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THE QUARTIC MOMENT PROBLEM

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ABSTRACT. In this paper, we consider the quartic moment problem suggested by Curto-Fialkow[6]. Given complex numbers $\gamma \equiv \gamma^{(4)}$: $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, we discuss the conditions for the existence of a positive Borel measure μ, supported in the complex plane $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j d\mu$ $(0 \le i+j \le 4)$. We obtain sufficient conditions for flat extension of the quartic moment matrix $M(2)$. Moreover, we examine the existence of flat extensions for nonsingular positive quartic moment matrices $M(2)$.

1. Introduction and preliminaries

Given a closed subset $K \subseteq \mathbb{C}$ and a doubly indexed finite sequence of complex numbers

(1.1)

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \cdots, \gamma_{0,2n}, \gamma_{1,2n-1}, \cdots, \gamma_{2n-1,1}, \gamma_{2n,0},$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \overline{\gamma}_{ij}$, the *truncated K-moment problem* entails finding a positive Borel measure μ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \ (0 \le i+j \le 2n) \text{ and } \operatorname{supp} \mu \subseteq K; \tag{1.2}$$

γ is called a *truncated moment sequence* (of order $2n$) and μ is called a *representing measure* for γ ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]).

For $n \ge 1$, let $m \equiv m(n) := (n+1)(n+2)/2$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the set of $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering:

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$$\underbrace{1}_{(1)}, \underbrace{Z, \bar{Z}}_{(2)}, \underbrace{Z^2, \bar{Z}Z, \bar{Z}^2}_{(3)}, \underbrace{Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3}_{(4)},$$
$$\cdots, \underbrace{Z^n, \bar{Z}Z^{n-1}, \cdots, \bar{Z}^{n-1}Z, \bar{Z}^n}_{(n+1)}.$$

For $0 \le i+j \le n, 0 \le l+k \le n$, we denote the entry in row $\bar{Z}^l Z^k$, column $\bar{Z}^i Z^j$ of A by $A_{(l,k)(i,j)}$.

Given the truncated moment sequence γ in (1.1) and $0 \le i, j \le n$, we define the $(i+1) \times (j+1)$ matrix $M[i,j]$ whose entries are the moments of order $i+j$:

$$M[i,j] := \begin{pmatrix} \gamma_{i,j} & \gamma_{i+1,j-1} & \cdots & \cdots & \gamma_{i+j,0} \\ \gamma_{i-1,j+1} & \gamma_{i,j} & \cdots & \cdots & \gamma_{i+j-1,1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \gamma_{0,i+j} & \gamma_{1,i+j-1} & \cdots & \cdots & \gamma_{j,i} \end{pmatrix}, \tag{1.3}$$

where we note that $M[i,j]$ has the Toeplitz-like property of being constant on each diagonal; in particular, $M[i,i]$ is a self-adjoint Toeplitz matrix. We now define the *moment matrix* $M(n) \equiv M(n)(\gamma)$ via the block decomposition $M(n) := (M[i,j])_{0 \le i,j \le n}$. For example, if $n=1$, the *quartic moment problem* for $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{pmatrix} M[0,0] & M[0,1] \\ M[1,0] & M[1,1] \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}, \tag{1.4}$$

and if $n=2$, the *quartic moment problem* for

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40}$$

corresponds to

$$\begin{aligned} M(2) &= \begin{pmatrix} M[0,0] & M[0,1] & M[0,2] \\ M[1,0] & M[1,1] & M[1,2] \\ M[2,0] & M[2,1] & M[2,2] \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix} \end{aligned} \tag{1.5}$$

Let $\mathcal{P}_n \subseteq \mathbb{C}[z,\bar{z}]$ denote the complex polynomials in $z, \bar{z}$ of total degree $\leq n$. For $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0\leq i+j\leq n} a_{ij}\bar{z}^i z^j$, let $\hat{p} \equiv (a_{00}, a_{01}, a_{10}, \cdots, a_{0n}, \cdots, a_{n0})^T \in \mathbb{C}^{m(n)}$. For $M \in \mathcal{M}_{m(n)}(\mathbb{C})$ and $p, q \in \mathcal{P}_n$, let $\langle p, q\rangle_M := (M\hat{p}, \hat{q})$. For $M \in \mathcal{M}_{m(n)}$, let $\mathcal{C}_M$ denote the column space of M, i.e.,

$$\mathcal{C}_M := \langle 1, Z, \bar{Z}, \cdots, Z^n, \cdots, \bar{Z}^n\rangle \subseteq \mathbb{C}^{m(n)}.$$

For $p \in \mathcal{P}_n$, $p(z,\bar{z}) \equiv \sum_{0\leq i+j\leq n} a_{ij}\bar{z}^i z^j$, we define

$$p(Z,\bar{Z}) := \sum_{0\leq i+j\leq n} a_{ij}\bar{Z}^i Z^j \in \mathcal{C}_M;$$

thus, for $p, q \in \mathcal{P}_n$, we have $M\hat{p} = p(Z,\bar{Z})$ and $\langle p,q\rangle_M = (M\hat{p},\hat{q}) = (p(Z,\bar{Z}),\hat{q})$. The basic connection between $M(n)(\gamma)$ and any representing measure μ is provided by the identity

$$\int f\bar{g}\,d\mu = (M(n)\hat{f},\hat{g}) \quad (f, g \in \mathcal{P}_n);$$

in particular,

$$(M(n)\hat{f},\hat{f}) = \int |f|^2 d\mu \geq 0,$$

so if γ admits a representing measure, then $M(n) \geq 0$.

We say that $M(n)$ is *recursively generated* if

(RG) $\qquad p, q, pq \in \mathcal{P}_n, \quad p(Z,\bar{Z}) = 0 \Longrightarrow (pq)(Z,\bar{Z}) = 0.$

Recall from [6, Proposition 3.1] that if μ is a representing measure for γ, then

(1.6)

for $p \in \mathcal{P}_n$, $p(Z,\bar{Z}) = 0 \Longleftrightarrow \text{supp}\,\mu \subseteq \mathcal{Z}(p) := \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}$.

It follows from [6, Corollary 3.5] that

(1.7)

if μ is a representing measure for γ, then card supp $\mu \geq$ rank $M(n)$.

Motivated by (1.6), the *variety* of γ is defined by [8]

$$\mathcal{V}(\gamma) := \bigcap_{\substack{p\in\mathcal{P}_n \\ p(Z,\bar{Z})=0}} \mathcal{Z}(p). \tag{1.8}$$

$\mathcal{V}(\gamma)$ is a closed (possibly empty) subset of the plane, and (1.6)–(1.7) imply that if μ is a presenting measure for γ, then

(1.9) $\quad$ supp $\mu \subseteq \mathcal{V}(\gamma)$ and rank $M(n) \leq$ card supp $\mu \leq$ card $\mathcal{V}(\gamma)$.

Let $\rho(\gamma) := \text{card}\,\mathcal{V}(\gamma) - \text{rank}\,M(n)(\gamma)$. It follows easily from (1.9) that the condition $\rho(\gamma) < 0$ is an obstruction to the existence of representing measures.

For a positive matrix A, an *extension* of A is a block matrix of the form

$$\tilde{A} := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}. \tag{1.10}$$

We first recall some elements of the theory of positive extensions of moment matrices.

PROPOSITION 1.1. ([8, Proposition 1.7], [6, Theorem 5.13])

(i) *Suppose $M(n)$ is positive and recursively generated. If $M(n+1)$ is a flat extension of $M(n)$, then $M(n+1)$ is positive and recursively generated.*

(ii) *γ has a* rank$M(n)$*-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$, i.e., $M(n)$ can be extended to a positive moment matrix $M(n+1)$ satisfying* rank$M(n+1) = $ rank$M(n)$.

Suppose the analytic columns of $M(n), 1, Z, \cdots, Z^i, \cdots, Z^n$ are dependent. Then there exists a minimal positive integer $r, 1 \leq r \leq n$, such that $Z^r \in \langle 1, \cdots, Z^{r-1} \rangle$; thus there exists unique scalars $\phi_j (j = 0, 1, \cdots, r-1)$ such that

$$Z^r = \phi_0 1 + \cdots + \phi_{r-1} Z^{r-1}.$$

In this case, γ has a unique representing measure,

$$\mu = \rho_0 \delta_{z_0} + \rho_1 \delta_{z_1} + \cdots + \rho_{r-1} \delta_{z_{r-1}},$$

whose support $\{z_0, \cdots, z_{r-1}\}$ consists of the r distinct roots of

$$z^r - (\phi_0 + \cdots + \phi_{r-1} z^{r-1})$$

and whose densities ρ_j satisfy the Vandermonde equation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_{r-1} \\ \vdots & \vdots & & \vdots \\ z_0^{r-1} & z_1^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \vdots \\ \gamma_{0,r-1} \end{pmatrix}$$

It is interesting to find a flat extension $M(n+1)$ of $M(n)$. In order to construct a flat extension $M(n+1)$ of $M(n)$, one makes use of the following Smul'jan's Theorem.

PROPOSITION 1.2. [12] *For $A \geq 0$, the following statements are equivalent:*

(i) $\tilde{A} \geq 0$;

(ii) *There exists W such that $AW = B$ and $C \geq W^*AW$.*

If $A \geq 0$ and $AW = B$, i.e., $\operatorname{Ran} B \subseteq \operatorname{Ran} A$, there is a unique flat extension of the form (1.10), which we denote by $[A; B]$. For $M(n) \geq 0$, we want to construct a positive flat extension of the form $M(n+1) = [M(n); B(n)]$.

Given $\gamma \equiv \gamma^{(2n)}$, we may define blocks $B_{0,n+1}, \cdots, B_{n-1,n+1}$ via the analogue of (1.4), that is, $B_{i,n+1} := (\gamma_{i+t-s,n+1+s-t})_{0 \leq s \leq i; 0 \leq t \leq n+1}$. Now, given a block $B[n, n+1] \in \mathcal{M}_{n+1,n+2}(\mathbb{C})$, let

$$B := \begin{pmatrix} B_{0,n+1} \\ \vdots \\ B_{n-1,n+1} \\ B[n, n+1] \end{pmatrix}$$

and denote the successive columns of B by $Z^{n+1}, \bar{Z}Z^n, \cdots, \bar{Z}^{n+1}$, and the successive rows by *1*, $Z, \bar{Z}, \cdots, Z^n, \cdots, \bar{Z}^n$. Following [7], we say that $B[n, n+1]$ is *symmetric* if, whenever $i + j = n, k + l = n + 1$, then $B_{(i,j),(k,l)} = B_{(j,i),(l,k)}$; we also say that $B[n, n+1]$ satisfies *normality* if, whenever $i + j = n, k + l = n + 1, j \geq 1$, and $l \geq 1$, then $B_{(i,j),(k,l)} = B_{(i+1,j-1),(k+1,l-1)}$. Recall from [7, (1.11)–(1.13)] that to construct a flat moment matrix extension $M(n+1)$ of $M(n) \geq 0$ it is necessary and sufficient to construct a block $B[n, n+1]$ such that

$B[n, n+1]$ is symmetric and normal;

Ran $B \subseteq$ Ran $M(n)$, that is, $B = M(n)W$ for some W;

$C := W^*M(n)W$ is Toeplitz, i.e., constant on diagonals.

If B is a moment matrix extension block satisfying Ran $B \subseteq$ Ran $M(n)$, then $[M(n); B]$ is a moment matrix if and only if the C block is Toeplitz. Let $C := (c_{ij})_{1 \leq i,j \leq n+1}$ in $[M(n); B(n)]$. In principle, establishing this property entails $(n+1)^2$ test of the form $c_{ij} = c_{i+1,j+1}$, but several factors serve to reduce this number:

(i) C is self-adjoint, so it suffices to consider the main diagonal and the lower diagonals;

(ii) C is symmetric block [6, Proposition 2.3], so within each diagonal elements that are symmetric with respect to the diagonal midpoint are equal;

(iii) Normal bands in the B block propagate to normal bands in the C block.

Conditions (i)–(iii) thus lead to a *reduced C-block test* for normality. From this test, we know that if $M(2) > 0$ (i.e., $M(2)$ is positive and invertible), then $[M(2); B(2)]$ is a flat extension of $M(2)$ if and only if $c_{11} = c_{22}$ and $c_{21} = c_{32}$.

CURTO-FIALKOW'S QUARTIC COMPLEX MOMENT PROBLEM. *If $M(2)(\gamma)$ is the positive moment matrix corresponding to the moment sequence*

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40},$$

does $M(2)(\gamma)$ have a representing measure ?

Related to the singular (i.e., $\det M(2) = 0$) quartic moment problem, Curto and Fialkow showed good results in [8] and [9], but the nonsingular case is still open.

THEOREM 1.3. [8, Theorem 1.10] *Suppose $M(2)(\gamma)$ is positive and recursively generated. Then γ has a* rank$M(2)$*-atomic representing measure in each of the following cases:*

(i) *$\{1, Z, \bar{Z}, Z^2\}$ is linearly dependent in $\mathcal{C}_{M(2)}$;*
(ii) *$\{1, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, and the moments γ_{ij} are all real, with the possible exception of γ_{04};*
(iii) *$\{1, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, and the reduced C-block test $c_{11} = c_{22}$ passes;*
(iv) *$\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$, $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, and the reduced C-block test $c_{11} = c_{22}$ passes for some choice of γ_{05}.*

THEOREM 1.4. [9, Theorem 1.3] *Suppose $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2\}$ is independent in $\mathcal{C}_{M(2)}$, and $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2, D \neq 0$. The following are equivalent:*

(i) *$\gamma^{(4)}$ admits a 4-atomic* (minimal) *representing measure;*
(ii) *$M(2)$ admits a flat extension $M(3)$;*
(iii) *there exists $\gamma_{23} \in \mathbb{C}$ such that $\bar{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}$.*

THEOREM 1.5. [9, Theorem 4.1] *Suppose $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. Then $\gamma^{(4)}$ admits a 5-atomic* (minimal) *representing measure if and only if there exists $\gamma_{23} \in \mathbb{C}$ such that the C-block of $[M(2); B(3)[\gamma_{23}]]$ satisfies $C_{21} = C_{32}$.*

THEOREM 1.6. [9, Theorem 1.5] *Suppose $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$. Then $\gamma^{(4)}$ admits a representing measure.*

In Section 2, we obtain some conditions for the positivity of $M(2)$ without any representing measure, which recapture a singular positive moment matrix $M(2)$ admitting no representing measure and hence having no flat extension $M(3)$(cf. [9]). In Section 3, by computer algebra (algorithm) independently, we show that if $M(2)$ is positive, $\{1, Z, \bar{Z}, Z^2\}$ is independent and $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, then $M(2)$ admits a flat extension $M(3)$ (Theorem 3.2); if $M(2)$ is positive, $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is independent and $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2\rangle$, then $M(2)$ admits a flat extension $M(3)$ (Theorem 3.4). Finally, in Section 4 we show that if $M(2) > 0$, $M(1) = I, B[1,2] = 0$ and the set S_t (see Proposition 4.3) is not empty, then $M(2)$ admits a flat extension $M(3)$ and hence a representing measure (Theorem 4.4).

Some of the calculations in this article were obtained throughout computer experiments using the software tool *Mathematica* ([13]).

2. Moment matrices with no representing measure

We will find a singular moment matrix $M(2)$ admitting no representing measure by illustrating $\rho(\gamma) < 0$ in this section. In other words, we want to find polynomials p and q in $\mathcal{P}_2$ such that

(1) $p(Z, \bar{Z}) = 0$, $q(Z, \bar{Z}) = 0$,

(2) card $(\mathcal{Z}(p) \cap \mathcal{Z}(q)) < 4$.

To do so we consider a truncated moment sequence $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = v, \gamma_{12} = u, \gamma_{04} = h, \gamma_{13} = w, \gamma_{22} = x$. Then the corresponding moment matrix is given by

$$(2.1) \qquad M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & u & \bar{u} & \bar{v} \\ 0 & 0 & 1 & v & u & \bar{u} \\ 0 & \bar{u} & \bar{v} & x & \bar{w} & \bar{h} \\ 1 & u & \bar{u} & w & x & \bar{w} \\ 0 & v & u & h & w & x \end{pmatrix}.$$

Then we first analyze the positivity of $M(2)$ in (2.1).

LEMMA 2.1. *Let $M(2)$ be a moment matrix as in (2.1). Then $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$ if and only if the following hold:*

(i) $x > |u|^2 + |v|^2$,

(ii) $x^2 - (1+3|u|^2+|v|^2)x + |u|^4 + |u|^2 + |v|^2 - |w|^2 + |u|^2|v|^2 + 2Re(u\bar{v}w + u^2\bar{w} - u^3\bar{v}) = 0$,

(iii) $x^2 - 2(|u|^2 + |v|^2)x + |u|^4 + |v|^4 - |h|^2 - 2|u|^2|v|^2 + 4Re(uv\bar{h}) = 0$.

Proof. Suppose $M(2) \geq 0$ and $\{\mathit{1}, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$. If we let

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & v \\ 0 & \bar{u} & \bar{v} & x \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & u & \bar{u} \\ 0 & 0 & 1 & v & u \\ 0 & \bar{u} & \bar{v} & x & \bar{w} \\ 1 & u & \bar{u} & w & x \end{pmatrix},$$

and

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & u & \bar{v} \\ 0 & 0 & 1 & v & \bar{u} \\ 0 & \bar{u} & \bar{v} & x & \bar{h} \\ 0 & v & u & h & x \end{pmatrix},$$

then $\det A > 0$, $\det B = 0$ and $\det C = 0$. Thus the properties (i), (ii) and (iii) are satisfied.

Conversely, suppose the properties (i), (ii), and (iii) are satisfied. Then by (i), $\{\mathit{1}, Z, \bar{Z}, Z^2\}$ is linearly independent. Also by (ii), (iii) and Extension Principle ([10]; [6, Proposition 3.9]), $\bar{Z}Z$ and $\bar{Z}^2$ are contained in the span of $\{\mathit{1}, Z, \bar{Z}, Z^2\}$. Moreover, the positivity of $M(2)$ immediately follows from the observation that $M(2)$ is a flat extension of the positive matrix A. □

THEOREM 2.2. *Let $M(2)$ be given as in (2.1). If $M(2)$ satisfies*

(i) $x > |u|^2 + |v|^2$,
(ii) $x^2-(1+3|u|^2+|v|^2)x+|u|^4+|u|^2+|v|^2-|w|^2+|u|^2|v|^2+2Re(u\bar{v}w+u^2\bar{w}-u^3\bar{v})=0$,
(iii) $x^2 - 2(|u|^2 + |v|^2)x + |u|^4 + |v|^4 - |h|^2 - 2|u|^2|v|^2 + 4Re(uv\bar{h}) = 0$,
(iv) $(\bar{h} - 2\bar{u}\bar{v})(x - |v|^2 - |u|^2) = (\bar{w} - u\bar{v} - \bar{u}^2)^2$,
(v) $(x - |v|^2 - |u|^2)^2 \neq (v\bar{h} - x\bar{u} + \bar{u}|u|^2 - \bar{u}|v|^2)(ux - u|u|^2v\bar{u}^2 - v\bar{w})$,

then $M(2)$ is positive but has no representing measure.

Proof. By Lemma 2.1, (i), (ii), and (iii) imply $M(2) \geq 0$ and $\{\mathit{1}, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$. Thus there is the unique $k_1, k_2, k_3, k_4, l_1, l_2, l_3, l_4$ in $\mathbb{C}$ such that

$$Z = k_1 \mathit{1} + k_2 Z + k_3 \bar{Z} + k_4 Z^2,$$
$$\bar{Z}^2 = l_1 \mathit{1} + l_2 Z + l_3 \bar{Z} + l_4 Z^2.$$

Indeed,

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & v \\ 0 & \bar{u} & \bar{v} & x \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \bar{u} \\ u \\ \bar{w} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{x\bar{u}+u^2\bar{v}-|v|^2\bar{u}-u\bar{w}}{x-|u|^2-|v|^2} \\ \frac{xu-u|u|^2+v\bar{u}^2-v\bar{w}}{x-|u|^2-|v|^2} \\ \frac{-\bar{u}^2-u\bar{v}+\bar{w}}{x-|u|^2-|v|^2} \end{pmatrix}$$

and

$$\begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 1 & v \\ 0 & \bar{u} & \bar{v} & x \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \bar{v} \\ \bar{u} \\ \bar{h} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{u\bar{h}-\bar{v}(x+|u|^2-|v|^2)}{x-|u|^2-|v|^2} \\ -\frac{v\bar{h}-\bar{u}(x-|u|^2+|v|^2)}{x-|u|^2-|v|^2} \\ -\frac{2\bar{u}\bar{v}-\bar{h}}{x-|u|^2-|v|^2} \end{pmatrix}.$$

Take

$$p(z, \bar{z}) := 1 + k_2 z + k_3 \bar{z} + k_4 z^2 - \bar{z}z \tag{2.2}$$

and

$$q(z, \bar{z}) := l_2 z + l_3 \bar{z} + l_4 z^2 - \bar{z}^2. \tag{2.3}$$

Then the polynomials p and q in $\mathcal{P}_2$ satisfy $p(Z, \bar{Z}) = 0$ and $q(Z, \bar{Z}) = 0$. From (2.2) and (2.3) we can obtain card $(\mathcal{Z}(p) \cap \mathcal{Z}(q)) < 4$ if $l_4 - k_4^2 = 0$ and $1 + k_3 l_3 \neq 0$. Note that $l_4 - k_4^2 = 0$ if and only if $(\bar{h} - 2\bar{u}\bar{v})(x - |v|^2 - |u|^2) = (\bar{w} - u\bar{v} - \bar{u}^2)^2$. Also $1 + k_3 l_3 \neq 0$ if and only if $(x - |v|^2 - |u|^2)^2 \neq (v\bar{h} - x\bar{u} + \bar{u}|u|^2 - \bar{u}|v|^2)(ux - u|u|^2 v\bar{u}^2 - v\bar{w})$. But since rank $M(2) = 4$ and card $(\mathcal{Z}(p) \cap \mathcal{Z}(q) < 4$, it follows $\rho(\gamma) < 0$. This completes the proof. □

It is not difficult to see that the conditions in Theorem 2.2 are independent each other. The following example illustrates that the condition (v) in Theorem 2.2 is essential for card $(\mathcal{Z}(p) \cap \mathcal{Z}(q)) < 4$.

We may capture an example that provides a singular quartic moment matrix admitting no representing measure.

EXAMPLE 2.3. If the truncated moment sequence γ of order 4 is given by

$$\gamma : 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 1,\ 1,\ 2,\ 1,\ 1,$$

then we have

(i) $M(2) \geq 0$,

(ii) rank $M(2) = 4$ and thus $M(2)$ satisfies the property (RG),

(iii) γ has no representing measure.

Proof. Observe

$$M(2)=\begin{pmatrix}1&0&0&0&1&0\\0&1&0&0&0&1\\0&0&1&1&0&0\\0&0&1&2&1&1\\1&0&0&1&2&1\\0&1&0&1&1&2\end{pmatrix}.$$

(i) This follows from a straightforward calculation.

(ii) Note that $\{1,Z,\bar{Z},Z^2\}$ is linearly independent. Thus evidently, $M(2)$ satisfies the property (RG). The rank assertion follows from a straightforward calculation.

(iii) Note that

$$\bar{Z}Z=1-\bar{Z}+Z^2 \quad\text{and}\quad \bar{Z}^2=Z-\bar{Z}+Z^2.$$

Let $p(z,\bar{z}):=1-\bar{z}+z^2-z\bar{z}$ and $q(z,\bar{z}):=z-\bar{z}+z^2-\bar{z}^2$. Then $p(Z,\bar{Z})=0$ and $q(Z,\bar{Z})=0$. Moreover,

$$\mathcal{Z}(p)\cap\mathcal{Z}(q)=\{z\in\mathbb{C}:z^3=1\}.$$

Observe that if μ is a representing measure for γ, then supp $\mu\subset\mathcal{Z}(p)\cap\mathcal{Z}(q)$ and card supp $\mu\geq$ rank $M(n)=4$. Then we have $\rho(\gamma)<0$, and that γ has no representing measure. □

3. Flat extensions of singular quartic moment matrices

In this section we examine the existence of flat extensions for singular quartic moment matrix $M(2)$. More precisely, we consider the following:

PROBLEM 3.1. *Assume $M(2)$ is positive and $\bar{Z}Z\in\langle 1,Z,\bar{Z}\rangle$. Does $M(2)$ admit a flat extension $M(3)$?*

We answer Problem 3.1 affirmatively below. If rank $M(2)\leq 3$, then $\{1,Z,\bar{Z},Z^2\}$ is linearly dependent in $\mathcal{C}_{M(2)}$, so that by Theorem 1.3 (i), the answer to Problem 3.1 is affirmative. Thus we will focus on the cases that $4\leq\operatorname{rank}M(2)\leq 5$. So we may assume that $\{1,Z,\bar{Z},Z^2\}$ is linearly independent in $\mathcal{C}_{M(2)}$. First, we consider the case that rank $M(2)=4$. To do this we introduce some notations. For a positive $N\times N$ matrix A, we denote by $[A]_k$ the compression of A to the first k rows and columns; similarly, the first k entries of a column C will be denoted by $[C]_k$. More generally, if $1\leq n_1<\cdots<n_k\leq N$ we let $[A]_{\{n_1,\cdots,n_k\}}$ denote the compression of A to the rows and columns indexed by $\{n_1,\cdots,n_k\}$.

THEOREM 3.2. *If*
(i) $M(2)$ *is positive,*
(ii) $\{1, Z, \bar{Z}, Z^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$,
(iii) $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$,
then $M(2)$ *has the unique flat extension* $M(3)$.

Proof. (Existence) Assume that $\bar{Z}Z = \alpha 1 + \beta Z + \theta \bar{Z}$. Since $p(Z, \bar{Z}) = 0$ implies $\bar{p}(Z, \bar{Z}) = 0$ ([6, Lemma 3.10]), we have that $\bar{Z}Z = \bar{\alpha} 1 + \bar{\theta} Z + \bar{\beta}\bar{Z}$, and hence $\alpha \in \mathbb{R}$ and $\theta = \bar{\beta}$. By Theorem 1.3 (iii), it suffices to show that $c_{11} = c_{22}$. Since $[M(2)]_4 > 0$, there exist the unique scalars k_1, k_2, k_3, k_4 such that

$$[M(2)]_4 \cdot (k_1, k_2, k_3, k_4)^T = (\gamma_{03}, \gamma_{13}, \gamma_{04}, \gamma_{23})^T.$$

Thus we have $c_{11} = (\bar{\gamma}_{03},\ \bar{\gamma}_{13},\ \bar{\gamma}_{04},\ \bar{\gamma}_{23})(k_1, k_2, k_3, k_4)^T$. But since $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$, we have $\bar{Z}Z^2 = \alpha Z + \beta Z^2 + \bar{\beta}\bar{Z}Z$, so that $c_{22} = \alpha\gamma_{22} + \beta\gamma_{23} + \bar{\beta}\bar{\gamma}_{23}$. Therefore the reduced C-block test consists of verifying that

$$\begin{aligned}(\alpha\gamma_{22} + {}&\beta\gamma_{23} + \bar{\beta}\bar{\gamma}_{23}) \\ &- (\bar{\gamma}_{03},\ \bar{\gamma}_{13},\ \bar{\gamma}_{04},\ \bar{\gamma}_{23})[M(2)]_4{}^{-1}(\gamma_{03}, \gamma_{13}, \gamma_{04}, \gamma_{23})^T \\ = {}&0.\end{aligned} \tag{3.1}$$

From a symbolic manipulation using *Mathematica* (see the algorithm below), we can show that (3.1) is equivalent to

$$(\alpha + |\beta|^2)\frac{\det[M(2)]_{\{1,2,3,4,6\}}}{\det[M(2)]_4} = 0, \tag{3.2}$$

which is true by the fact that $\operatorname{rank} M(2) = 4$.

(Uniqueness) Observe that the relation $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$ must induce in a proposed flat extension $M(3)$ the relation $\bar{Z}Z^2 = \alpha Z + \beta Z^2 + \bar{\beta}\bar{Z}Z$, so the entries γ_{23} and γ_{14} are fully determined; namely,

$$\gamma_{23} = \alpha\gamma_{12} + \beta\gamma_{13} + \bar{\beta}\gamma_{22} \quad \text{and} \quad \gamma_{14} = \alpha\gamma_{03} + \beta\gamma_{04} + \bar{\beta}\gamma_{13}.$$

But since $\gamma_{05} = (\gamma_{02}, \gamma_{03}, \gamma_{12}, \gamma_{04})(k_1, k_2, k_3, k_4)^T$, the block B is uniquely determined. Consequently, the flat extension $M(3)$ is unique.

Algorithm for the proof of (3.2):

I. Put $\gamma_{00} := 1, \gamma_{01} := w, \gamma_{11} := x, \gamma_{02} := u, \gamma_{03} := p, \gamma_{12} := q$, $\gamma_{04} := s$;

II. Calculate

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \left(= \begin{pmatrix} \alpha \\ \beta \\ \bar{\beta} \end{pmatrix} \right) := M(1)^{-1} \begin{pmatrix} \gamma_{11} \\ \bar{\gamma}_{12} \\ \gamma_{12} \end{pmatrix};$$

III. Put

$\gamma_{22} = a\gamma_{11} + b\gamma_{12} + c\bar{\gamma}_{12}$

$d_1(=\gamma_{13}) = a\gamma_{02} + b\gamma_{03} + c\gamma_{12}, \qquad d_2(=\bar{\gamma}_{13}) = a\bar{\gamma}_{02} + c\bar{\gamma}_{03} + b\bar{\gamma}_{12},$

$e_1(=\gamma_{23}) = a\gamma_{12} + c\gamma_{22} + bd_1, \qquad e_2(=\bar{\gamma}_{23}) = a\bar{\gamma}_{12} + b\gamma_{22} + cd_2;$

IV. Put

$$A := (a\gamma_{22} + bd_1 + cd_2) - (\bar{\gamma}_{03}, e_2, \bar{\gamma}_{04}, d_2)[M(2)]_4^{-1}(\gamma_{03}, e_1, \gamma_{04}, d_1)^T,$$

$$B := (a + bc)\frac{\det\ [M(2)]_{\{1,2,3,4,6\}}}{\det\ [M(2)]_4};$$

V. Factor $[A - B]$. Then we have $A - B = 0$. □

With a concrete example we will find the flat extension and the representing measure for $M(2)$ in Theorem 3.2.

Example 3.3. Let $\gamma_{00} = 1, \gamma_{01} = \frac{i}{2}, \gamma_{02} = -\frac{1}{4}, \gamma_{11} = \frac{5}{4}, \gamma_{03} = -\frac{i}{2}, \gamma_{12} = i, \gamma_{04} = \frac{51}{31}, \gamma_{13} = -\frac{25}{32}$, and $\gamma_{22} = \frac{59}{32}$. Then

$$M(2) = \begin{pmatrix} 1 & \frac{i}{2} & -\frac{i}{2} & -\frac{1}{4} & \frac{5}{4} & -\frac{1}{4} \\ -\frac{i}{2} & \frac{5}{4} & -\frac{1}{4} & i & -i & \frac{i}{2} \\ \frac{i}{2} & -\frac{1}{4} & \frac{5}{4} & -\frac{i}{2} & i & -i \\ -\frac{1}{4} & -i & \frac{i}{2} & \frac{59}{32} & -\frac{25}{32} & \frac{51}{32} \\ \frac{5}{4} & i & -i & -\frac{25}{32} & \frac{59}{32} & -\frac{25}{32} \\ -\frac{1}{4} & -\frac{i}{2} & i & \frac{51}{32} & -\frac{25}{32} & \frac{59}{32} \end{pmatrix} \geq 0,$$

rank $M(2) = 4$ and $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$, where $\alpha = \frac{7}{8}$ and $\beta = -\frac{3i}{8}$. By Theorem 3.2, there exists the unique flat extension $M(3)$ of $M(2)$ with $\gamma_{23} = \frac{119i}{64}$, $\gamma_{14} = -\frac{85i}{64}$ and $\gamma_{05} = \frac{145i}{64}$. In fact,

$$M(3) = \begin{pmatrix} M(2) & B \\ B^* & C \end{pmatrix},$$

where

$$B := \begin{pmatrix} -\frac{i}{2} & i & -i & \frac{i}{2} \\ -\frac{25}{32} & \frac{59}{32} & -\frac{25}{32} & \frac{51}{32} \\ \frac{51}{32} & -\frac{25}{32} & \frac{59}{32} & -\frac{25}{32} \\ \frac{119i}{64} & -\frac{119i}{64} & \frac{85i}{64} & -\frac{145i}{64} \\ -\frac{85i}{64} & \frac{119i}{64} & -\frac{119i}{64} & \frac{85}{64} \\ \frac{145i}{64} & -\frac{85i}{64} & \frac{119i}{64} & -\frac{119i}{64} \end{pmatrix}$$

and

$$C := \begin{pmatrix} \frac{385}{128} & -\frac{481}{256} & \frac{351}{128} & -\frac{601}{256} \\ -\frac{481}{256} & \frac{385}{128} & -\frac{481}{256} & \frac{351}{128} \\ \frac{351}{128} & -\frac{481}{256} & \frac{385}{128} & -\frac{481}{256} \\ -\frac{601}{256} & \frac{351}{128} & -\frac{481}{256} & \frac{385}{128} \end{pmatrix}.$$

To obtain the unique representing measure μ, we proceed as follows, remembering [6, Theorem 4.7]. Since $M(3)$ is flat, it follows from [6, Theorem 5.4] that $M(3)$ has a unique flat extension $M(4)$, which is recursively generated by Proposition 1.1 (i). Thus the relation $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$ implies $\bar{Z}Z^3 = \alpha Z^2 + \beta Z^3 + \bar{\beta}\bar{Z}Z^2$, so that $\gamma_{34} = \alpha\gamma_{23} + \beta\gamma_{24} + \bar{\beta}\gamma_{33} = \frac{7085i}{2048}$. Now observe that $[M(3)]_{\text{anal}} := [M(3)]_{\{1,2,4,7\}}$ is positive and invertible: for, if $Z^3 \in \langle 1, Z, Z^2\rangle$ in $\mathcal{C}_{[M(3)]_{\text{anal}}}$ then $Z^3 \in \langle 1, Z, Z^2\rangle$ in $\mathcal{C}_{M(3)}$ by Extension Principle [10]. So (1.6) forces that the number of atoms of any representing measure is less than or equals to 3, which is contradict to Theorem 3.2. Therefore $Z^4 \in \langle 1, Z, Z^2, Z^3\rangle$ in $\mathcal{C}_{M(4)}$ because rank$M(4) = 4$. Thus this relation gives rise to the polynomial equation

$$z^4 - \frac{5i}{4}z^3 - \frac{9}{16}z^2 + \frac{23i}{64}z - \frac{119}{128} = 0, \tag{3.3}$$

which, by (1.6), is satisfied by every point in the support of μ. Let z_0, z_1, z_2 and z_3 be the roots of the equation in (3.3). Then they are atoms of μ. From [6, Theorem 4.7], we obtain the densities of μ, say ρ_0, ρ_1, ρ_2 and ρ_3, solving the equation

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ z_0 & z_1 & z_2 & z_3 \\ z_0^2 & z_1^2 & z_2^2 & z_3^2 \\ z_0^3 & z_1^3 & z_2^3 & z_3^3 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \end{pmatrix}.$$

By a straightforward calculation, we obtain $z_0 = -1 + \frac{i}{4}, z_1 = 1 + \frac{i}{4}, z_2 = \frac{1}{8}(3 - \sqrt{65})i, z_3 = \frac{1}{8}(3 + \sqrt{65})i$ and $\rho_0 = \rho_1 = \frac{1}{4}, \rho_2 = \frac{\sqrt{65}-3}{4\sqrt{65}}, \rho_3 = \frac{\sqrt{65}+3}{4\sqrt{65}}$; therefore $\mu = \sum_{0 \le i \le 3} \rho_i \delta_{z_i}$.

We now consider the case that rank $M(2) = 5$ with a weaker condition than the dependence assumption in Problem 3.1.

THEOREM 3.4. *If*
(i) $M(2)$ *is positive,*
(ii) $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$, *and*
(iii) $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2\rangle$,
then $M(2)$ *has a flat extension* $M(3)$.

Proof. We first claim

$$\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2\rangle \Longrightarrow \bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}, \ (\alpha \in \mathbb{R}, \beta \in \mathbb{C}).$$

To see this, suppose $\bar{Z}Z = a1 + bZ + c\bar{Z} + dZ^2$. Since $p(Z, \bar{Z}) = 0$ implies $\bar{p}(Z, \bar{Z}) = 0$, it follows that $\bar{Z}Z = \bar{a}1 + \bar{b}\bar{Z} + \bar{c}Z + \bar{d}\bar{Z}^2$. Thus we have

$$(a-\bar{a})1+(b-\bar{c})Z+(c-\bar{b})\bar{Z}+dZ^2-\bar{d}\bar{Z}^2=0.$$

Since $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is linearly independent, $a=\bar{a}, b=\bar{c}, d=0$, which proves our claim. By Theorem 1.3 (iv), $M(2)$ admits a flat extension $M(3)$ if and only if there exists γ_{05} for which

$$\begin{aligned}(\bar{\gamma}_{03}, \bar{\gamma}_{13}, \bar{\gamma}_{04}, \bar{\gamma}_{23}, \bar{\gamma}_{05})[M(2)]^{-1}_{\{1,2,3,4,6\}}(\gamma_{03}, {}_{13}\gamma, \gamma_{04}, \gamma_{23}, \gamma_{05})^T \\ = \alpha\gamma_{22}+\beta\gamma_{23}+\bar{\beta}\bar{\gamma}_{23},\end{aligned} \tag{3.4}$$

where

$$\gamma_{23}=\alpha\gamma_{12}+\bar{\beta}\gamma_{22}+\beta\gamma_{13}.$$

Write $M := [M(2)]_{\{1,2,3,4,6\}}$. Then

$$(\det M)\cdot M^{-1}=\begin{pmatrix} M_{11} & -M_{21} & M_{31} & -M_{41} & M_{51} \\ -M_{12} & M_{22} & -M_{32} & M_{42} & -M_{52} \\ M_{13} & -M_{23} & M_{33} & -M_{43} & M_{53} \\ -M_{14} & M_{24} & -M_{34} & M_{44} & -M_{54} \\ M_{15} & -M_{25} & M_{35} & -M_{45} & M_{55} \end{pmatrix},$$

where M_{ij} denotes the determinant of the cofactor of M with respect to (i,j). Put $y := \gamma_{05}$. Then (3.4) is equivalent to the equation:

$$\begin{aligned} M_{55}|y|^2 &+ (\bar{\gamma}_{03}M_{51}-\bar{\gamma}_{13}M_{52}+\bar{\gamma}_{04}M_{53}-\bar{\gamma}_{23}M_{54})y \\ &+ (\gamma_{03}M_{15}-\gamma_{13}M_{25}+\gamma_{04}M_{35}-\gamma_{23}M_{45})\bar{y} \\ &= (\det M)(\alpha\gamma_{22}+\beta\gamma_{23}+\bar{\beta}\bar{\gamma}_{23})-(|\gamma_{03}|^2M_{11}+|\gamma_{13}|^2M_{22} \\ &+ |\gamma_{04}|^2M_{33}+|\gamma_{23}|^2M_{44})+2\mathrm{Re}(\gamma_{03}\bar{\gamma}_{13}M_{12}-\gamma_{03}\bar{\gamma}_{04}M_{13} \\ &+ \gamma_{13}\bar{\gamma}_{04}M_{23}+\gamma_{03}\bar{\gamma}_{23}M_{14}-\gamma_{13}\bar{\gamma}_{23}M_{24}+\gamma_{04}\bar{\gamma}_{23}M_{34}).\end{aligned} \tag{3.5}$$

Since M is a self-adjoint matrix, it follows that $M_{ij}=\bar{M}_{ji}$, so that the equation (3.5) should be of the form $A|y|^2+By+\bar{B}\bar{y}+C=0$ $(A>0)$. Therefore if

$$\begin{aligned}\Delta_{M(2)} := &\ (\text{the right-hand side of (3.5)}) \\ &+ (M_{55})^{-1}(|\gamma_{03}M_{15}-\gamma_{13}M_{25}+\gamma_{04}M_{35}-\gamma_{23}M_{45}|^2)\end{aligned}$$

is positive, then the solution of the equation (3.5) forms a circle. But a direct simplification using *Mathematica* (see the algorithm below) shows that

$$\Delta_{M(2)}=(M_{55})^{-1}(\alpha+|\beta|^2)(\det M)^2. \tag{3.6}$$

It thus suffices to show that $\alpha + |\beta|^2 > 0$. Indeed, in $\mathcal{C}_{M(2)}$, since $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$, we have $\gamma_{11} = \alpha\gamma_{00} + \beta\gamma_{01} + \bar{\beta}\bar{\gamma}_{01}$, so that $\alpha = (\gamma_{00})^{-1}(\gamma_{11} - \beta\gamma_{01} - \bar{\beta}\bar{\gamma}_{01})$, which gives

$$\begin{aligned}\alpha + |\beta|^2 &= (\gamma_{00})^{-1}(\gamma_{11} - \beta\gamma_{01} - \bar{\beta}\bar{\gamma}_{01}) + |\beta|^2 \\ &= (\gamma_{00})^{-1}(|\sqrt{\gamma_{00}}\beta - \gamma_{01}/\sqrt{\gamma_{00}}|^2) + \frac{\gamma_{00}\gamma_{11} - |\gamma_{01}|^2}{\gamma_{00}^2} \\ &= (\gamma_{00})^{-1}(|\sqrt{\gamma_{00}}\beta - \gamma_{01}/\sqrt{\gamma_{00}}|^2) + \frac{\det[M(2)]_{\{1,2\}}}{\gamma_{00}^2} > 0.\end{aligned}$$

Algorithm for the proof of (3.6):

I. Put $\gamma_{00} := 1, \gamma_{01} := w, \gamma_{11} := x, \gamma_{02} := u, \gamma_{03} := p, \gamma_{12} := q,$ $\gamma_{04} := s$;

II. Calculate
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \left(= \begin{pmatrix} \alpha \\ \beta \\ \bar{\beta} \end{pmatrix} \right) := M(1)^{-1} \begin{pmatrix} \gamma_{11} \\ \bar{\gamma}_{12} \\ \gamma_{12} \end{pmatrix};$$

III. Put
$$\begin{aligned}&\gamma_{22} = a\gamma_{11} + b\gamma_{12} + c\bar{\gamma}_{12},\\ &d_1(=\gamma_{13}) = a\gamma_{02} + b\gamma_{03} + c\gamma_{12}, \qquad d_2(=\bar{\gamma}_{13}) = a\bar{\gamma}_{02} + c\bar{\gamma}_{03} + b\bar{\gamma}_{12},\\ &e_1(=\gamma_{23}) = a\gamma_{12} + c\gamma_{22} + bd_1, \qquad e_2(=\bar{\gamma}_{23}) = a\bar{\gamma}_{12} + b\gamma_{22} + cd_2;\end{aligned}$$

IV. Define M and $M_{ij} (1 \le i, j \le 5)$ as in the proof of Theorem 3.4;

V. Put
$$\begin{aligned}A := &(\det M)(a\gamma_{22} + be_1 + ce_2)\\ &- (\gamma_{03}\bar{\gamma}_{03}M_{11} + d_1 d_2 M_{22} + \gamma_{04}\bar{\gamma}_{04}M_{33} + e_1 e_2 M_{44})\\ &+ (\gamma_{03}d_2M_{12} - \gamma_{03}\bar{\gamma}_{04}M_{13} + d_1\bar{\gamma}_{04}M_{23} + \gamma_{03}e_2M_{14}\\ &\qquad - d_1e_2M_{24} + \gamma_{04}e_2M_{34})\\ &+ (d_1\bar{\gamma}_{03}M_{21} - \gamma_{04}\bar{\gamma}_{03}M_{31} + \gamma_{04}d_2M_{32} + \bar{\gamma}_{03}e_1M_{41}\\ &\qquad - d_2e_1M_{42} + \bar{\gamma}_{04}e_1M_{43})\\ &+ (\gamma_{03}M_{15} - d_1M_{25} + \gamma_{04}M_{35} - e_1M_{45})\\ &\qquad \times (\bar{\gamma}_{03}M_{51} - d_2M_{52} + \bar{\gamma}_{04}M_{53} - e_2M_{54})/M_{55},\\ B := &(a + bc)(\det M)^2/M_{55};\end{aligned}$$

VI. Factor $[A - B]$. Then we have $A - B = 0$. □

REMARK 3.5. (i) Let $p(z, \bar{z}) := z\bar{z} - \alpha - \beta z - \bar{\beta}\bar{z}$. Then $z \in \mathcal{Z}(p)$ if and only if $|z - \bar{\beta}|^2 = \alpha + |\beta|^2$. By the above result, $\alpha + |\beta|^2 > 0$.

Thus the atoms of representing measure for γ are contained in the circle $|z-\bar{\beta}| = \sqrt{\alpha + |\beta|^2}$.

(ii) Note that Theorem 3.4 shows that if $\det M(2)_{\{1,2,3,4,6\}} = 0$ (i.e., rank $M(2) = 4$), then the circle in the proof of Theorem 3.4 reduces to the point $(\gamma_{03}M_{15} - \gamma_{13}M_{25} + \gamma_{04}M_{35} - \gamma_{23}M_{45})/M_{55}$, and hence γ_{05} is uniquely determined, as we showed in Theorem 3.2.

The following example shows that Theorems 3.2 and 3.4 are still true even though $\beta = 0$ and, in addition, gives a quartic moment matrix whose atoms of the corresponding measure lie on the same circle regardless of rank.

EXAMPLE 3.6. First, we reconstruct a positive moment matrix $M(2)$ with rank $M(2) = 4$ that has a positive flat extension $M(3)$. Let $\gamma_{00} = 1, \gamma_{01} = 1-i, \gamma_{02} = -2i, \gamma_{11} = 4, \gamma_{03} = 0, \gamma_{12} = 4-4i, \gamma_{04} = 8, \gamma_{13} = -8i$ and $\gamma_{22} = 16$. Then

$$M(2) = \begin{pmatrix} 1 & 1-i & 1+i & -2i & 4 & 2i \\ 1+i & 4 & 2i & 4-4i & 4+4i & 0 \\ 1-i & -2i & 4 & 0 & 4-4i & 4+4i \\ 2i & 4+4i & 0 & 16 & 8i & 8 \\ 4 & 4-4i & 4+4i & -8i & 16 & 8i \\ -2i & 0 & 4-4i & 8 & -8i & 16 \end{pmatrix} \geq 0$$

and rank $M(2) = 4$ and $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$, where $\alpha = 4$ and $\beta = 0$. By Theorem 3.2, there exists a unique flat extension $M(3)$ of $M(2)$ with $\gamma_{23} = 16 - 16i$, $\gamma_{14} = 0$ and $\gamma_{05} = 18 - 16i$. In fact,

$$M(3) = \begin{pmatrix} M(2) & B \\ B^* & C \end{pmatrix},$$

where

$$B := \begin{pmatrix} 0 & 4-4i & 4+4i & 0 \\ -8i & 16 & 8i & 8 \\ 8 & -8i & 16 & 8i \\ 16-16i & 16+16i & 0 & 16-16i \\ 0 & 16-16i & 16+16i & 0 \\ 16-16i & 0 & 16-16i & 16+16i \end{pmatrix}$$

and

$$C := \begin{pmatrix} 64 & 32i & 32 & 64i \\ -32i & 64 & 32i & 32 \\ 32 & -32i & 64 & 32i \\ -64i & 32 & -32i & 64 \end{pmatrix}$$

The same argument as Example 3.3 gives rise to the polynomial equation

$$z^4 + 4iz^2 - 16 = 0.$$

By a straightforward calculation, we obtain

$$\begin{aligned} z_0 &\approx -0.517638 + 1.93185i, & \rho_0 &\approx 0.04588,\\ z_1 &\approx 0.517638 - 1.93185i, & \rho_1 &\approx 0.45412,\\ z_2 &\approx -1.93185 + 0.517638i, & \rho_2 &\approx 0.04588,\\ z_3 &\approx 1.93185 - 0.517638i, & \rho_3 &\approx 0.45412. \end{aligned}$$

Therefore $\mu = \sum_{0\le i\le 3} \rho_i \delta_{z_i}$.

Continuing this process we construct a positive moment matrix $M(2)$ with rank $M(2) = 5$. Let $\gamma_{04} = 4 + 2i$. Then

$$M(2) = \begin{pmatrix} 1 & 1-i & 1+i & -2i & 4 & 2i\\ 1+i & 4 & 2i & 4-4i & 4+4i & 0\\ 1-i & -2i & 4 & 0 & 4-4i & 4+4i\\ 2i & 4+4i & 0 & 16 & 8i & 4-2i\\ 4 & 4-4i & 4+4i & -8i & 16 & 8i\\ -2i & 0 & 4-4i & 4+2i & -8i & 16 \end{pmatrix} \ge 0$$

and $\bar{Z}Z = \alpha \mathit{1} + \beta Z + \bar{\beta}\bar{Z}$, where $\alpha = 4$ and $\beta = 0$. By Theorem 3.4, there exists a flat extension $M(3)$ of $M(2)$ with $\gamma_{23} = 16 - 16i$ and $\gamma_{14} = 0$. By the relation $\bar{Z}Z^2 = \alpha Z + \beta Z^2 + \bar{\beta}\bar{Z}Z$, we also have $\gamma_{33} = 64$ and $\gamma_{24} = -32i$. To make a flat extension $M(3)$ of $M(2)$, we must determine γ_{05}. But from the proof of Theorem 3.4, γ_{05} is contained in the circle

$$|\gamma_{05} - (17 - 9i)| = 6. \tag{3.7}$$

If we choose γ_{05} on the circle (3.7), then the remaining entries of $M(3)$ are fully determined by the choice of γ_{05}. More precisely, if $\gamma_{05} = 17 - 3i$, then $\gamma_{15} = 16 + 8i$ and $\gamma_{06} = 37 - 36i$. Therefore $M(3)$ is of the form

$$M(3) = \begin{pmatrix} M(2) & B\\ B^* & C \end{pmatrix},$$

where

$$B := \begin{pmatrix} 0 & 4-4i & 4+4i & 0\\ -8i & 16 & 8i & 4-2i\\ 4+2i & -8i & 16 & 8i\\ 16-16i & 16+16i & 0 & 17+3i\\ 0 & 16-16i & 16+16i & 0\\ 17-3i & 0 & 16-16i & 16+16i \end{pmatrix}$$

and

$$C := \begin{pmatrix} 64 & 32i & 16-8i & 37+36i \\ -32i & 64 & 32i & 16-8i \\ 16+8i & -32i & 64 & 32i \\ 37-36i & 16+8i & -32i & 64 \end{pmatrix}.$$

To get the associated representing measure μ, we proceed as follows, remembering [6, Theorem 4.7]. Since $M(3)$ is flat, [6, Theorem 5.4] gives that $M(3)$ has a unique flat extension $M(4)$, which in turn has a unique flat extension $M(5)$, which is recursively generated by Proposition 1.1 (i). Thus the relation $\bar{Z}Z = \alpha 1 + \beta Z + \bar{\beta}\bar{Z}$ implies $\bar{Z}Z^3 = \alpha Z^2 + \beta Z^3 + \bar{\beta}\bar{Z}Z^2$, so that $\gamma_{34} = \alpha\gamma_{23} + \beta\gamma_{24} + \bar{\beta}\gamma_{33} = 64 - 64i$. Now observe that $A = [M(4)]_{\text{anal}} := [M(4)]_{\{1,2,4,7,11\}}$ is positive and invertible: indeed,

$$A = \begin{pmatrix} 1 & 1-i & -2i & 0 & 4+2i \\ 1+i & 4 & 4-4i & -8i & 0 \\ 2i & 4+4i & 16 & 16-16i & -32i \\ 0 & 8i & 16+16i & 64 & 64-64i \\ 4-2i & 0 & 32i & 64+64i & 256 \end{pmatrix}$$

is positive and invertible. Therefore $Z^5 \in \langle 1, Z, Z^2, Z^3, Z^4 \rangle$ in $\mathcal{C}_{M(5)}$ because rank $M(4) = 5$. Thus this relation gives rise to the polynomial equation

$$z^5 + \left(\frac{1}{2} + \frac{3i}{2}\right) z^4 + 5iz^3 - 10z^2 - (12+4i)z - 32i = 0.$$

By a straightforward calculation, we obtain

$$\begin{aligned} z_0 &\approx 0.822876 - 1.82288i, & \rho_0 &\approx 0.44991, \\ z_1 &\approx -1.82288 + 0.822876i, & \rho_1 &\approx 0.03285, \\ z_2 &\approx 0.27601 - 1.87401i, & \rho_2 &\approx 0.07636, \\ z_3 &\approx -1.36214 + 2.48805i, & \rho_3 &\approx 0.04680, \\ z_4 &\approx 1.58613 - 1.11405i, & \rho_4 &\approx 0.39408. \end{aligned}$$

Therefore $\mu = \sum_{0 \le i \le 4} \rho_i \delta_{z_i}$. Both the atoms of rank 4 and the atoms of rank 5 are in the same circle $|z| = 2$.

4. Flat extensions of nonsingular quartic moment matrices

In this section, we consider the existence of flat extensions for nonsingular quartic moment matrices. In Section 1, we have known that if

$M(2) > 0$, then $[M(2); B(2)]$ is a flat extension of $M(2)$ if and only if $c_{11} = c_{22}$ and $c_{21} = c_{32}$. Note that in $B(2)$, we have

$Z^3 = (\gamma_{03}, \gamma_{13}, \gamma_{04}, \gamma_{23}, \gamma_{14}, \gamma_{05})^T$ and $\bar{Z}Z^2 = (\gamma_{12}, \gamma_{22}, \gamma_{13}, \bar{\gamma}_{23}, \gamma_{23}, \gamma_{14})^T$.

Then we get

$$\begin{aligned} c_{11} &= (\bar{\gamma}_{03}, \bar{\gamma}_{13}, \bar{\gamma}_{04}, \bar{\gamma}_{23}, \bar{\gamma}_{14}, \bar{\gamma}_{05})\, M(2)^{-1}\, Z^3, \\ c_{22} &= (\bar{\gamma}_{12}, \gamma_{22}, \bar{\gamma}_{13}, \gamma_{23}, \bar{\gamma}_{23}, \bar{\gamma}_{14})\, M(2)^{-1}\, \bar{Z}Z^2, \\ c_{21} &= (\bar{\gamma}_{12}, \gamma_{22}, \bar{\gamma}_{13}, \gamma_{23}, \bar{\gamma}_{23}, \bar{\gamma}_{14})\, M(2)^{-1}\, Z^3, \\ c_{32} &= (\gamma_{12}, \gamma_{13}, \gamma_{22}, \gamma_{14}, \gamma_{23}, \bar{\gamma}_{23})\, M(2)^{-1}\, \bar{Z}Z^2. \end{aligned}$$

PROPOSITION 4.1. *Let $M(2) > 0$. Then $M(2)$ admits a flat extension $M(3)$ if and only if there exist $\gamma_{23}, \gamma_{14}, \gamma_{05} \in \mathbb{C}$ satisfying*

$$(4.1)\qquad \begin{aligned} &(\bar{\gamma}_{03}, \bar{\gamma}_{13}, \bar{\gamma}_{04}, \bar{\gamma}_{23}, \bar{\gamma}_{14}, \bar{\gamma}_{05})\, M\, (\gamma_{03}, \gamma_{13}, \gamma_{04}, \gamma_{23}, \gamma_{14}, \gamma_{05})^T \\ &= (\bar{\gamma}_{12}, \gamma_{22}, \bar{\gamma}_{13}, \gamma_{23}, \bar{\gamma}_{23}, \bar{\gamma}_{14})\, M\, (\gamma_{12}, \gamma_{22}, \gamma_{13}, \bar{\gamma}_{23}, \gamma_{23}, \gamma_{14})^T \end{aligned}$$

and

$$(4.2)\qquad \begin{aligned} &(\bar{\gamma}_{12}, \gamma_{22}, \bar{\gamma}_{13}, \gamma_{23}, \bar{\gamma}_{23}, \bar{\gamma}_{14})\, M\, (\gamma_{03}, \gamma_{13}, \gamma_{04}, \gamma_{23}, \gamma_{14}, \gamma_{05})^T \\ &= (\gamma_{12}, \gamma_{13}, \gamma_{22}, \gamma_{14}, \gamma_{23}, \bar{\gamma}_{23})\, M\, (\gamma_{12}, \gamma_{22}, \gamma_{13}, \bar{\gamma}_{23}, \gamma_{23}, \gamma_{14})^T, \end{aligned}$$

where $M := (\det M(2))\, M(2)^{-1}$.

Observe that the above system consists of two equations in $\gamma_{23}, \gamma_{14}, \gamma_{05}$. Thus we may put $\gamma_{23} = 0$ for the solution of the above system. In fact, M is the adjoint matrix of moment matrix $M(2)$. So, we let

$$M = \begin{pmatrix} c_{11} & c_{21} & c_{31} & c_{41} & c_{51} & c_{61} \\ c_{12} & c_{22} & c_{32} & c_{42} & c_{52} & c_{62} \\ c_{13} & c_{23} & c_{33} & c_{43} & c_{53} & c_{63} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{54} & c_{64} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{65} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix},$$

where c_{ij} is the cofactor of the i-th row j-th column entry of $M(2)$, $i, j = 1, 2, 3, 4, 5, 6$. It is easy to see that $c_{ij} = \bar{c}_{ji}$. Thus from (4.2), we have

$$(4.3)\qquad (c_{66}\bar{\gamma}_{14} + \theta_1)\gamma_{05} = c_{64}\gamma_{14}^2 - |\gamma_{14}|^2 c_{56} + \theta_2\gamma_{14} + \theta_3\bar{\gamma}_{14} + \theta_4,$$

where

$$\begin{aligned}
\theta_1 &= \bar{\gamma}_{12}c_{61} + \gamma_{22}c_{62} + \bar{\gamma}_{13}c_{63},\\
\theta_2 &= \gamma_{12}(c_{14} + c_{61}) + \gamma_{22}(c_{24} + c_{63}) + \gamma_{13}(c_{62} + c_{34})\\
&\quad - \bar{\gamma}_{12}c_{51} - \gamma_{22}c_{52} - \bar{\gamma}_{13}c_{53},\\
\theta_3 &= -\gamma_{03}c_{16} - \gamma_{13}c_{26} - \gamma_{04}c_{36},\\
\theta_4 &= (\gamma_{12}c_{11} + \gamma_{13}c_{12} + \gamma_{22}c_{13})\gamma_{12} + (\gamma_{12}c_{21} + \gamma_{22}c_{23})\gamma_{22}\\
&\quad + (\gamma_{12}c_{31} + \gamma_{13}c_{32} + \gamma_{22}c_{33} - \bar{\gamma}_{12}c_{21} - \bar{\gamma}_{13}c_{23})\gamma_{13}\\
&\quad - (\bar{\gamma}_{12}c_{11} + \gamma_{22}c_{12} + \bar{\gamma}_{13}c_{13})\gamma_{03} - (\bar{\gamma}_{12}c_{31} + \gamma_{22}c_{32} + \bar{\gamma}_{13}c_{33})\gamma_{04},
\end{aligned}$$

and from (4.1) we have

$$\begin{aligned}
&c_{66}|\gamma_{05}|^2 + 2\mathrm{Re}(\nu_1\gamma_{05}) + 2\mathrm{Re}(\gamma_{05}\bar{\gamma}_{14}c_{65})\\
&\qquad + (c_{55} - c_{66})|\gamma_{14}|^2 + 2\mathrm{Re}(\nu_2\gamma_{14}) + \nu_3 = 0,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\nu_1 &= \bar{\gamma}_{03}c_{61} + \bar{\gamma}_{13}c_{62} + \bar{\gamma}_{04}c_{63},\\
\nu_2 &= \bar{\gamma}_{03}c_{51} + \bar{\gamma}_{13}c_{52} + \bar{\gamma}_{04}c_{53} - \bar{\gamma}_{12}c_{61} - \gamma_{22}c_{62} + \bar{\gamma}_{13}c_{63},\\
\nu_3 &= (|\gamma_{03}|^2 - |\gamma_{12}|^2)c_{11} + (|\gamma_{13}|^2 - \gamma_{22}^2)c_{22} + (|\gamma_{04}|^2 - |\gamma_{13}|^2)c_{33}\\
&\quad + 2\mathrm{Re}[(\gamma_{13}\bar{\gamma}_{03} - \gamma_{22}\bar{\gamma}_{12})c_{21} + (\gamma_{04}\bar{\gamma}_{03} - \gamma_{13}\bar{\gamma}_{12})c_{31}\\
&\qquad + (\gamma_{04}\bar{\gamma}_{13} - \gamma_{22}\gamma_{13})c_{32}].
\end{aligned}$$

Thus, if there exist $\gamma_{14}, \gamma_{05} \in \mathbb{C}$ satisfying (4.3) and (4.4), then $M(2)$ admits a flat extension $M(3)$.

In the sequel, we consider the case that $M(1) = I$ and $\gamma_{12} = \gamma_{03} = 0$. Thus $M(2)$ is of the form

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma_{22} & \gamma_{31} & \gamma_{40}\\ 1 & 0 & 0 & \gamma_{13} & \gamma_{22} & \gamma_{31}\\ 0 & 0 & 0 & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}. \tag{4.5}$$

PROPOSITION 4.2. *Let $M(2)$ be a moment matrix as in (4.5). Then $M(2) > 0$ if and only if*

(i) $\gamma_{22} > 0$,
(ii) $\gamma_{22}{}^2 - \gamma_{22} - |\gamma_{13}|^2 > 0$,
(iii) $\gamma_{22}{}^3 - \gamma_{22}{}^2 - \gamma_{22}(|\gamma_{04}|^2 + 2|\gamma_{13}|^2) + |\gamma_{04}|^2 + 2Re(\gamma_{04}\gamma_{31}{}^2) > 0$.

Proof. For the moment matrix $M(2)$ in (4.5), it is positive and invertible if and only if $\det[M(2)]_4 > 0, \det[M(2)]_5 > 0$, and $\det M(2) > 0$. □

From (4.3) we have

$$\gamma_{05} = \frac{c_{64}{\gamma_{14}}^2 - c_{56}|\gamma_{14}|^2 + c_{22}c_{56}}{c_{66}\bar{\gamma}_{14}}. \tag{4.6}$$

Substituting (4.6) into (4.4) we can obtain

$$\begin{aligned}(|c_{46}|^2 - |c_{56}|^2 + c_{66}(c_{55} - c_{66}))|\gamma_{14}|^4 + 2c_{22}\mathrm{Re}(c_{46}c_{56}\bar{\gamma}_{14}^2) \\ - c_{22}c_{55}c_{66}|\gamma_{14}|^2 + {c_{22}}^2|c_{56}|^2 = 0.\end{aligned} \tag{4.7}$$

PROPOSITION 4.3. *Let $a \in \mathbb{C}$, $c \in \mathbb{R}$ and $b > 0$. Then the complex equation $|z|^2 - b\,|z - a| = c$ has a solution if and only if the set*

$$S_t := \{t \in \mathbb{R}^+ \mid c \le t^2,\ c - |a|b \le t^2 - bt \le |a|b + c,\ |a|b + c \le t^2 + bt\} \tag{4.7}$$

is not empty.

Proof. We know that the equation $|z|^2 - b\,|z - a| = c$ has a solution if and only if there is an $\omega \ge 0$ such that the following two circles

$$|z| = \sqrt{\omega} \quad \text{and} \quad |z - a| = \frac{\omega - c}{b}$$

intersect. This is equivalent to

$$|\sqrt{\omega} - \frac{\omega - c}{b}| \le |a| \le \sqrt{\omega} + \frac{\omega - c}{b}.$$

If we let $t := \sqrt{\omega}$, then we have our conclusion. □

THEOREM 4.4. *Let $M(2)$ be a positive and nonsingular moment matrix as in (4.5) and let*

$$a := \frac{c_{22}c_{64}c_{65}}{|c_{64}|^2 - |c_{65}|^2 + c_{66}(c_{55} - c_{66})},$$
$$b := \frac{{c_{22}}^2|c_{65}|^2}{|c_{64}|^2 - |c_{65}|^2 + c_{66}(c_{55} - c_{66})},$$
$$c := \frac{{c_{22}}^2|c_{65}|(|c_{64}|^2 - 1)}{|c_{64}|^2 - |c_{65}|^2 + c_{66}(c_{55} - c_{66})}.$$

If the following set S_t as (4.7) is not empty, then $M(2)$ admits a flat extension $M(3)$.

EXAMPLE 4.5. Assume $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = 0, \gamma_{12} = 0, \gamma_{04} = 1 + i, \gamma_{13} = 2 + i$ and $\gamma_{22} = 4$. Then

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 2-i & 1-i \\ 1 & 0 & 0 & 2+i & 4 & 2-i \\ 0 & 0 & 0 & 1+i & 2+i & 4 \end{pmatrix},$$

which is positive and invertible. In fact, by Mathematica, the eigenvalues of $M(2)$ are

$$1, \quad 1, \quad 2, \quad 8, \quad \frac{1}{2}(3-\sqrt{5}), \quad \frac{1}{2}(3+\sqrt{5}).$$

A straightforward calculation gives that $a = -3 - 5i, b = 98$, and $c = -510$. But the set S_t is empty. So, we can't know whether $M(2)$ admits a flat extension $M(3)$ or not by Theorem 4.4.

EXAMPLE 4.6. Assume $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = 0, \gamma_{12} = 0, \gamma_{04} = 0, \gamma_{13} = 0$ and $\gamma_{22} = 2$. Then

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

which is positive and invertible. In fact, by Mathematica, the eigenvalues of $M(2)$ are

$$1, \quad 1, \quad 2, \quad 2, \quad \frac{1}{2}(3-\sqrt{5}), \quad \frac{1}{2}(3+\sqrt{5}).$$

Also a straightforward calculation gives that $a = 0, b = 8$ and $c = 0$, so $S_t = \{0, 8\}$. Thus by Theorem 4.4, $M(2)$ admits a flat extension $M(3)$. We now give a flat extension and the corresponding representing measure. First of all, from (4.7), we obtain $|\gamma_{14}|^2 = 8$. We choose $\gamma_{14} = 2 + 2i$. Then the extension $[M(2); B(2)]$ is

$$M(3) := [M(2); B(2)] = \begin{pmatrix} M(2) & B \\ B^* & C \end{pmatrix},$$

where

$$B := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2-2i & 0 \\ 2+2i & 0 & 0 & 2-2i \\ 0 & 2+2i & 0 & 0 \end{pmatrix}$$

and

$$C := \begin{pmatrix} 8 & 0 & 0 & -8i \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 8i & 0 & 0 & 8 \end{pmatrix}.$$

To get the associated representing measure μ, we proceed as follows, remembering [6, Theorem 4.7]. Since $M(3)$ is flat, [6, Theorem 5.4] gives that $M(3)$ has a unique flat extension $M(4)$, which in turn has a unique flat extension $M(5)$, which in turn has a unique flat extension $M(6)$, which is recursively generated by Proposition 1.1 (i). Now observe that $A = [M(5)]_{\text{anal}} := [M(5)]_{\{1,2,4,7,11,16\}}$ is positive and invertible: indeed,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2+2i & 0 \\ 0 & 0 & 2 & 0 & 0 & 12+12i \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 2-2i & 0 & 0 & 40 & 0 \\ 0 & 0 & 12-12i & 0 & 0 & 208 \end{pmatrix}$$

is positive and invertible. In fact, by Mathematica, the eigenvalues of A are

$$1,\ 4,\ \frac{1}{2}(41-\sqrt{1553}),\ \frac{1}{2}(41+\sqrt{1553}),\ \frac{128}{105+\sqrt{10897}},\ 105+\sqrt{10897}.$$

Therefore we have that $Z^6 \in \langle 1, Z, Z^2, Z^3, Z^4, Z^5 \rangle$ in $\mathcal{C}_{M(6)}$ because rank $A = 6$. Thus this relation gives rise to the polynomial equation

$$z^6 - (8+8i)z^3 - 8i = 0.$$

By a straightforward calculation, we obtain

$$\begin{aligned}
z_0 &\approx 0.618034 - 0.618034i, & \rho_0 &\approx 0.315738,\\
z_1 &\approx -1.61803 - 1.61803i, & \rho_1 &\approx 0.0175955,\\
z_2 &\approx -0.84425 - 0.226216i, & \rho_2 &\approx 0.315738,\\
z_3 &\approx 2.21028 + 0.592242i, & \rho_3 &\approx 0.0175955,\\
z_4 &\approx -0.592242 - 2.21028i, & \rho_4 &\approx 0.0175955,\\
z_5 &\approx 0.226216 + 0.84425i, & \rho_5 &\approx 0.315738.
\end{aligned}$$

Therefore $\mu = \sum_{0\le i\le 5} \rho_i \delta_{z_i}$.

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