ on the nilmanifold $\mathcal{H}/N$, where N is a normal nilpotent subgroup of π isomorphic to Γ_k.

PROPOSITION 2.2. *$\mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes Aut(\mathcal{H})$ imbeds into $\mathrm{Aff}(\mathbb{R}^3) \subset \mathrm{GL}(4,\mathbb{R})$ by*

$$\left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} p & q \\ r & s \end{bmatrix}\right)\right) \longmapsto$$

$$\left(\begin{bmatrix} ps - qr & \frac{k}{2}(pv^* - ru^*) & \frac{k}{2}(qv^* - su^*) & \frac{kxy}{2} - kz \\ 0 & p & q & x \\ 0 & r & s & y \\ 0 & 0 & 0 & 1 \end{bmatrix}\right),$$

where k is any nonzero real number, $u^* = 2u + x$ and $v^* = 2v + y$.

From now on, we shall use the presentations of almost Bieberbach groups in [2, pp. 155–164]. In the following presentations, we shall use $t_i \in \mathrm{Aff}(\mathcal{H})$ $(i = 1, 2, 3)$ whose explicit representations into $\mathrm{Aff}(\mathbb{R}^3)$ are $\lambda(a), \lambda(b)$ and $\lambda(c)$, respectively. That is, for some $k > 0$,

$$\lambda(a) = \left(\begin{bmatrix} 1 & 0 & -\frac{k}{2} & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right), \quad \lambda(b) = \left(\begin{bmatrix} 1 & \frac{k}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right),$$

$$\lambda(c) = \left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right),$$

and by Proposition 2.2, we get

$$t_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad t_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

$$t_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

respectively, where I is the identity in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$.

The following is the list (see [1]) for 15 kinds of the 3-dimensional almost Bieberbach groups imbedded in $\mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes (\mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R}))$. In each presentation, π_k means Seifert bundle type k, n is any positive integer, and t_3 is central except π_3 and π_4. Note that t_1 and t_2 are fixed, but k in t_3 varies for each $\pi_{i,j}$. For example, $k = n$ for π_1; $k = 2n$ for π_2, etc.

$\pi_1 = \langle t_1, t_2, t_3 \mid [t_2, t_1] = t_3^n \rangle.$

$\pi_2 = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{2n}, \alpha^2 = t_3, \alpha t_1 \alpha^{-1} = t_1^{-1}, \alpha t_2 \alpha^{-1} = t_2^{-1} \rangle,$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{4n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right).$$

$$\begin{aligned} \pi_3 = \langle t_1, t_2, t_3, \alpha \mid & [t_2, t_1] = t_3^{2n},\ [t_3, t_1] = [t_3, t_2] = 1,\ \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ & \alpha t_1 \alpha^{-1} = t_1,\ \alpha t_2 = t_2^{-1} \alpha t_3^{-n},\ \alpha^2 = t_1 \rangle, \end{aligned}$$

$$\alpha = \left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right)$$

$$\begin{aligned} \pi_4 = \langle t_1, t_2, t_3, \alpha, \beta \mid & [t_2, t_1] = t_3^{4n}, [t_3, t_1] = [t_3, t_2] = [\alpha, t_3] = 1, \\ & \beta t_3 \beta^{-1} = t_3^{-1}, \alpha t_1 = t_1^{-1} \alpha t_3^{2n},\ \alpha t_2 = t_2^{-1} \alpha t_3^{-2n}, \\ & \alpha^2 = t_3,\ \beta^2 = t_1,\ \beta t_1 \beta^{-1} = t_1,\ \beta t_2 = t_2^{-1} \beta t_3^{-2n}, \\ & \alpha\beta = t_1^{-1} t_2^{-1} \beta \alpha t_3^{-(2n+1)} \rangle, \end{aligned}$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{8n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right),$$

$$\beta = \left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \right).$$

$$\pi_{5,1} = \langle t_1, t_2, t_3, \alpha \,|\, [t_2, t_1] = t_3^{4n-2},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^4 = t_3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{4(4n-2)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

$$\pi_{5,2} = \langle t_1, t_2, t_3, \alpha \,|\, [t_2, t_1] = t_3^{4n},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^4 = t_3^3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{3}{16n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

$$\pi_{5,3} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{4n},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^4 = t_3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{16n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

$$\pi_{6,1} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1},\ \alpha^3 = t_3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{9n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \right)$$

$$\pi_{6,2} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1},\ \alpha^3 = t_3^2 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{2}{9n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \right)$$

$$\pi_{6,3} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n-2},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1},\ \alpha^3 = t_3^2 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{2}{9n-6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \right)$$

$$\pi_{6,4} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n-1},\ \alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1},\ \alpha^3 = t_3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{9n-3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right) \right)$$

$$\pi_{7,1} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n},\ \alpha t_1 \alpha^{-1} = t_1 t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^6 = t_3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{36n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

$$\pi_{7,2} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n-2},\ \alpha t_1 \alpha^{-1} = t_1 t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^6 = t_3 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{36n-12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

$$\pi_{7,3} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n},\ \alpha t_1 \alpha^{-1} = t_1 t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^6 = t_3^5 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{5}{36n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

$$\pi_{7,4} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{6n-4},\ \alpha t_1 \alpha^{-1} = t_1 t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^6 = t_3^5 \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{5}{36n-24} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right) \right)$$

Let $(G, \mathcal{N}_p)$ be a free affine action of a finite abelian group G on the nilmanifold $\mathcal{N}_p$. Then $\mathcal{N}_p/G$ is an infra-nilmanifold. Let $\pi = \pi_1(\mathcal{N}_p/G)$, and $\Gamma_p = \pi_1(\mathcal{N}_p)$. Then π is an almost Bieberbach group. In fact, since the covering projection $\mathcal{N}_p \to \mathcal{N}_p/G$ is regular, Γ_p is a normal subgroup of π.

Definition 2.3. Let $\pi \subset \mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of π. We say that (N_1, π) is *affinely conjugate* to (N_2, π), denoted by $N_1 \sim N_2$, if there exists an element $(t, T) \in \mathrm{Aff}(\mathcal{H})$ such that $(t,T)\pi(t,T)^{-1} = \pi$ and $(t,T)N_1(t,T)^{-1} = N_2$.

From now on, we shall denote the normalizer of π by $N_{\mathrm{Aff}(\mathcal{H})}(\pi)$.

Our classification problem of free finite group actions $(G, \mathcal{N}_p)$ with $\pi_1(\mathcal{N}_p/G) \cong \pi$ can be solved by two steps:

(I) Find all normal nilpotent subgroups N of π each of which is isomorphic to Γ_p, and classify (N, π) up to affine conjugacy.

(II) Realize the action of G on $\mathcal{H}/N$ as an action of G on $\mathcal{H}/\Gamma_p = \mathcal{N}_p$.

For the first part (I), we need the following lemma([1, Lemma 2.4.]), and for the second part (II) "Realization", we just follow Procedure of [1].

LEMMA 2.4. *Any normal nilpotent subgroup N of π has a set of generators of the form*

$$\langle t_1^{d_1} t_2^{m} t_3^{n_1},\ t_2^{d_2} t_3^{n_2},\ t_3^{d_3} \rangle,$$

where $0 \le m < d_2$, $0 \le n_i < d_3$, $i = 1, 2$.

The second part (II) "Realization" can be done by the following procedure. Let π be an almost Bieberbach group and N be a normal subgroup of π with $G = \pi/N$ finite. To describe the natural affine action of G on the nilmanifold $\mathcal{H}/N$ as an action of G on $\mathcal{N}_p$, we must make the nilmanifold $\mathcal{H}/N$ the nilmanifold $\mathcal{N}_p$ whose fundamental group is Γ_p and describe the action on the universal covering level. In other words, the action of G should be defined on $\mathcal{H}$ as affine maps (this is really explaining the liftings of a set of generators of G in π), and simply say that our action is the affine action modulo the lattice Γ_p. It is quite easy to achieve this. Find an automorphism $B \in \mathrm{Aut}(\mathcal{H})$ which maps N onto Γ_p. Therefore, the conjugation by $B \in \mathrm{Aff}(\mathcal{H})$ maps π into another almost Bieberbach group in such a way that N maps onto Γ_p. Suppose $\{\alpha_1, \cdots, \alpha_k\}$ generates the quotient group G when projected down via $\pi \to G$, then $\{B\alpha_1 B^{-1}, \cdots, B\alpha_k B^{-1}\}$ describes the action of G on the nilmanifold $\mathcal{N}_p$.

3. Free actions of finite abelian groups G on the nilmanifold

In this section, we shall find all possible finite abelian groups acting freely(up to topological conjugacy) on the 3-dimensional nilmanifold $\mathcal{N}_p$. The following proposition is a working criterion for determining all normal nilpotent subgroups of each almost Bieberbach group π which are isomorphic to Γ_p.

PROPOSITION 3.1. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_p. Then N can be represented by a set of generators*

$$N = \langle t_1^{d_1} t_2^{m} t_3^{n_1},\ t_2^{d_2} t_3^{n_2},\ t_3^{\frac{K d_1 d_2}{p}} \rangle,$$

where d_1, d_2 *are divisors of* p; K *is determined by* $t_3^K = [t_2, t_1]$;
$0 \le m < d_2$, $0 \le n_i < \frac{K d_1 d_2}{p}$ $(i = 1, 2)$.

Proof. Let $\Delta = \pi \cap \mathcal{H}$ be pure translations of π. Then

$$\Delta = \langle t_1, t_2, t_3 \rangle$$

is the maximal normal nilpotent subgroup of π. For every π, Δ contains the relations

$$[t_2,\, t_1] = t_3^K, \quad [t_3,\, t_1] = [t_3,\, t_2] = 1,$$

where K is a positive integer. Let N be a normal nilpotent subgroup of π. Then N is a subgroup of Δ and can be represented by a set of generators

$$\langle t_1^{d_1} t_2^m t_3^{n_1},\, t_2^{d_2} t_3^{n_2},\, t_3^{d_3} \rangle$$

by Lemma 2.4. Note that

$$[t_2^{d_2} t_3^{n_2},\, t_1^{d_1} t_2^m t_3^{n_1}] = [t_2^{d_2},\, t_1^{d_1}] = [t_2,\, t_1]^{d_1 d_2} = t_3^{K d_1 d_2}.$$

Since N is isomorphic to Γ_p, we have $d_3 = \frac{K d_1 d_2}{p}$. By the normality of N in π, the following two relations

$$t_2 (t_1^{d_1} t_2^m t_3^{n_1}) t_2^{-1} = (t_1^{d_1} t_2^m t_3^{n_1})(t_3^{d_3})^{\frac{K d_1}{d_3}},$$
$$t_1 (t_2^{d_2} t_3^{n_2}) t_1^{-1} = (t_2^{d_2} t_3^{n_2})(t_3^{d_3})^{-\frac{K d_2}{d_3}}$$

show that d_1, d_2 are divisors of p. This completes the proof by Lemma 2.4. □

Remark that if $p = 1$, then we obtain the following result, which is the same as Proposition 3.1 of [1].

COROLLARY 3.2. *Let N be a normal nilpotent subgroup of an almost Bieberbach group π and isomorphic to Γ_1. Then N can be represented by a set of generators*

$$N = \langle t_1 t_3^{n_1},\, t_2 t_3^{n_2},\, t_3^K \rangle,$$

where $0 \leq n_i < K$, $i = 1, 2$, *and K is determined by* $t_3^K = [t_2,\, t_1]$.

Now we shall deal with 15 distinct almost Bieberbach groups up to Seifert local invariant. This, as in other parts of calculations, was done by the program Mathematica[14] and hand-checked. From now on, we shall denote affine conjugacy classes by AC classes and use the following notation.

NOTATION: ${}^{\xi}\langle \alpha_1, \cdots, \alpha_k \rangle$ means the subgroup generated by conjugations of $\alpha_1, \cdots, \alpha_k$ by ξ.

THEOREM 3.3. (π_2) *Table 1 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups* G *on* $\mathcal{N}_p$ *which yield an orbit manifold homeomorphic to* $\mathcal{H}/\pi_2$.

Table 1

Group G	Genertors	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{4n}{p}}$	$\langle\alpha\rangle$	$\frac{2n}{p}\in\mathbb{N}$	$N_1=\langle t_1,\ t_2,\ t_3^{\frac{2n}{p}}\rangle$
	$^{\xi_2}\langle\alpha\rangle$	$\frac{n}{p}\in\mathbb{N},\ p\neq 1$	$N_2=\langle t_1, t_2t_3^{\frac{n}{p}}, t_3^{\frac{2n}{p}}\rangle$
	$^{\xi_3}\langle\alpha\rangle$		$N_3=\langle t_1t_3^{\frac{n}{p}}, t_2t_3^{\frac{n}{p}}, t_3^{\frac{2n}{p}}\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_{\frac{8n}{p}}$	$^{\eta_1}\langle t_1,\ \alpha\rangle$	$\frac{4n}{p}\in\mathbb{N}, p\in 2\mathbb{N}$	$L_1=\langle t_1^2,\ t_2,\ t_3^{\frac{4n}{p}}\rangle$
	$^{\eta_2}\langle t_1,\ \alpha\rangle$	$\frac{2n}{p}\in\mathbb{N}, p\in 2\mathbb{N}+2$	$L_2=\langle t_1^2,\ t_2t_3^{\frac{2n}{p}}, t_3^{\frac{4n}{p}}\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_{\frac{16n}{p}}$	$^{\zeta}\langle t_1,\ t_2,\ \alpha\rangle$	$\frac{8n}{p}\in\mathbb{N}, p\in 4\mathbb{N}$	$N=\langle t_1^2,\ t_2^2,\ t_3^{\frac{8n}{p}}\rangle$

where I *is the identity in* $\mathcal{H}$, *and*

$$\xi_2=\left(\mathrm{I},\left(\begin{bmatrix}\frac{1}{2p}\\0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\right),\qquad \xi_3=\left(\mathrm{I},\left(\begin{bmatrix}\frac{1}{2p}\\-\frac{1}{2p}\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\right),$$

$$\eta_1=\left(\mathrm{I},\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}\right)\right),\qquad \eta_2=\left(\mathrm{I},\left(\begin{bmatrix}\frac{1}{2p}\\0\end{bmatrix},\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}\right)\right),$$

$$\zeta=\left(\mathrm{I},\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}\right)\right).$$

Proof. Recall that

$$\pi_2=\langle\, t_1,t_2,t_3,\alpha \mid [t_2,t_1]=t_3^{2n},\ [t_3,\alpha]=[t_3,t_1]=[t_3,t_2]=1,$$
$$\alpha^2=t_3,\ \alpha t_1\alpha^{-1}=t_1^{-1},\ \alpha t_2\alpha^{-1}=t_2^{-1}\,\rangle,$$

$$\alpha=\left(\begin{bmatrix}1&0&-\frac{1}{4n}\\0&1&0\\0&0&1\end{bmatrix},\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\right)\right).$$

Let N be a normal nilpotent subgroup of π_2 isomorphic to Γ_p such that π_2/N is abelian. Then

$$N=\langle\, t_1^{d_1}t_2^{m}t_3^{\ell},\ t_2^{d_2}t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}}\,\rangle,\quad \left(0\le m<d_2,\ 0\le \ell,r<\frac{2nd_1d_2}{p}\right)$$

by Proposition 3.1. Since $[\pi_2,\pi_2]=\langle t_1^2,\ t_2^2,\ t_3^{2n}\rangle\subset N$, the possible pairs of $(d_1,\,d_2)$ are $(1,1)$, $(2,1)$, $(1,2)$, and $(2,2)$.

(i) When $d_1=d_2=1$: We have

$$N=\langle t_1t_3^{\ell},\ t_2t_3^{r},\ t_3^{\frac{2n}{p}}\rangle,\quad \left(0\le \ell,r<\frac{2n}{p},\ \frac{2n}{p}\in\mathbb{N}\right).$$

If n is not a multiple of p, then there exists only one normal nilpotent subgroup

$$N_1=\langle t_1,\ t_2,\ t_3^{\frac{2n}{p}}\rangle.$$

Since N is a normal subgroup of π_2, if n is a multiple of p, then the following two relations

$$\alpha(t_1t_3^{\ell})\alpha^{-1}=(t_1t_3^{\ell})^{-1}t_3^{2\ell}\ \in N,$$

$$\alpha(t_2t_3^{r})\alpha^{-1}=(t_2t_3^{r})^{-1}t_3^{2r}\ \in N$$

show that $\ell=0$ or $\frac{n}{p}$, and $r=0$ or $\frac{n}{p}$. Thus the possible normal nilpotent subgroups are

$$N_1=\langle t_1,\ t_2,\ t_3^{\frac{2n}{p}}\rangle,\qquad N_2=\langle t_1,\ t_2t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}}\rangle,$$
$$N_3=\langle t_1t_3^{\frac{n}{p}},\ t_2t_3^{\frac{n}{p}},\ t_3^{\frac{2n}{p}}\rangle,\qquad N_4=\langle t_1t_3^{\frac{n}{p}},\ t_2,\ t_3^{\frac{2n}{p}}\rangle.$$

To calculate the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$, let $(t,T)\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$. Since $TAT^{-1}=A$ for $\alpha=(a,A)\in\pi_2$ and $a\in\mathcal{Z}(\mathcal{H})$, we have

$$(t,T)(a,A)(t,T)^{-1}(a,A)^{-1}=t\cdot A(t^{-1}).$$

Therefore

$$(t,T)\in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)\ \text{ if and only if }\ t\cdot A(t^{-1})\in\Delta=\pi\cap\mathcal{H}.$$

Thus we obtain

$$N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$$
$$=\left\{\left(\begin{bmatrix}1&\frac{r}{2}&*\\0&1&\frac{s}{2}\\0&0&1\end{bmatrix},\left(\begin{bmatrix}0\\0\end{bmatrix},\ T\right)\right)\ \middle|\ T\in\mathrm{GL}(2,\mathbb{Z}),\ r,\ s\in\mathbb{Z}\right\}.$$

It is not hard to see $N_2 \sim N_4$ by using

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

Note that

$$\mu_1 = \left(\begin{bmatrix} 1 & -\frac{1}{2p} & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathrm{I}\right)\right),$$

$$\mu_2 = \left(\begin{bmatrix} 1 & -\frac{1}{2p} & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right)$$

are the only ones that can conjugate N_1 onto N_2. However, when $p > 1$,

$$\mu_i \notin N_{\mathrm{Aff}(\mathcal{H})}(\pi_2) \quad (i = 1, 2).$$

Thus we proved that N_1 is not affinely conjugate to N_2. Similarly we can prove that N_1 (or N_2) is not affinely conjugate N_3. Note that when $p = 1$, N_i $(i = 1, 2, 3)$ are affinely conjugate each other [1, Theorem 3.3. (Type 2)]. Therefore if $\frac{n}{p} \in \mathbb{N}$ and $p \neq 1$, there exist 3 affinely non-conjugate normal nilpotent subgroups N_i $(i = 1, 2, 3)$ of π_2.

(ii) When $d_1 = 2,\ d_2 = 1$: Let L be a normal nilpotent subgroup of π_2 isomorphic to Γ_p such that π_2/L is abelian. Then by Proposition 3.1,

$$L = \langle t_1^2 t_3^\ell,\ t_2 t_3^r,\ t_3^{\frac{4n}{p}} \rangle, \quad \left(0 \le \ell, r < \frac{4n}{p},\ \ p \in 2\mathbb{N}\right).$$

Since $t_1^2 \in [\pi_2, \pi_2] \subset L$, we have $\ell = 0$. By the normality of L, the following relation

$$\alpha(t_2 t_3^r)\alpha^{-1} = (t_2 t_3^r)^{-1} t_3^{2r} \ \in L$$

shows that $r = 0$ or $\frac{2n}{p}$, whenever $\frac{2n}{p} \in \mathbb{N}$. Therefore the possible normal nilpotent subgroups are

$$L_1 = \langle t_1^2,\ t_2,\ t_3^{\frac{4n}{p}} \rangle, \qquad L_2 = \langle t_1^2,\ t_2 t_3^{\frac{2n}{p}},\ t_3^{\frac{4n}{p}} \rangle.$$

Remark that if $p=2$, then $L_1 \sim L_2$ by using

$$\left(\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \left(\begin{bmatrix}0\\0\end{bmatrix}, \mathrm{I}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

However if $p \in 2\mathbb{N}+2$ and $\frac{2n}{p} \in \mathbb{N}$, then there does not exist $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2)$ whose conjugation maps L_1 onto L_2. Thus L_1 is not affinely conjugate to L_2.

(iii) When $d_1 = 1$, $d_2 = 2$: By Proposition 3.1,

$$L = \langle t_1 t_2^m t_3^\ell,\ t_2^2 t_3^r,\ t_3^{\frac{4n}{p}} \rangle, \quad \left(0 \le m < 2,\ 0 \le \ell, r < \frac{4n}{p},\ p \in 2\mathbb{N}\right).$$

Since $t_2^2 \in [\pi_2, \pi_2] \subset L$, we have $r = 0$. By the normality of L, the following relation

$$\alpha(t_1 t_2^m t_3^\ell)\alpha^{-1} = (t_1 t_2^m t_3^\ell)^{-1} t_3^{2\ell - 2mn} \ \in L$$

shows that $\ell = 0$ or $\frac{2n}{p}$, whenever $\frac{2n}{p} \in \mathbb{N}$. Thus the possible normal nilpotent subgroups are

$$L_3 = \langle t_1,\ t_2^2,\ t_3^{\frac{4n}{p}} \rangle, \qquad L_4 = \langle t_1 t_3^{\frac{2n}{p}},\ t_2^2,\ t_3^{\frac{4n}{p}} \rangle,$$
$$L_5 = \langle t_1 t_2,\ t_2^2,\ t_3^{\frac{4n}{p}} \rangle, \qquad L_6 = \langle t_1 t_2 t_3^{\frac{2n}{p}},\ t_2^2,\ t_3^{\frac{4n}{p}} \rangle.$$

It is easy to see that $L_3 \sim L_5$ and $L_4 \sim L_6$ by using

$$\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 1\end{bmatrix}, \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

Note that if $p \in 2\mathbb{N}+2$, we can see that L_3 is not affinely conjugate to L_4. However we know that $L_1 \sim L_3$ and $L_2 \sim L_4$ by using

$$\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_2).$$

Therefore if $\frac{2n}{p} \in \mathbb{N}$ and $p \in 2\mathbb{N}+2$, then there exist 2 affinely non-conjugate normal nilpotent subgroups L_1 and L_2 of π_2.

(iv) When $d_1 = d_2 = 2$: Since $[\pi_2, \pi_2] = \langle t_1^2,\ t_2^2,\ t_3^{2n}\rangle \subset N$, if $p \in 4\mathbb{N}$ and $\frac{8n}{p} \in \mathbb{N}$, then there exists only one normal nilpotent subgroup

$$N = \langle t_1^2,\ t_2^2,\ t_3^{\frac{8n}{p}}\rangle.$$

The realization of the action of $G = \pi_2/N_i$ on the nilmanifold $\mathcal{H}/N_i$, as an affine action on the nilmanifold $\mathcal{N}_p$, is easy provided that we follow the "Realization" procedure. For example, the normal nilpotent subgroup N_1 of π_2 is equal to Γ_p itself. Therefore $G = \pi_2/N_1 \cong \mathbb{Z}_{\frac{4n}{p}}$ acts on $\mathcal{N}_p$. Thus the generator α of the group G can be obtained from $\alpha \in \pi_2$. However we observe that N_2 in π_2 is not equal to Γ_p, but isomorphic to Γ_p. To obtain an action of $G = \pi_2/N_2$ on $\mathcal{N}_p$, one has to conjugate the representation of π_2 so that N_2 becomes Γ_p by means of an automorphism $\xi_2 \in \mathrm{Aut}(\mathcal{H})$, where

$$\xi_2 = \left(\mathrm{I},\ \left(\begin{bmatrix} \frac{1}{2p} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) \in \mathrm{Aff}(\mathcal{H}).$$

Thus we can see that $\xi_2\, N_2\, \xi_2^{-1} = \Gamma_p$, and

$$\xi_2\, \alpha\, \xi_2^{-1} = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{4n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} \frac{1}{p} \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right)$$

describes an action of $\pi_2/N_2 \cong \mathbb{Z}_{\frac{4n}{p}}$ on $\mathcal{N}_p$. That is, it acts on $\mathcal{H}$ by

$$\xi_2\, \alpha\, \xi_2^{-1} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x & z - \frac{y}{p} - \frac{1}{4n} \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly for the case of N_3 in π_2, using an automorphism

$$\xi_3 = \left(\mathrm{I},\ \left(\begin{bmatrix} \frac{1}{2p} \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) \in \mathrm{Aff}(\mathcal{H}),$$

we can see that $\xi_3\, N_3\, \xi_3^{-1} = \Gamma_p$, and

$$\xi_3\, \alpha\, \xi_3^{-1} = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{4n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} \frac{1}{p} \\ -\frac{1}{p} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right)$$

describes an action of $\pi_2/N_3 \cong \mathbb{Z}_{\frac{4n}{p}}$ on $\mathcal{N}_p$. That is, it acts on $\mathcal{H}$ by

$$\xi_3\,\alpha\,\xi_3^{-1}\cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x & z-\frac{x}{p}-\frac{y}{p}-\frac{1}{4n} \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}$$

For the case of L_2, since $G=\pi_2/L_2 \cong \mathbb{Z}_2\times\mathbb{Z}_{\frac{8n}{p}}$ is generated by the images of t_1 and α, and L_2 is not equal to Γ_p, using an automorphism

$$\eta_2 = \left(\mathrm{I}, \left(\begin{bmatrix} \frac{1}{2p} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\right)\right) \in \mathrm{Aff}(\mathcal{H}),$$

we can see that $\eta_2\,L_2\,\eta_2^{-1}=\Gamma_p$, and

$$\eta_2\,t_1\,\eta_2^{-1} = \left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right),$$

$$\eta_2\,\alpha\,\eta_2^{-1} = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{8n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right).$$

The other cases can be done similarly. □

THEOREM 3.4. (π_3) *Table 2 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$.*

Table 2

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{p}{n}}$	$^{\xi}\langle\alpha\rangle$	$\frac{p}{2n}\in\mathbb{N}$	$N=\langle t_1^{\frac{p}{2n}},\ t_2,\ t_3\rangle$
$\mathbb{Z}_{\frac{2p}{n}}\times\mathbb{Z}_2$	$^{\eta_1}\langle\alpha,t_3\rangle$	$\frac{p}{n}\in\mathbb{N},\ n\in 2\mathbb{N}$	$N_1=\langle t_1^{\frac{p}{n}},\ t_2,\ t_3^2\rangle$
	$^{\eta_2}\langle\alpha,t_3\rangle$		$N_2=\langle t_1^{\frac{p}{n}}t_3,\ t_2,\ t_3^2\rangle$
$\mathbb{Z}_{\frac{p}{2n}}\times\mathbb{Z}_2$	$^{\eta_3}\langle\alpha,t_2\rangle$	$\frac{p}{4n}\in\mathbb{N}$	$L_1=\langle t_1^{\frac{p}{4n}},\ t_2^2,\ t_3\rangle$
	$^{\eta_4}\langle\alpha,t_2\rangle$		$L_2=\langle t_1^{\frac{p}{4n}}t_2,\ t_2^2,\ t_3\rangle$

$\mathbb{Z}_{\frac{p}{n}} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	${}^{\zeta_1}\langle \alpha, t_2, t_3 \rangle$	$\frac{p}{2n} \in \mathbb{N}, n \in 2\mathbb{N}$	$K_1 = \langle t_1^{\frac{p}{2n}},\ t_2^2,\ t_3^2 \rangle$
	${}^{\zeta_2}\langle \alpha, t_2, t_3 \rangle$		$K_2 = \langle t_1^{\frac{p}{2n}} t_2,\ t_2^2,\ t_3^2 \rangle$
	${}^{\zeta_3}\langle \alpha, t_2, t_3 \rangle$		$K_3 = \langle t_1^{\frac{p}{2n}} t_3,\ t_2^2,\ t_3^2 \rangle$
	${}^{\zeta_4}\langle \alpha, t_2, t_3 \rangle$	$\frac{p}{2n} \in \mathbb{N}, n \in 2\mathbb{N}-1$	$K_4 = \langle t_1^{\frac{p}{2n}},\ t_2^2 t_3,\ t_3^2 \rangle$
	${}^{\zeta_5}\langle \alpha, t_2, t_3 \rangle$		$K_5 = \langle t_1^{\frac{p}{2n}} t_2,\ t_2^2 t_3,\ t_3^2 \rangle$
	${}^{\zeta_6}\langle \alpha, t_2, t_3 \rangle$	$\frac{p}{2n} \in \mathbb{N}, p \in 2\mathbb{N}+2$	$K_6 = \langle t_1^{\frac{p}{2n}} t_3,\ t_2^2 t_3,\ t_3^2 \rangle$

where

$$\xi = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ 0 & 1 \end{bmatrix} \right) \right), \qquad \eta_1 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{n}{p} & 0 \\ 0 & 1 \end{bmatrix} \right) \right),$$

$$\eta_2 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{n}{p} & 0 \\ 0 & 1 \end{bmatrix} \right) \right), \qquad \eta_3 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4n}{p} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \right),$$

$$\eta_4 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{4n}{p} & 0 \\ -\frac{2n}{p} & \frac{1}{2} \end{bmatrix} \right) \right), \qquad \zeta_1 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \right),$$

$$\zeta_2 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ -\frac{n}{p} & \frac{1}{2} \end{bmatrix} \right) \right), \qquad \zeta_3 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \right),$$

$$\zeta_4 = \left(\mathrm{I}, \left(\begin{bmatrix} \frac{1}{2p} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \right), \qquad \zeta_5 = \left(\mathrm{I}, \left(\begin{bmatrix} \frac{1}{2p} \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ -\frac{n}{p} & \frac{1}{2} \end{bmatrix} \right) \right),$$

$$\zeta_6 = \left(\mathrm{I}, \left(\begin{bmatrix} \frac{1}{2p} \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} \frac{2n}{p} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \right) \right).$$

Proof. Let N be a normal nilpotent subgroup of π_3 isomorphic to Γ_p such that π_3/N is abelian. Then by Proposition 3.1,

$$N = \langle\, t_1^{d_1} t_2^{m} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{2nd_1d_2}{p}} \,\rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{2nd_1d_2}{p} \right).$$

Since $[\pi_3,\pi_3]=\langle t_2^2 t_3^n,\ t_3^2 \rangle \subset N$, d_2 must be 1 or 2, and $\frac{2nd_1d_2}{p}$ must be 1 or 2.

(i) When $d_2 = 1$, $d_1 = \frac{p}{2n} \in \mathbb{N}$: There exists only one normal nilpotent subgroup

$$N = \langle t_1^{\frac{p}{2n}},\ t_2,\ t_3\rangle, \quad \left(\frac{p}{2n} \in \mathbb{N}\right).$$

(ii) When $d_2 = 1$, $d_1 = \frac{p}{n} \in \mathbb{N}$: We have

$$N = \langle\, t_1^{\frac{p}{n}} t_3^{\ell},\ t_2 t_3^{r},\ t_3^2\,\rangle, \quad \left(0 \le \ell, r < 2,\ \ \frac{p}{n} \in \mathbb{N}\right).$$

By the normality of N, the following two relations

$$\alpha(t_1^{\frac{p}{n}} t_3^{\ell})\alpha^{-1} = (t_1^{\frac{p}{n}} t_3^{\ell}) t_3^{-2\ell} \ \in N,$$

$$\alpha(t_2 t_3^{r})\alpha^{-1} = (t_2 t_3^{r})^{-1} t_3^{n} \in N$$

show that n must be an even integer. Therefore the possible normal nilpotent subgroups are

$$N_1 = \langle\, t_1^{\frac{p}{n}},\ t_2,\ t_3^2\,\rangle, \qquad N_2 = \langle\, t_1^{\frac{p}{n}} t_3,\ t_2,\ t_3^2\,\rangle,$$
$$N_3 = \langle\, t_1^{\frac{p}{n}},\ t_2 t_3,\ t_3^2\,\rangle, \qquad N_4 = \langle\, t_1^{\frac{p}{n}} t_3,\ t_2 t_3,\ t_3^2\,\rangle.$$

Now we shall show that N_1 is not affinely conjugate to N_2, but $N_1 \sim N_3$ and $N_2 \sim N_4$. First, we need to calculate the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_3)$. Let

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_3).$$

Since the holonomy group of π_3 is $\mathbb{Z}_2 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle$, one can see that the normalizer of $\mathbb{Z}_2$ in $\mathrm{GL}(2, \mathbb{Z})$ is $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$, where $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{Z}_2$. Thus it is not hard to see that

$$v = 0, \quad y = k/2 \ \ (k \in \mathbb{Z}),$$

$x + u$ must be a multiple of $1/2n$, and z must be a multiple of $1/4n$.

It is easy to see that $N_1 \sim N_3$ and $N_2 \sim N_4$ by using $\mu_n \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_3)$, where

$$\mu_n = \left(\begin{bmatrix} 1 & -\frac{1}{2n} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right).$$

Note that

$$\mu_p = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & \frac{1}{2p} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -x \\ 0 \end{bmatrix}, T \right) \right), \quad T \in \mathbb{Z}_2 \rtimes \mathbb{Z}_2, \quad x,\, z \in \mathbb{R},$$

are the only ones that can conjugate N_1 onto N_2. However, since $n \in 2\mathbb{N}$ and $\frac{p}{n} \in \mathbb{N}$, p must be an even integer. Therefore

$$\mu_p \notin N_{\mathrm{Aff}(\mathcal{H})}(\pi_3).$$

Thus we proved that N_1 is not affinely conjugate to N_2.

(iii) When $d_2 = 2$, $d_1 = \frac{p}{4n} \in \mathbb{N}$: By Proposition 3.1,

$$L = \langle t_1^{\frac{p}{4n}} t_2^m,\ t_2^2,\ t_3 \rangle, \quad (0 \le m < 2).$$

Thus the possible normal nilpotent subgroups are

$$L_1 = \langle\, t_1^{\frac{p}{4n}},\ t_2^2,\ t_3\, \rangle, \qquad L_2 = \langle\, t_1^{\frac{p}{4n}} t_2,\ t_2^2,\ t_3\, \rangle.$$

Note that for any $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_3)$, the conjugation by μ cannot map t_1 onto $t_1 t_2$. That is,

$$\mu t_1^{\frac{p}{4n}} \mu^{-1} = \left(\begin{bmatrix} 1 & \pm\frac{p}{4n} & \mp\frac{p}{4n} y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathrm{I} \right) \right) \neq t_1^{\frac{p}{4n}} t_2.$$

Thus we proved that L_1 is not affinely conjugate to L_2.

(iv) When $d_2 = 2$, $d_1 = \frac{p}{2n} \in \mathbb{N}$: By Proposition 3.1,

$$K = \langle t_1^{\frac{p}{2n}} t_2^m t_3^\ell,\ t_2^2 t_3^r,\ t_3^2 \rangle, \quad \left(0 \le m < 2,\ 0 \le \ell,\, r < 2,\ \tfrac{p}{2n} \in \mathbb{N}\right).$$

Since $t_2^2 t_3^n \in [\pi_3, \pi_3] \subset K$, if $n \in 2\mathbb{N}$, then $r = 0$. Therefore the possible normal nilpotent subgroups are

$$K_1 = \langle\, t_1^{\frac{p}{2n}},\ t_2^2,\ t_3^2\, \rangle, \qquad K_2 = \langle\, t_1^{\frac{p}{2n}} t_2,\ t_2^2,\ t_3^2\, \rangle,$$

$$K_3 = \langle\, t_1^{\frac{p}{2n}} t_3,\ t_2^2,\ t_3^2\, \rangle, \qquad K_2' = \langle\, t_1^{\frac{p}{2n}} t_2 t_3,\ t_2^2,\ t_3^2\, \rangle.$$

If $n \in 2\mathbb{N} - 1$, since $t_2^2 t_3 \in [\pi_3, \pi_3] \subset K$, then $r = 1$. Thus we have

$$K_4 = \langle\, t_1^{\frac{p}{2n}},\ t_2^2 t_3,\ t_3^2\, \rangle, \qquad K_5 = \langle\, t_1^{\frac{p}{2n}} t_2,\ t_2^2 t_3,\ t_3^2\, \rangle,$$

$$K_6 = \langle\, t_1^{\frac{p}{2n}} t_3,\ t_2^2 t_3,\ t_3^2\, \rangle, \qquad K_5' = \langle\, t_1^{\frac{p}{2n}} t_2 t_3,\ t_2^2 t_3,\ t_3^2\, \rangle.$$

It is easy to see that $K_2 \sim K_2'$ and $K_5 \sim K_5'$ by the conjugation by $\mu_n \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_3)$. Note that $K_4 \sim K_6$ only when $p = 2$. It is not hard to see that K_i $(1 \le i \le 6)$ is not affinely conjugate to each other except for such one case by applying above methods.

The realization of the free action of the group G on $\mathcal{N}_p$ can be done easily by using the method in Theorem 3.3. □

THEOREM 3.5. (π_4) *Table 3 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_4$.*

Table 3

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\langle \beta, \alpha \rangle$	$p = 4n$	$N_1 = \langle t_1,\ t_2,\ t_3 \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	${}^{\eta}\langle \beta, \alpha \rangle$	$p = 2n$	$N_2 = \langle t_1,\ t_2 t_3,\ t_3^2 \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	${}^{\zeta_1}\langle \beta, t_2, \alpha \rangle$	$p = 8n$	$L_1 = \langle t_1 t_2,\ t_2^2,\ t_3 \rangle$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$	${}^{\zeta_2}\langle \beta, t_2, \alpha \rangle$	$p = 4n$	$L_2 = \langle t_1 t_2 t_3,\ t_2^2,\ t_3^2 \rangle$

where

$$\eta = \left(\mathrm{I}, \left(\begin{bmatrix} \frac{1}{4n} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right), \qquad \zeta_1 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)\right),$$

$$\zeta_2 = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ \frac{1}{4} - \frac{1}{8n} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right)\right).$$

Proof. Recall that

$$\begin{aligned} \pi_4 = \langle t_1, t_2, t_3, \alpha, \beta \mid\ & [t_2, t_1] = t_3^{4n},\ [t_3, t_1] = [t_3, t_2] = [\alpha, t_3] = 1, \\ & \beta t_3 \beta^{-1} = t_3^{-1},\ \alpha t_1 = t_1^{-1} \alpha t_3^{2n},\ \alpha t_2 = t_2^{-1} \alpha t_3^{-2n}, \\ & \alpha^2 = t_3,\ \beta^2 = t_1,\ \beta t_1 \beta^{-1} = t_1,\ \beta t_2 = t_2^{-1} \beta t_3^{-2n}, \\ & \alpha\beta = t_1^{-1} t_2^{-1} \beta \alpha t_3^{-(2n+1)} \rangle, \end{aligned}$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{8n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right),$$

$$\beta = \left(\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{8} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)\right)$$

Let N be a normal nilpotent subgroup of π_4 isomorphic to Γ_p such that π_4/N is abelian. Then by Proposition 3.1,

$$N = \langle t_1^{d_1} t_2^{m} t_3^{\ell},\ t_2^{d_2} t_3^{r},\ t_3^{\frac{4nd_1d_2}{p}} \rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{4nd_1d_2}{p}\right).$$

Since $[\pi_4,\pi_4]=\langle t_1^2,\ t_2^2,\ t_3^2,\ t_1t_2t_3\rangle \subset N$, d_1 must be 1, and d_2 must be 1 or 2.

If $d_1 = d_2 = 1$ and $p = 4n$, then there exists only one normal nilpotent subgroup

$$N_1 = \langle t_1,\ t_2,\ t_3\rangle.$$

If $d_1 = d_2 = 1$ and $p = 2n$, since $t_1t_2t_3 \in N$, then there exist two normal nilpotent subgroups

$$N_2 = \langle t_1,\ t_2t_3,\ t_3^2\rangle, \qquad N_3 = \langle t_1t_3,\ t_2,\ t_3^2\rangle.$$

However, it is easy to see that $N_2 \sim N_3$ by using $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_4)$, where

$$\mu = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)\right).$$

If $d_1 = 1$, $d_2 = 2$ and $p = 8n$, then there exists only one normal nilpotent subgroup

$$L_1 = \langle t_1t_2,\ t_2^2,\ t_3\rangle.$$

If $d_1 = 1$, $d_2 = 2$ and $p = 4n$, then there exists only one normal nilpotent subgroup

$$L_2 = \langle t_1t_2t_3,\ t_2^2,\ t_3^2\rangle.$$

The realization of the free action of the group G on $\mathcal{N}_p$ can be done easily by using the method in Theorem 3.3. □

According to Theorems 3.4 and 3.5, if $p = 1$, then we obtain the following result, which is the same as Theorem 3.2 of [1], where $\mathcal{N} = \mathcal{H}/\Gamma_1$ is called the *standard nilmanifold.*

COROLLARY 3.6. *There does not exist any finite group acting freely* (up to topological conjugacy) *on the standard nilmanifold $\mathcal{N}$ which yields an orbit manifold homeomorphic to $\mathcal{H}/\pi_3$ or $\mathcal{H}/\pi_4$.*

THEOREM 3.7. (π_5) (A) *Table 4 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{5,1}$ or $\mathcal{H}/\pi_{5,3}$.*

Table 4

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{4K}{p}}$	$\langle\alpha\rangle$	$\frac{K}{p}\in\mathbb{N}$	$N_1=\langle t_1,\ t_2,\ t_3^{\frac{K}{p}}\rangle$
	${}^{\xi}\langle\alpha\rangle$	$(*)\frac{K}{2p}\in\mathbb{N}$	$N_2=\langle t_1t_3^{\frac{K}{2p}},\ t_2t_3^{\frac{K}{2p}},\ t_3^{\frac{K}{p}}\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_{\frac{8K}{p}}$	${}^{\eta}\langle\ t_2,\alpha\rangle$	$\frac{2K}{p}\in\mathbb{N},\ \ p\geq 2$	$L=\langle t_1t_2,\ t_2^2,\ t_3^{\frac{2K}{p}}\rangle$

where $K=4n-2$ *for the case of* $\pi_{5,1}$ *or* $K=4n$ *for the case of* $\pi_{5,3}$, *and*

$$\xi=\left(\mathrm{I},\ \left(\begin{bmatrix}\frac{1}{2p}\\-\frac{1}{2p}\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\right),\qquad \eta=\left(\mathrm{I},\ \left(\begin{bmatrix}0\\\frac{1}{4}\end{bmatrix},\begin{bmatrix}1&0\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\right)\right).$$

Here $(*)$: *for the case of* $K=4n$, N_1 *is affinely conjugate to* N_2 *only when* $p=1$.

(B) *Table 5 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups* G *on* $\mathcal{N}_p$ *which yield an orbit manifold homeomorphic to* $\mathcal{H}/\pi_{5,2}$.

Table 5

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{16n}{p}}$	$\langle\alpha^{-1}t_3\rangle$	$\frac{4n}{p}\in\mathbb{N}$	$N_1=\langle t_1,\ t_2,\ t_3^{\frac{4n}{p}}\rangle$
	${}^{\xi}\langle\alpha^{-1}t_3\rangle$	$\frac{2n}{p}\in\mathbb{N},\ \ p\geq 2$	$N_2=\langle t_1t_3^{\frac{2n}{p}},\ t_2t_3^{\frac{2n}{p}},\ t_3^{\frac{4n}{p}}\rangle$
$\mathbb{Z}_2\times\mathbb{Z}_{\frac{32n}{p}}$	${}^{\eta}\langle\ t_2,\alpha^{-1}t_3\rangle$	$\frac{8n}{p}\in\mathbb{N},\ \ p\geq 2$	$L=\langle t_1t_2,\ t_2^2,\ t_3^{\frac{8n}{p}}\rangle$

Proof. Recall that

$$\pi_{5,i}=\langle t_1,t_2,t_3,\alpha\mid [t_2,t_1]=t_3^K,\ [t_3,\alpha]=[t_3,t_1]=[t_3,t_2]=1,$$
$$\alpha t_1\alpha^{-1}=t_2,\ \alpha t_2\alpha^{-1}=t_1^{-1},\ \alpha^4=t_3^m\rangle,$$

$$\alpha=\left(\begin{bmatrix}1&0&-\frac{m}{4K}\\0&1&0\\0&0&1\end{bmatrix},\ \left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0&-1\\1&0\end{bmatrix}\right)\right),$$

where $1 \le i \le 3$, $K = 4n-2$ for the case of $\pi_{5,1}$, or $K = 4n$ for the cases of $\pi_{5,2}$ and $\pi_{5,3}$, and $m = 3$ for the case of $\pi_{5,2}$, or $m = 1$ otherwise.

Let N be a normal nilpotent subgroup of $\pi_{5,i}(i = 1, 2, 3)$ and isomorphic to Γ_p such that $\pi_{5,i}/N$ is abelian. Then by Proposition 3.1,

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{Kd_1d_2}{p}}\,\rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{Kd_1d_2}{p}\right).$$

Since $[\pi_{5,i}, \pi_{5,i}] = \langle t_2 t_1^{-1},\ t_1^{-1} t_2^{-1},\ t_3^K\rangle = \langle t_1 t_2,\ t_2^2,\ t_3^K\rangle \subset N$, d_1 must be 1 and d_2 must be 1 or 2.

(i) When $d_1 = d_2 = 1$: We have

$$N = \langle t_1 t_3^\ell,\ t_2 t_3^r,\ t_3^{\frac{K}{p}}\rangle, \quad \left(0 \le \ell, r < \frac{K}{p},\ \frac{K}{p} \in \mathbb{N}\right).$$

Since N is a normal subgroup of $\pi_{5,i}$, if K is a multiple of p, then the following two relations

$$\alpha(t_1 t_3^\ell)\alpha^{-1} = (t_2 t_3^r) t_3^{\ell - r} \in N,$$
$$\alpha(t_2 t_3^r)\alpha^{-1} = (t_1 t_3^\ell)^{-1} t_3^{\ell + r} \in N$$

show that $\ell = r = 0$ or $\frac{K}{2p}$ with the condition $\frac{K}{2p} \in \mathbb{N}$. Thus the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2,\ t_3^{\frac{K}{p}}\rangle, \qquad N_2 = \langle t_1 t_3^{\frac{K}{2p}},\ t_2 t_3^{\frac{K}{2p}},\ t_3^{\frac{K}{p}}\rangle.$$

Recall that ([1, Theorem 3.3.(Type 5)]) the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_{5,i})$ is of the form

$$\mu = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix},\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right),$$

where $x = p/2,\ y = q/2\ \ (p,\, q \in \mathbb{Z}),\ z \in \mathbb{R}$, and x^2 must be multiple of m/K, and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{Z}_4 \rtimes \mathbb{Z}_2 = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\rangle.$$

Note that

$$\mu = \left(\begin{bmatrix} 1 & -\frac{1}{2p} & z \\ 0 & 1 & \frac{1}{2p} \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ T\right)\right),\ T \in \mathbb{Z}_4 \rtimes \mathbb{Z}_2,$$

are the only ones that can conjugate N_1 onto N_2. Therefore if p=1 and $K = 4n$, then N_1 is affinely conjugate to N_2, but when $K = 4n - 2$, N_1 is not affinely conjugate to N_2 because $x^2 = 1/4$ is not a multiple of $1/(4n-2)$. If $p > 1$, then $\mu \notin N_{\mathrm{Aff}(\mathcal{H})}(\pi_{5,i})$. Thus N_1 is not affinely conjugate to N_2.

(ii) When $d_1 = 1,\ d_2 = 2$: By Proposition 3.1,

$$L = \langle\ t_1 t_2 t_3^{\ell},\ t_2^2 t_3^r,\ t_3^{\frac{2K}{p}}\ \rangle,$$

where $0 \le \ell,\, r < \frac{2K}{p}$. However, $[\pi_{5,i},\, \pi_{5,i}] = \langle t_1 t_2,\, t_2^2,\, t_3^K\rangle \subset N$ implies that $\ell = r = 0$ and $p \geq 2$. Thus there exist only one normal nilpotent subgroup

$$L = \langle\ t_1 t_2,\ t_2^2,\ t_3^{\frac{2K}{p}}\ \rangle.$$

The realization of the action of $G = \pi_{5,i}/N$ on the nilmanifold on $\mathcal{N}_p$ is easy if we follow the procedure of Theorem 3.3. We shall deal with only the case of $\pi_{5,2}$. Since $\alpha^4 = t_3^3$, it will be nice if we can find $\beta \in \pi_{5,2}$ such that $\beta^4 = t_3$. In fact, $\beta = \alpha^{-1} t_3$ is such one. Since N_1 is equal to Γ_p itself, the group $G{=}\pi_{5,2}/N_1 \cong \mathbb{Z}_{\frac{4K}{p}} \cong \mathbb{Z}_{\frac{16n}{p}}$ is generated by the image of β, where

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{3}{16n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \right) \in \pi_{5,2},$$

and

$$\beta = \alpha^{-1} t_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{16n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \right)$$

However N_2 in $\pi_{5,2}$ is not equal to Γ_p, but isomorphic to Γ_p. Therefore we must conjugate the representation of $\pi_{5,2}$ so that N_2 becomes Γ_p by means of an automorphism

$$\xi = \left(\mathrm{I},\ \left(\begin{bmatrix} \frac{1}{2p} \\ -\frac{1}{2p} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) \in \mathrm{Aff}(\mathcal{H}).$$

Thus we can see that $\xi\, N_2\, \xi^{-1} = \Gamma_p$, and

$$\xi\, \beta\, \xi^{-1} = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{16n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} \frac{1}{p} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \right)$$

describes an action of $\pi_{5,2}/N_2 \cong \mathbb{Z}_{\frac{16n}{p}}$ on $\mathcal{N}_p$.

THEOREM 3.8. *($\pi_6 - a$) Table 6 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{6,i}$* $(i = 1, 2)$.

Table 6

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{9n}{p}}$	$\langle\hat{\alpha}\rangle$	$\frac{3n}{p} \in \mathbb{N}$	$N_1 = \langle\, t_1,\ t_2,\ t_3^{\frac{3n}{p}}\,\rangle$
	${}^{\xi_2}\langle\hat{\alpha}\rangle$	$\frac{n}{p} \in \mathbb{N},\ \ p \geq 2$	$N_2 = \langle\, t_1 t_3^{\frac{n}{p}},\ t_2 t_3^{\frac{n}{p}},\ t_3^{\frac{3n}{p}}\,\rangle$
	${}^{\xi_3}\langle\hat{\alpha}\rangle$	$\frac{n}{p} \in \mathbb{N},\ \ p \geq 3$	$N_3 = \langle\, t_1 t_3^{\frac{2n}{p}},\ t_2 t_3^{\frac{2n}{p}},\ t_3^{\frac{3n}{p}}\,\rangle$
$\mathbb{Z}_3 \times \mathbb{Z}_{\frac{27n}{p}}$	${}^{\eta}\langle t_2, \hat{\alpha}\rangle$	$\frac{9n}{p} \in \mathbb{N},\ p \in 3\mathbb{N}$	$L = \langle\, t_1 t_2^2,\ t_2^3,\ t_3^{\frac{9n}{p}}\,\rangle$

where $\hat{\alpha} = \alpha$ *for the case of* $\pi_{6,1}$, *or* $\hat{\alpha} = \alpha^{-1}t_3$ *for the case of* $\pi_{6,2}$, *and*

$$\xi_2 = \left(\mathrm{I},\ \left(\begin{bmatrix} \frac{1}{3p} \\ -\frac{1}{3p} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right), \qquad \xi_3 = \left(\mathrm{I},\ \left(\begin{bmatrix} \frac{2}{3p} \\ -\frac{2}{3p} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right),$$

$$\eta = \left(\mathrm{I},\ \left(\begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}\right)\right).$$

Proof. Recall that

$$\pi_{6,i} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{3n},\ [t_3, \alpha] = [t_3, t_1] = [t_3, t_2] = 1,$$
$$\alpha t_1 \alpha^{-1} = t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1} t_2^{-1},\ \alpha^3 = t_3^m \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{m}{9n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\right)\right),$$

where $i = 1, 2$, $m = 1$ for the case of $\pi_{6,1}$, and $m = 2$ for the case of $\pi_{6,2}$.

Let N be a normal nilpotent subgroup of $\pi_{6,i}$ and isomorphic to Γ_p. Then by Proposition 3.1,

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{3nd_1d_2}{p}}\,\rangle, \quad \left(0 \leq m < d_2,\ 0 \leq \ell, r < \frac{3nd_1d_2}{p}\right).$$

Since $\pi_{6,i}/N$ is abelian, the following relation

$$[\pi_{6,i}, \pi_{6,i}] = \langle t_2 t_1^{-1},\ t_1^{-1} t_2^{-2},\ t_3^{3n}\rangle = \langle t_1 t_2^{-1},\ t_2^3,\ t_3^{3n}\,\rangle \subset N$$

induces that d_1 must be 1 and d_2 must be 1 or 3.

First we shall deal with the case of $d_1 = d_2 = 1$. Then we have

$$N = \langle t_1 t_3^{\ell},\ t_2 t_3^{r},\ t_3^{\frac{3n}{p}} \rangle, \quad \left(0 \le \ell, r < \frac{3n}{p},\ \frac{3n}{p} \in \mathbb{N}\right).$$

By the normality of N, the following two relations

$$\alpha(t_1 t_3^{\ell})\alpha^{-1} = t_2 t_3^{\ell} = (t_2 t_3^{r}) t_3^{\ell - r} \in N,$$
$$\alpha(t_2 t_3^{r})\alpha^{-1} = (t_1 t_3^{\ell})^{-1}(t_2 t_3^{r})^{-1} t_3^{\ell + 2r} \in N$$

show that $\ell = r = 0$, $\frac{n}{p}$ or $\frac{2n}{p}$, whenever $\frac{n}{p} \in \mathbb{N}$. Thus the possible normal nilpotent subgroups are

$$N_1 = \langle t_1,\ t_2,\ t_3^{\frac{3n}{p}} \rangle,\ N_2 = \langle t_1 t_3^{\frac{n}{p}},\ t_2 t_3^{\frac{n}{p}},\ t_3^{\frac{3n}{p}} \rangle,\ N_3 = \langle t_1 t_3^{\frac{2n}{p}},\ t_2 t_3^{\frac{2n}{p}},\ t_3^{\frac{3n}{p}} \rangle.$$

We shall show that if $\frac{n}{p} \in \mathbb{N}$ and $p \ge 3$, then these are not affinely conjugate each other. By applying the method used in Theorem 3.3, we can find the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i})$:

$$\mu(x, y, z, u, v) = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right),$$

where $z \in \mathbb{R}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{Z}_6 \rtimes \mathbb{Z}_2 = \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle,$$

and if $ad - bc = 1$, then

$$x = (2r - s)/3,\ y = (r + s)/3 \ \ (r,\ s \in \mathbb{Z});$$

if $ad - bc = -1$, then

$$x = (p + q)/3,\ y = (2p - q)/3 \ \ (p,\ q \in \mathbb{Z}).$$

Note that $(u, v) \in \mathrm{Aut}(\mathcal{H})$ can be evaluated respectively by the elements of $\mathbb{Z}_6 \rtimes \mathbb{Z}_2$. More precisely, (u, v) is either $(0, 1/2)$, $(-1/2, 0)$, $(0, 0)$, $(-1/3, 1/3)$, $(-1/3, -1/6)$, or $(1/6, 1/3)$. For example, we can find

$$\mu(x, y, z, -\frac{1}{3}, \frac{1}{3}) = \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix},\ \left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i}).$$

It is interesting to see that if $p=1$, then $N_1 \sim N_2$ and $N_2 \sim N_3$ by the conjugation by $\mu(-\frac{1}{3}, \frac{1}{3}, 0, 0, 0)$, which also induces that if $p=2$, then $N_1 \sim N_3$. It is not hard to see that there does not exist $\mu \in N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i})$ which conjugates N_1 onto N_2. For example,

$$\mu_* = \left(\begin{bmatrix} 1 & -\frac{1}{3p} & z \\ 0 & 1 & \frac{1}{3p} \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, T \right) \right), \quad T = \mathrm{I} \text{ or } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

are the ones that can conjugate both N_1 onto N_2 and N_2 onto N_3. However, when $p > 1$,

$$\mu_* \notin N_{\mathrm{Aff}(\mathcal{H})}(\pi_{6,i}) \quad (i = 1, 2).$$

Thus we proved that N_1(or N_2, respectively) is not affinely conjugate to N_2(or N_3, respectively). Similarly we can prove that N_1 is not affinely conjugate to N_3 except for $p=2$. Therefore if $\frac{n}{p} \in \mathbb{N}$ and $p \geq 3$, then there exist 3 affinely non-conjugate normal nilpotent subgroups N_i $(i = 1,2,3)$ of $\pi_{6,i}$.

For the case of $d_1 = 1$ and $d_2 = 3$, if $\frac{9n}{p} \in \mathbb{N}$ and $p \in 3\mathbb{N}$, then it is easy to see that there exists only one normal nilpotent subgroup L isomorphic to Γ_p, where

$$L = \langle\, t_1 t_2^2,\ t_2^3,\ t_3^{\frac{9n}{p}}\, \rangle.$$

The realization of the free action of the group G on $\mathcal{N}_p$ can be done easily by using the method in Theorem 3.3. □

THEOREM 3.9. *($\pi_6 - b$) Table 7 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{6,i}$ $(i = 3, 4)$.*

Table 7

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{3K}{p}}$	$\langle\hat{\alpha}\rangle$	$\frac{K}{p} \in \mathbb{N}$	$N = \langle\, t_1,\ t_2,\ t_3^{\frac{K}{p}}\, \rangle$
$\mathbb{Z}_3 \times \mathbb{Z}_{\frac{9K}{p}}$	${}^{\eta}\langle t_2, \hat{\alpha}\rangle$	$\frac{3K}{p} \in \mathbb{N},\ p \in 3\mathbb{N}$	$L = \langle\, t_1 t_2^2,\ t_2^3,\ t_3^{\frac{3K}{p}}\, \rangle$

where $K = 3n-2$ and $\hat{\alpha} = \alpha^{-1} t_3$ for the case of $\pi_{6,3}$, or $K = 3n-1$ and $\hat{\alpha} = \alpha$ for the case of $\pi_{6,4}$, and

$$\eta = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \right) \right).$$

Proof. Let N be a normal nilpotent subgroup of $\pi_{6,i}$ ($i = 3$, 4), and isomorphic to Γ_p. In the proof of Theorem 3.8, when $d_1 = d_2 = 1$, we have

$$N = \langle t_1 t_3^{\ell},\ t_2 t_3^{r},\ t_3^{\frac{K}{p}} \rangle, \quad \left(0 \le \ell, r < \frac{K}{p},\ \frac{K}{p} \in \mathbb{N}\right).$$

By the normality of N, the following two relations

$$\alpha(t_1 t_3^{\ell})\alpha^{-1} = t_2 t_3^{\ell} = (t_2 t_3^{r}) t_3^{\ell - r} \in N,$$
$$\alpha(t_2 t_3^{r})\alpha^{-1} = (t_1 t_3^{\ell})^{-1} (t_2 t_3^{r})^{-1} t_3^{\ell + 2r} \in N$$

show that $\ell = r = 0$, because $\frac{K}{3p}$ cannot be an integer for $K = 3n - 1$ or $3n - 2$. Thus there exists only one normal nilpotent subgroup

$$N = \langle t_1,\ t_2,\ t_3^{\frac{K}{p}} \rangle.$$

The remaining work can be done similarly as Theorem 3.8. □

THEOREM 3.10. (π_7) *Table 8 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_{7,i}$, ($i = 1, 2, 3, 4$).*

Table 8

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{\frac{6K}{p}}$	$\langle \hat{\alpha} \rangle$	$\frac{K}{p} \in \mathbb{N}$	$N = \langle t_1,\ t_2,\ t_3^{\frac{K}{p}} \rangle$

where $K = 6n$ and $\hat{\alpha} = \alpha$ for the case of $\pi_{7,1}$, $K = 6n - 2$ and $\hat{\alpha} = \alpha$ for the case of $\pi_{7,2}$, $K = 6n$ and $\hat{\alpha} = \alpha^{-1} t_3$ for the case of $\pi_{7,3}$, or $K = 6n - 4$ and $\hat{\alpha} = \alpha^{-1} t_3$ for the case of $\pi_{7,4}$.

Proof. Recall that

$$\pi_{7,i} = \langle t_1, t_2, t_3, \alpha \mid [t_2, t_1] = t_3^{K},\ [t_3, \alpha] = [t_3, t_1] = [t_3, t_2] = 1,$$
$$\alpha t_1 \alpha^{-1} = t_1 t_2,\ \alpha t_2 \alpha^{-1} = t_1^{-1},\ \alpha^6 = t_3^{m} \rangle,$$

$$\alpha = \left(\begin{bmatrix} 1 & 0 & -\frac{m}{6K} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \right) \right),$$

where $1 \le i \le 4$, $m = 1$ for the cases of $\pi_{7,1}$ and $\pi_{7,2}$, and $m = 5$ otherwise.

Let N be a normal nilpotent subgroup of $\pi_{7,i}$ and isomorphic to Γ_p such that $\pi_{7,i}/N$ is abelian. Then

$$N = \langle\, t_1^{d_1} t_2^m t_3^\ell,\ t_2^{d_2} t_3^r,\ t_3^{\frac{Kd_1d_2}{p}}\,\rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{Kd_1d_2}{p}\right)$$

by Proposition 3.1. Since $[\pi_{7,i}, \pi_{7,i}] = \langle t_1^{-1}t_2^{-1},\ t_2,\ t_3^K\rangle = \langle t_1,\ t_2,\ t_3^K\rangle \subset N$, d_1 and d_2 must be 1. Then we have

$$N = \langle t_1 t_3^\ell,\ t_2 t_3^r,\ t_3^{\frac{K}{p}}\rangle, \quad \left(0 \le \ell, r < \frac{K}{p},\ \frac{K}{p} \in \mathbb{N}\right).$$

By the normality of N,

$$\begin{aligned} \alpha(t_1t_3^\ell)\alpha^{-1} &= t_1t_2t_3^r = (t_1t_3^\ell)(t_2t_3^r)t_3^{-r} \in N, \\ \alpha(t_2t_3^r)\alpha^{-1} &= t_1^{-1}t_3^r = (t_1t_3^\ell)^{-1}t_3^{\ell+r} \in N, \end{aligned}$$

show that $\ell = r = 0$. Therefore there exists only one normal nilpotent subgroup

$$N = \langle t_1,\ t_2,\ t_3^{\frac{K}{p}}\rangle\ ,$$

isomorphic to Γ_p.

The realization of the action of $G = \pi_{7,i}/N$, $(1 \le i \le 4)$ on $\mathcal{H}/N$ can be done by the similar method of the other cases. □

THEOREM 3.11. (π_1) *Table 9 gives a complete list of all free actions*(up to topological conjugacy) *of finite abelian groups G on $\mathcal{N}_p$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\pi_1$.*

Table 9

Group G	Generators	AC classes of normal nilpotent subgroups	
$\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \mathbb{Z}_{\frac{nd_1d_2}{p}}$	${}^{\xi}\langle t_1, t_2, t_3\rangle$	$\frac{nd_1d_2}{p} \in \mathbb{N}$, $p \ge d_1d_2,\ d_1 \ge d_2$	$N = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{nd_1d_2}{p}}\rangle$

where

$$\xi = \left(\mathrm{I}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{bmatrix}\right)\right)$$

Proof. Recall that

$$\pi_1 = \langle t_1, t_2, t_3, \mid [t_2, t_1] = t_3^n,\ \ [t_3, t_1] = [t_3, t_2] = 1\rangle.$$

Let N be a normal nilpotent subgroup of π_1 isomorphic to Γ_p such that π_1/N is abelian. Then

$$N = \langle\, t_1^{d_1} t_2^m t_3^{\ell},\ t_2^{d_2} t_3^r,\ t_3^{\frac{nd_1d_2}{p}}\,\rangle, \quad \left(0 \le m < d_2,\ 0 \le \ell, r < \frac{nd_1d_2}{p}\right)$$

by Proposition 3.1.

Note that $[\pi_1,\pi_1]=\langle\, t_3^n\,\rangle \subset N$, and for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Z})$,

$$N_{\mathrm{Aff}(\mathcal{H})}(\pi_1) = \left\{ \left(\begin{bmatrix} 1 & x & * \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \,\middle|\, x, y, u, v \in \mathbb{R} \right\},$$

where $n(cx + cu - ay - av + \frac{1}{2}ac)$, $n(dx + du - by - bv + \frac{1}{2}bd) \in \mathbb{Z}$.
It needs some calculations to see that N is affinely conjugate to

$$N_m = \langle t_1^{d_1} t_2^m,\ t_2^{d_2},\ t_3^{\frac{nd_1d_2}{p}}\rangle$$

using the conjugation by

$$\left(\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & \frac{mr}{nd_1d_2} \\ 0 & 0 & 1 \end{bmatrix}, \left(\begin{bmatrix} \frac{r}{nd_2} \\ -\frac{\ell}{nd_1} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right),$$

where $0 \le m < q = \gcd(d_1,\, d_2)$, which can be obtained by using other representation of N_m and $\mathrm{GL}(2,\mathbb{Z}) \in \mathrm{Aut}(\mathcal{H})$. If $q = 1$, then N_m is affinely conjugate to $N_0 = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{nd_1d_2}{p}}\rangle$. Note that

$$N_0 \sim \langle t_1^{d_2},\ t_2^{d_1},\ t_3^{\frac{nd_1d_2}{p}}\rangle$$

by using $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Aut}(\mathcal{H})$. However if $q \ge 2$ and $\gcd(m, q) = c$, then

$$N_m \sim \langle t_1^c,\ t_2^{\frac{d_1d_2}{c}},\ t_3^{\frac{nd_1d_2}{p}}\rangle$$

by using $\mathrm{GL}(2,\mathbb{Z}) \in \mathrm{Aut}(\mathcal{H})$. That is, π_1/N_m is isomorphic to either $\mathbb{Z}_d \times \mathbb{Z}_{\frac{dn}{p}}$ for the case of $c = 1$ or $\mathbb{Z}_c \times \mathbb{Z}_{\frac{d_1d_2}{c}} \times \mathbb{Z}_{\frac{nd_1d_2}{p}}$ for the case of

$c \geq 2$. Recall that c is a common divisor of d_1 and d_2. Therefore there exists only one normal nilpotent subgroup

$$N = N_0 = \langle t_1^{d_1},\ t_2^{d_2},\ t_3^{\frac{nd_1d_2}{p}} \rangle,$$

where $\frac{nd_1d_2}{p} \in \mathbb{N},\ p \geq d_1d_2,\ d_1 \geq d_2.$ □

According to previous Theorems, if $p = 1$, then we obtain the following result which is the same as the Theorem 3.3 of [1].

COROLLARY 3.12. *Suppose G is a finite group acting freely*(up to topological conjugacy) *on the standard nilmanifold $\mathcal{N}$. Then G is cyclic, and it is one of the following :*

Table 10

π	*Group G*	*AC classes of normal nilpotent subgroups isomorphic to* Γ_1
π_1	$\mathbb{Z}_n = \langle t_3 \rangle$	$N = \langle t_1, t_2, t_3^n \rangle$.
π_2	$\mathbb{Z}_{4n} = \langle \alpha \rangle$	$N = \langle t_1, t_2, t_3^{2n} \rangle$.
$\pi_{5,1}$	$\mathbb{Z}_{16n-8} = \langle \alpha \rangle$	$N_1 = \langle t_1, t_2, t_3^{4n-2} \rangle$,
	$\mathbb{Z}_{16n-8} = {}^{\xi}\langle \alpha \rangle$	$N_2 = \langle t_1t_3^{2n-1}, t_2t_3^{2n-1}, t_3^{4n-2} \rangle$.
$\pi_{5,2}$	$\mathbb{Z}_{16n} = \langle \alpha^{-1}t_3 \rangle$	$N = \langle t_1, t_2, t_3^{4n} \rangle$.
$\pi_{5,3}$	$\mathbb{Z}_{16n} = \langle \alpha \rangle$	$N = \langle t_1, t_2, t_3^{4n} \rangle$.
$\pi_{6,1}$	$\mathbb{Z}_{9n} = \langle \alpha \rangle$	$N = \langle t_1, t_2, t_3^{3n} \rangle$.
$\pi_{6,2}$	$\mathbb{Z}_{9n} = \langle \alpha^{-1}t_3 \rangle$	$N = \langle t_1, t_2, t_3^{3n} \rangle$.
$\pi_{6,3}$	$\mathbb{Z}_{9n-6} = \langle \alpha^{-1}t_3 \rangle$	$N = \langle t_1, t_2, t_3^{3n-2} \rangle$.
$\pi_{6,4}$	$\mathbb{Z}_{9n-3} = \langle \alpha \rangle$	$N = \langle t_1, t_2, t_3^{3n-1} \rangle$.
$\pi_{7,1}$	$\mathbb{Z}_{36n} = \langle \alpha \rangle$	$N = \langle t_1, t_2, t_3^{6n} \rangle$.
$\pi_{7,2}$	$\mathbb{Z}_{36n-12} = \langle \alpha \rangle$	$N = \langle t_1, t_2, t_3^{6n-2} \rangle$.
$\pi_{7,3}$	$\mathbb{Z}_{36n} = \langle \alpha^{-1}t_3 \rangle$	$N = \langle t_1, t_2, t_3^{6n} \rangle$.
$\pi_{7,4}$	$\mathbb{Z}_{36n-24} = \langle \alpha^{-1}t_3 \rangle$	$N = \langle t_1, t_2, t_3^{6n-4} \rangle$.

where $\xi = \left(\mathrm{I}, \left(\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\right) \in \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$.

The following example shows that a free action of a cyclic group on the nilmanifold $\mathcal{N}_2$ has more affinely non-conjugate classes than that on the nilmanifold $\mathcal{N}_1$ by comparing the example given in [1].

EXAMPLE. Assume $\mathbb{Z}_{48}$ acts freely on the nilmanifold $\mathcal{N}_2 = \mathcal{H}/\Gamma_2$. Then there exist 10 distinct topological conjugacy classes of free actions. In fact, there exist two distinct free actions in each infra-nilmanifold whose $\pi_1(\mathcal{N}_2/\mathbb{Z}_{48})$ is $\pi_{5,2}$ or $\pi_{5,3}$, three in π_2, and one in π_1, $\pi_{6,4}$ or $\pi_{7,2}$.

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