

## OBSERVATIONS ABOUT ALGEBRAIC DEPENDENCE OF DIRICHLET SERIES

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ABSTRACT. Certain observations about formal Dirichlet series whose coefficients are arithmetic functions are made. These include for example algebraic independence and representation as infinite series of logarithms.

### 1. Introduction

A (formal) Dirichlet series is an expression of the form

$$Z(s) = \sum_{n=1}^{\infty} \frac{z(n)}{n^s}, \quad z(n) \in \mathbb{C}.$$

The set  $(\mathcal{D}, +, \cdot)$  of all Dirichlet series equipped with addition and multiplication is isomorphic, through the map  $Z \leftrightarrow z$ , to the set of all (complex-valued) arithmetic functions  $(\mathcal{A}, +, *)$  equipped with addition and convolution, and is indeed a unique factorization domain ([1, 2]). Through this isomorphism, any algebraic relations from one setting have corresponding counterparts in the other, which allows us to refer to both interchangeably, and we often do so without further ado.

In Shapiro-Sparner[5], a systematic investigation of algebraic independence of Dirichlet series is made. A thorough study of this paper leads us to observations which either extend or simplify certain results in [5]. These observations include:

(i) A Dirichlet series  $Z(s)$ , with function  $z$  non-vanishing at infinitely many prime values of  $n$ , does not satisfy any algebraic differential equation.

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This extends the case  $z(n) = 1$ , for all  $n \in \mathbb{N}$ , of Riemann zeta function, in [5] and is indeed an old result of Ostrowski[4].

(ii) For a normalized Dirichlet series  $Z$  which is multiplicative at two distinct primes belonging to its support, if another Dirichlet series  $F$  is  $\mathbb{C}$ -algebraically dependent on  $Z$ , then  $F$  can be uniquely represented as a power series in  $\log Z$ .

(iii) For a normalized Dirichlet series  $Z$  with infinite support,  $[\text{supp}(z)]$ , if two normalized Dirichlet series are multiplicative over an infinite subset of  $[\text{supp}(z)]$  and are  $\mathbb{C}$ -algebraically dependent on  $Z$ , then one is a rational power of the other.

Observations (ii) and (iii) are slight extensions of those in [5] where “multiplicative at primes” is replaced by “multiplicative”, while their proofs given below clarify and simplify certain obscurities in [5].

The last observation, (iv) involves two results: the former is the algebraic independence of most commonly encountered arithmetic functions, viz. units, while the latter reveals relationships between norms of two dependent arithmetic functions.

Before proceeding to the proofs, let us first collect relevant definitions and related results.

Let  $I \in \mathcal{A}$  be defined by  $I(1) = 1$  and  $I(n) = 0$  for  $n > 1$ . Then  $I$  is the convolution identity of  $\mathcal{A}$ .

Let  $\mathcal{E}$  be a subring of  $\mathcal{A}$ . For  $r > 1$ ,  $f_1, f_2, \dots, f_r \in \mathcal{A}$  are *algebraically dependent* over  $\mathcal{E}$  if there exists  $P \in \mathcal{E}[x_1, \dots, x_r] \setminus \{0\}$  such that

$$P(f_1, \dots, f_r) = \sum_{(i)} a_{(i)} * f_1^{i_1} * \dots * f_r^{i_r} = 0,$$

and is said to be *algebraically independent* otherwise.

Let  $Z$  be a Dirichlet series. A Dirichlet series  $F$  is  $\mathbb{C}$ -*algebraically dependent on  $Z$* , written  $F \in \overline{\mathbb{C}[Z]}$ , if  $F$  and  $Z$  are algebraically dependent over  $\mathbb{C}$ , and  $F$  is *properly  $\mathbb{C}$ -algebraically dependent on  $Z$*  if  $F \in \overline{\mathbb{C}[Z]} \setminus \mathbb{C} := \overline{\mathbb{C}[Z]}^*$ .

A *derivation*  $d$  over  $\mathcal{A}$  is a mapping of  $\mathcal{A}$  into itself satisfying

$$d(f * g) = df * g + f * dg, \quad d(c_1f + c_2g) = c_1df + c_2dg$$

for all  $f, g \in \mathcal{A}$  and  $c_1, c_2 \in \mathbb{C}$ .

Let  $p$  be a prime. Define the  *$p$ -basic derivation* over  $\mathcal{A}$  by  $d_p f(n) = f(np)v_p(np)$ , where  $v_p(n)$  denotes the exponent of highest power of  $p$  dividing  $n$ .

We may also regard derivation  $d$  over  $\mathcal{A}$ , also as a derivation over  $\mathcal{D}$  via  $dF = \sum_{n=1}^{\infty} \frac{(df)(n)}{n^s}$ .

We shall make use of the following results from [5].

LEMMA 1. *Let  $\mathcal{E}$  be a subring of  $\mathcal{A}$ . If  $f \in \mathcal{A}$  is such that there exists a derivation  $d$  over  $\mathcal{A}$  which annihilates all of  $\mathcal{E}$  and  $d(f) \neq 0$ , then  $f$  is not algebraic over  $\mathcal{E}$ .*

Given  $f_1, \dots, f_r \in \mathcal{A}$  and derivations  $d_1, \dots, d_r$  over  $\mathcal{A}$ , the *Jacobian* of the  $f_i$  relative to the  $d_i$  is the determinant  $J(f_1, \dots, f_r/d_1, \dots, d_r) = \det(d_i(f_j))$ .

THEOREM 1. *Let  $f_1, \dots, f_r \in \mathcal{A}$  and  $d_1, \dots, d_r$  be distinct derivations over  $\mathcal{A}$  which annihilate all elements of the subring  $\mathcal{E}$ . If  $J(f_1, \dots, f_r/d_1, \dots, d_r) \neq 0$ , then  $f_1, \dots, f_r$  are algebraically independent over  $\mathcal{E}$ .*

THEOREM 2. *Let  $z \in \mathcal{A}$ ,  $z(1) = 1$ , with  $Z$  being its corresponding Dirichlet series. Assume that  $Z - 1$  is not an  $l$ -th powers of a Dirichlet series for any  $l > 1$ . If  $f \in \mathcal{A}$  is  $\mathbb{C}$ -algebraically dependent on  $z$ , then its corresponding Dirichlet series can uniquely be written under the form*

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_{\nu}}{\nu!} (\log Z)^{\nu},$$

where  $\phi_{\nu} \in \mathbb{C}$ .

## 2. Observation 1

Our first observation is the following :

THEOREM 3. *Let  $\xi \in \mathcal{A}$  be such that  $\xi(p) \neq 0$  for infinitely many primes  $p$ . Let  $\mathcal{E}$  be a subring of  $\mathcal{A}$  having the property that given any finite subset  $\mathcal{E}^* \subseteq \mathcal{E}$ , for all sufficiently large primes  $p$ , the derivations  $d_p$  annihilate all of  $\mathcal{E}^*$ . Then for any sequence of complex numbers  $(r_i)_{i \geq 1}$ , with distinct real parts, and any sequence of integers  $(t_j)_{j \geq 1}$  (not necessarily distinct), the functions*

$$f_{ij}(n) = \xi(n)n^{r_i}(\log n)^{t_j}$$

are algebraically independent over  $\mathcal{E}$ .

*Proof.* Suppose that the assertion is false, i.e., there is a finite subset of  $\{f_{ij}\}$  which are algebraically dependent over  $\mathcal{E}$ . For ease of writing, we may assume that this set is  $\{f_{11}, \dots, f_{kl}\}$ . Let  $\mathcal{E}^* (\subset \mathcal{E})$  be the finite

set of all coefficients in this algebraic relation. By hypothesis, for all sufficiently large primes  $p$ , each  $d_p$  annihilates all of  $\mathcal{E}^*$ , and so each  $d_p$  annihilates all of  $\mathcal{E}' = \langle \mathcal{E}^* \rangle$ , the subring of  $\mathcal{E}$  generated by  $\mathcal{E}^*$ . Thus  $f_{11}, \dots, f_{kl}$  are algebraically dependent over  $\mathcal{E}'$ . If we can choose primes  $p_{ij}$  among these so that

$$J(f_{11}, \dots, f_{kl}/d_{p_{11}}, \dots, d_{p_{kl}}) \neq 0,$$

then Theorem 1 implies that  $f_{11}, \dots, f_{kl}$  are algebraically independent over  $\mathcal{E}'$ , which is a contradiction and the desired result will follow.

We may assume without loss of generality that  $-s \leq t_j \leq s$  for all  $j \in \{1, \dots, l\}$ , where  $s$  is a fixed positive integer, and rewrite the above set as  $\{f_{ij} \mid i \in \{1, \dots, k\}, j \in \{-s, \dots, s\}\}$  instead of  $\{f_{11}, \dots, f_{kl}\}$ . Let  $T = (2s + 1)k$ . For any sequence of sufficiently large primes,  $p_1 > p_2 > \dots > p_T$ , each  $\xi(p_i) \neq 0$ , we have

$$\begin{aligned} J(n) &:= J(f_{1,-s}, \dots, f_{1,s}, \dots, f_{k,-s}, \dots, f_{k,s}/d_{p_1}, \dots, d_{p_T})(n) \\ &= \det(d_{p_m}(f_{ij}))(n) \\ &= \det(f_{ij}(np_m)v_{p_m}(np_m)) \\ &= \det(\xi(np_m)(np_m)^{r_i}(\log np_m)^j v_{p_m}(np_m)), \end{aligned}$$

where  $m = 1, \dots, T; i \in \{1, \dots, k\}; j \in \{-s, \dots, s\}$ .

Putting  $n = 1$ , we have

$$J(1) = \det(\xi(p_m)p_m^{r_i}(\log p_m)^j) = \xi(p_1) \cdots \xi(p_T) \det(p_m^{r_i}(\log p_m)^j),$$

and consider

$$J^* = \frac{J(1)}{\xi(p_1) \cdots \xi(p_T)} = \det(p_m^{r_i}(\log p_m)^j).$$

Note that a typical term in the expansion of the determinant defining  $J^*$  is of the form

$$t(\vec{p}, \vec{r}, \vec{j}) := \pm p_1^{r_1 \mu_1} (\log p_1)^{j_1} p_2^{r_2 \mu_2} (\log p_2)^{j_2} \cdots p_T^{r_T \mu_T} (\log p_T)^{j_T},$$

where  $\mu_1, \dots, \mu_T \in \{1, \dots, k\}; j_1, \dots, j_T \in \{-s, \dots, s\}$ .

We may assume that  $Re(r_1) > Re(r_2) > \dots > Re(r_k)$ . In the first row, the column which has the unique largest absolute value is  $p_1^{r_1}(\log p_1)^s$ , so we exchange the first column with this column. In the second row, we consider the column which has the next unique largest absolute value (after the first column) and exchange the second column with this column. Continuing this process. We claim that in the final determinant, by choosing  $p_1 > p_2 > \dots > p_T$  sufficiently large the term

with largest absolute value is the main diagonal term

$$Y := a_{11}a_{22} \cdots a_{TT} = p_1^{r_1}(\log p_1)^s p_2^{(r)_2}(\log p_2)^{(s)_2} \cdots p_T^{(r)_T}(\log p_T)^{(s)_T},$$

where  $(r)_i, (s)_i$  denote the diagonal exponents. Let

$$a_j := a_{1j_1}a_{2j_2} \cdots a_{Tj_T} = p_1^{\alpha_1}(\log p_1)^{\beta_1} \cdots p_T^{\alpha_T}(\log p_T)^{\beta_T}$$

be any term in the determinant expansion. There are three possibilities.

- (i) If  $r_1 \neq \alpha_1$ , so that  $Re(r_1) > Re(\alpha_1)$ , then choosing  $p_1$  sufficiently large in comparison with other  $p_i$ 's, we see that  $p_1^{r_1} \gg p_1^{\alpha_1}$  which leads to  $|Y| > |a_j|$ .
- (ii) If  $r_1 = \alpha_1, s > \beta_1$ , then as in (i),  $(\log p_1)^s \gg (\log p_1)^{\beta_1}$  and so  $|Y| > |a_j|$ .
- (iii) If  $r_1 = \alpha_1, s = \beta_1$  (i.e., both terms arise from the expansion of the (1,1) term),

repeating the same arguments as above we see that the next largest term must come from the main diagonal.

Furthermore, we can even choose the primes  $p_1 > \dots > p_T$  so large that

$$\left| \frac{t(\vec{p}, \vec{i}, \vec{j})}{Y} \right| < \frac{1}{T!} \quad \text{for each } t(\vec{p}, \vec{i}, \vec{j}) \neq Y.$$

Thus  $\frac{J^*}{Y} = 1 + ((T! - 1)\text{terms each with absolute value } < \frac{1}{T!}) \neq 0$ .

This shows that there are sets of primes such that  $J^* \neq 0$ , yielding  $J(1) \neq 0$ , as required. □

Theorem 3 reduces to Theorem 3.3 of [5] when  $\xi(n) = I(n)$ . By the same proof as in Theorem 3 we also have the following result:

**THEOREM 3'.** *Let  $\xi \in \mathcal{A}$  be such that  $\xi(p) \neq 0$  for all sufficiently large primes  $p$ . Let  $\mathcal{E}$  be a subring of  $\mathcal{A}$  having the property that given any finite subset  $\mathcal{E}^* \subseteq \mathcal{E}$ , there are infinitely many primes  $p$ , whose derivations  $d_p$  annihilate all of  $\mathcal{E}^*$ . Then for any sequence of complex numbers  $(r_i)_{i \geq 1}$ , with distinct real parts, and any sequence of integers  $(t_j)_{j \geq 1}$  (not necessarily distinct), the functions*

$$f_{ij}(n) = \xi(n)n^{r_i}(\log n)^{t_j}$$

are algebraically independent over  $\mathcal{E}$ .

Since for each prime  $p$ ,  $d_p$  annihilates all elements of  $\mathbb{C}$ , from Theorem 3, we easily deduce

COROLLARY 1. Let  $\xi \in \mathcal{A}$  be such that  $\xi(p) \neq 0$  for infinitely many primes  $p$ . Let  $(r_i)_{i \geq 1}$  be a sequence of complex numbers with distinct real parts, and  $(t_j)_{j \geq 1}$  a sequence of integers (not necessarily distinct). Then the functions

$$f_{ij}(n) = \xi(n)n^{r_i}(\log n)^{t_j},$$

for all distinct  $(r_i, t_j)$ , are algebraically independent over  $\mathbb{C}$ .

COROLLARY 2. Let  $\Xi(s) = \sum_{n=1}^{\infty} \frac{\xi(n)}{n^s}$ , where  $\xi \in \mathcal{A}$  is such that  $\xi(p) \neq 0$  for infinitely many primes  $p$ . Let  $r_i, i = 1, \dots, L$  be complex numbers with distinct real parts, and  $m_j, j = 1, \dots, L$  any nonnegative integers. Then the functions

$$\Xi^{(m_j)}(s - r_i), \quad i, j \in \{1, \dots, L\}$$

are algebraically independent over  $\mathbb{C}$ .

*Proof.* This follows readily from Corollary 1, noting that

$$\Xi^{(m)}(s - r) = \sum_{n=1}^{\infty} \frac{(-1)^m \xi(n)}{n^{s-r}} (\log n)^m. \quad \square$$

A rephrasing of Corollary 2 is:

COROLLARY 3. Let  $\Xi(s) = \sum_{n=1}^{\infty} \frac{\xi(n)}{n^s}$ , where  $\xi \in \mathcal{A}$  is such that  $\xi(p) \neq 0$  for infinitely many primes  $p$ . Then  $\Xi(s)$  does not satisfy any nontrivial algebraic differential difference equation over  $\mathbb{C}$ .

### 3. Observation 2

Let  $\mathcal{A}_1$  be the subset of  $\mathcal{A}$  consisting  $f \in \mathcal{A}$  which  $f(1) = 1$ . Let  $p$  be a prime. We say that  $z$  is *multiplicative at  $p$*  (also referred to as *locally multiplicative*), written  $z \in \mathcal{M}_p$ , if

$$z(mp^\alpha) = z(m)z(p^\alpha),$$

for each  $\alpha, m \in \mathbb{N}$ ,  $\text{g.c.d.}(m, p) = 1$ .

Note that multiplicative functions are multiplicative at  $p$ , for each prime  $p$ .

For  $f \in \mathcal{A}$ , define the *support of  $f$*  to be  $\text{supp}(f) = \{n \in \mathbb{N} : f(n) \neq 0\}$  and define  $[\text{supp}(f)]$  to be the smallest set of primes which generate a subsemigroup of the positive integers containing  $\text{supp}(f)$ .

The proof of the next lemma is taken from Lemma 7.1 in [5], while the condition is weakened.

LEMMA 2. Let  $z \in \mathcal{A}_1$  be such that  $[\text{supp}(z)]$  contains at least two primes. Let  $p, q \in [\text{supp}(z)]$ ,  $p \neq q$ . If  $z \in \mathcal{M}_p \cap \mathcal{M}_q$ , then there does not exist an integer  $l > 1$  such that

$$Z = 1 + H^l,$$

for any Dirichlet series  $H$ .

*Proof.* Suppose that  $Z = 1 + H^l$  for some  $l > 1$  and some Dirichlet series  $H$  with corresponding  $h \in \mathcal{A}$ . Then

$$d_p Z = lH^{l-1}d_p H.$$

Since  $z \in \mathcal{M}_p$ , we have

$$\begin{aligned} d_p Z &= \sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} \sum_{\substack{m=1 \\ (p,m)=1}}^{\infty} \frac{z(m)}{m^s} \\ &= Z \left( \sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} \right) / \left( \sum_{a=0}^{\infty} \frac{z(p^a)}{(p^a)^s} \right). \end{aligned}$$

Now  $H$  and  $1 + H^l$  being relatively prime implies  $H$  divides

$$\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s},$$

i.e.,

$$(1) \quad \sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} = HG_p,$$

for some Dirichlet series  $G_p$  whose arithmetic counterpart is  $g_p$ .

If  $\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} = 0$ , then  $z(p^a) = 0$  for all  $a \geq 1$ , and so  $z(p^a m) = 0$  for all  $a, m \in \mathbb{N}$ . This yields  $p \notin [\text{supp}(z)]$ , which is a contradiction. Thus  $\sum_{a=0}^{\infty} \frac{(a+1)z(p^{a+1})}{(p^a)^s} \neq 0$ . Since  $h(1) = 0$ , then let  $n, m$  both  $> 1$  be the smallest integers such that both  $h(n)$  and  $g_p(m)$  are nonzero (if  $m$  exists).

The coefficient of  $(nm)^{-s}$  on the right side of (1) being nonzero gives  $nm = p^c$  for some  $c > 0$ . Thus  $n = p^a$  for some  $a > 0$  (if  $m$  does not exist, then  $g_p = I$ , so  $n = p^c$ ). Since  $n$  depends only on  $H$ , if this also holds for  $q$ , then  $p^a = n = q^b$  for some  $a, b > 0$ , yielding a contradiction. Consequently, this can only hold for  $p$ , and so  $z(q^b) = 0$  for all  $b \geq 1$ . By

local multiplicativity  $z(q^b m) = 0$  for all  $b, m \in \mathbb{N}$ , implying  $q \notin [\text{supp}(z)]$ , a contradiction.  $\square$

Lemma 2 and Theorem 2 together give the result of our second observation.

**THEOREM 4.** *Let  $z \in \mathcal{A}_1$  be such that  $[\text{supp}(z)]$  contains at least two primes. Assume that  $z \in \mathcal{M}_p \cap \mathcal{M}_q$  for some  $p, q \in [\text{supp}(z)]$ ,  $p \neq q$ . If  $f \in \mathcal{A}_1$  is  $\mathbb{C}$ -algebraically dependent over  $z$ , then we have uniquely the representation*

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_{\nu}}{\nu!} (\log Z)^{\nu},$$

where  $\phi_{\nu} \in \mathbb{C}$ .

Theorem 4 slightly improves Theorem 7.2 of [5] by weakening the “multiplicative” condition to that of “local multiplicative at two primes”.

### 4. Observation 3

To establish our third observation, we show first that dependent functions have the same set of support generators.

**LEMMA 3.** *Let  $z \in \mathcal{A} \setminus \{0\}$ . For  $f \in \mathcal{A}$ , if  $f$  is properly  $\mathbb{C}$ -algebraically dependent over  $z$ , then  $[\text{supp}(f)] = [\text{supp}(z)]$ .*

*Proof.* Since  $f \in \overline{\mathbb{C}[z]}^*$ , then  $z \in \overline{\mathbb{C}[f]}^*$ . First we prove that  $[\text{supp}(z)] \subseteq [\text{supp}(f)]$ . Suppose not. There is a  $p \in [\text{supp}(z)] \setminus [\text{supp}(f)]$ , and so  $z(pm) \neq 0$  for some  $m \in \mathbb{N}$ . From  $d_p z(m) = z(pm) \nu_p(pm) \neq 0$ , we get  $d_p z \neq 0$ . Since  $p \notin [\text{supp}(f)]$ , then  $f(np) = 0$  for all  $n \in \mathbb{N}$ , implying  $d_p f(n) = f(np) \nu_p(np) = 0$  for all  $n \in \mathbb{N}$ , and so  $d_p f = 0$ . Therefore  $d_p^k f = 0$  for all  $k \in \mathbb{N}$ , which induces  $d_p g = 0$  for all  $g \in \mathbb{C}[f]$ . By Lemma 1,  $z \notin \overline{\mathbb{C}[f]}$ , and so  $f \notin \overline{\mathbb{C}[z]}$ , which is a contradiction. The other inclusion  $[\text{supp}(f)] \subseteq [\text{supp}(z)]$  is proved similarly.  $\square$

Note that if  $f \in \mathcal{A} \setminus \{0\}$  is (properly)  $\mathbb{C}$ -algebraic over  $z \in \mathcal{A} \setminus \{0\}$  and  $[\text{supp}(z)]$  is infinite then  $[\text{supp}(f)]$  is also infinite.

Our third observation is the next theorem which strengthens Theorem 7.3 of [5] by lessening the “multiplicative” condition to that of “local multiplicative” and the proof given here corrects certain gaps in the original proof of [5].



**THEOREM 5.** *Let  $z \in \mathcal{A}_1$  be such that  $[\text{supp}(z)]$  is infinite. Assume that there is an infinite subset  $S \subseteq [\text{supp}(z)]$  such that  $z \in \bigcap_{p \in S} \mathcal{M}_p$ .*

*Let  $f \in \mathcal{A}_1$  be  $\mathbb{C}$ -algebraically dependent over  $z$ . If  $f \in \bigcap_{p \in S} \mathcal{M}_p$ , then  $f = z^c$ , where  $c$  is rational.*

*Proof.* Let  $p \in S$ . Then  $z \in \mathcal{M}_p$  and  $z(p^a m) \neq 0$  for some  $a, m \in \mathbb{N}$ , and  $(p, m) = 1$ . Thus  $0 \neq z(p^a m) = z(p^a)z(m)$ , i.e.,  $z(p^a) \neq 0$ . Let  $a_p$  be the smallest such positive value of  $a$ . Let  $F$  be the corresponding Dirichlet series of  $f$ . Since  $f \in \overline{\mathbb{C}[z]}$ , by Theorem 4,

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_{\nu}}{\nu!} (\log Z)^{\nu},$$

where  $\phi_{\nu} \in \mathbb{C}$ . Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} &= \sum_{\nu=0}^{\infty} \frac{\phi_{\nu}}{\nu!} (\log Z)^{\nu} = \sum_{\nu=0}^{\infty} \phi_{\nu} \left( \sum_{j=\nu}^{\infty} s(j, \nu) \frac{(Z-1)^j}{j!} \right) \\ &= \sum_{j=0}^{\infty} (Z-1)^j \sum_{\nu \leq j} \phi_{\nu} \frac{s(j, \nu)}{j!}, \end{aligned}$$

where  $s(j, \nu)$  is the Stirling numbers of the first kind ([3], pp. 282). Since  $S$  is infinite, for any  $p_1, \dots, p_k \in S$ , we have

$$\begin{aligned} f(p_1^{a_{p_1}} \dots p_k^{a_{p_k}}) &= \sum_{j=1}^{\infty} \sum_{n_1 \dots n_j = p_1^{a_{p_1}} \dots p_k^{a_{p_k}}} z(n_1) \dots z(n_j) \sum_{\nu \leq j} \phi_{\nu} \frac{s(j, \nu)}{j!} \\ &= z(p_1^{a_{p_1}} \dots p_k^{a_{p_k}}) \sum_{j=1}^k \sum_{\substack{n_1 \dots n_j = T_1 \dots T_k \\ T_i = p_i^{a_{p_i}}}} 1 \sum_{\nu \leq j} \phi_{\nu} \frac{s(j, \nu)}{j!}. \end{aligned}$$

$$\begin{aligned} \frac{f(p_1^{a_{p_1}} \dots p_k^{a_{p_k}})}{z(p_1^{a_{p_1}} \dots p_k^{a_{p_k}})} &= \sum_{\nu=1}^k \phi_{\nu} \sum_{j=\nu}^k s(j, \nu) S(k, j) \\ &= \sum_{\nu=1}^k \phi_{\nu} \delta_{k\nu} = \phi_k, \end{aligned}$$

where  $S(k, j)$  is the Stirling numbers of the second kind ([3], pp. 150, pp. 281).

Thus for all primes  $p \in S$ ,  $\frac{f(p^{ap})}{z(p^{ap})} = \phi_1 = c$ , a constant. Since  $f, z \in \mathcal{M}_{p_1} \cap \dots \cap \mathcal{M}_{p_k}$ , then

$$\phi_k = \frac{f(p_1^{ap_1} \dots p_k^{ap_k})}{z(p_1^{ap_1} \dots p_k^{ap_k})} = \frac{f(p_1^{ap_1}) \dots f(p_k^{ap_k})}{z(p_1^{ap_1}) \dots z(p_k^{ap_k})} = \phi_1^k = c^k,$$

and so

$$F = \sum_{\nu=0}^{\infty} \frac{\phi_\nu}{\nu!} (\log Z)^\nu = \sum_{\nu=0}^{\infty} \frac{c^\nu (\log Z)^\nu}{\nu!} = \exp(c \log Z) = Z^c.$$

Since  $F \in \overline{\mathbb{C}[Z]}$ , there are  $a_{rj} \in \mathbb{C}$ , not all zero, such that

$$\begin{aligned} 0 &= \sum_{r,j} a_{rj} Z^r F^j = \sum_{r,j} a_{rj} Z^{r+cj} \\ &= \sum_{r,j} a_{rj} \sum_{\nu=0}^{\infty} \frac{(r+cj)^\nu}{\nu!} (\log Z)^\nu. \end{aligned}$$

Equating the coefficients of  $(p_1^{ap_1} \dots p_k^{ap_k})^{-s}$ , where  $p_i \in S$ , using the same reasoning as before we obtain

$$\sum_{r,j} a_{rj} (r+cj)^k = 0.$$

If  $c$  is complex or irrational, then  $r+cj$  are all distinct not equal to zero, and this implies  $a_{rj} = 0$  for all  $r, j$ , a contradiction. Hence  $c$  is rational and  $f = z^c$ . □

Using Theorem 5, an improvement of Theorem 7.4 in [5] is as follows:

**THEOREM 6.** *Let  $z \in \mathcal{A}_1$  be such that  $[\text{supp}(z)]$  is an infinite set. Assume that  $f_1, f_2 \in \mathcal{A}_1$  are properly  $\mathbb{C}$ -algebraically dependent over  $z$ . If there exists an infinite subset  $S \subseteq [\text{supp}(z)]$  such that  $f_1, f_2 \in \bigcap_{p \in S} \mathcal{M}_p$ ,*

*then  $f_2$  is a rational power of  $f_1$ .*

*Proof.* Since  $f_1, f_2 \in \overline{\mathbb{C}[z]}^*$ , by Lemma 3,  $[\text{supp}(f_1)] = [\text{supp}(z)] = [\text{supp}(f_2)]$ . Since  $f_2 \in \overline{\mathbb{C}[z]}^*$  and  $z \in \overline{\mathbb{C}[f_1]}^*$ , then  $f_2 \in \overline{\mathbb{C}[f_1]}^*$ . By Theorem 5,  $f_2 = f_1^c$  for some rational  $c$ . □

**5. Observation 4**

For a fixed prime  $p$ , we define the derivation  $D_p$  over  $\mathcal{D}$  by

$$D_p F = \sum_{n=1}^{\infty} \frac{f(n)v_p(n)}{(n/p)^s}, \text{ where } F = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Note that the derivation  $D_p$  is indeed the  $p$ -basic derivation  $d_p$ , for,

$$\begin{aligned} D_p F &= \sum_{n=1}^{\infty} \frac{f(n)v_p(n)}{(n/p)^s} = \sum_{n=1}^{\infty} \frac{f(np)v_p(np)}{(np/p)^s} + \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{f(n)v_p(n)}{(n/p)^s} \\ &= \sum_{n=1}^{\infty} \frac{f(np)v_p(np)}{n^s} = \sum_{n=1}^{\infty} \frac{d_p f(n)}{n^s} = d_p F. \end{aligned}$$

The norm,  $Nf$ , of a function  $f \in \mathcal{A}$  is defined as

$$Nf = \min \{n \in \mathbb{N} \mid f(n) \neq 0\} \text{ if } f \neq 0, \text{ and } Nf = \infty \text{ if } f = 0.$$

Clearly,  $N(f * g) = (Nf)(Ng)$ ,  $N(f + g) \geq \min\{Nf, Ng\}$ , and the units of  $\mathcal{A}$  are those functions whose norms are equal to 1.

LEMMA 4. *If  $f_1, \dots, f_r \in \mathcal{A}$  are such that for all sets of  $r$  distinct primes  $p_1, \dots, p_r$ , we have  $J(f_1, \dots, f_r/p_1, \dots, p_r) = 0$ , then  $\det(v_{p_i}(Nf_j)) = 0$ .*

*Proof.* This is a special case of Lemma 8.8 in [5]. □

Our last observation gives interesting information about dependence of non-units and norms of elements in  $\mathcal{A}$ .

THEOREM 7. *The set of nonzero non-unit arithmetic functions whose norms are pairwise relatively prime is algebraically independent over  $\mathbb{C}$ .*

*Proof.* Let  $r \in \mathbb{N}$  and  $f_1, \dots, f_r$  be nonzero non-unit arithmetic functions whose norms are pairwise relatively prime. Then  $Nf_i > 1$  for all  $i = 1, \dots, r$ . Note that for each prime  $p$ ,  $d_p$  annihilates all of  $\mathbb{C}$ . Suppose that  $f_1, \dots, f_r$  are algebraically dependent over  $\mathbb{C}$ . By Theorem 1, for all sets of primes  $p_1, \dots, p_r$ , we have

$$J(f_1, \dots, f_r/d_{p_1}, \dots, d_{p_r}) = 0.$$

By Lemma 4,  $\det(v_{p_i}(Nf_j)) = 0$ . Thus there exist integers  $\alpha_1, \dots, \alpha_r$ , not all zero, such that for all primes  $p$ ,

$$\sum_{j=1}^r \alpha_j v_p(Nf_j) = 0,$$

and so

$$0 = \sum_{j=1}^r \alpha_j v_p(Nf_j) = v_p((Nf_1)^{\alpha_1} \dots (Nf_r)^{\alpha_r}).$$

Then  $(Nf_1)^{\alpha_1} \dots (Nf_r)^{\alpha_r} = 1$ . Since  $Nf_1, \dots, Nf_r$  are pairwise relatively prime, this is impossible.  $\square$

For an  $f \in \mathcal{A}$ , let  $n'$  be the smallest integer greater than  $Nf$  such that  $f(n') \neq 0$ . Define  $N_1f = n'$ . If  $n'$  does not exist, define  $N_1f = Nf$ .

**THEOREM 8.** *Let  $f, g \in \mathcal{A}$  be nonzero such that  $(Nf)(N_1g) \neq (N_1f)(Ng)$ . If  $f$  and  $g$  are algebraically dependent over  $\mathbb{C}$ , then*

(i) *there exist integers  $x_1, x_2$ , not both zero, such that*

$$(Nf)^{x_1}(Ng)^{x_2} = 1;$$

(ii) *there exist integers  $y_1, y_2$ , not both zero, such that*

$$(Nf)^{y_1}(N_1g)^{y_2} = 1; \text{ and}$$

(iii) *there exist integers  $z_1, z_2$ , not both zero, such that*

$$(N_1f)^{z_1}(Ng)^{z_2} = 1.$$

*Proof.* For ease of writing, let  $Nf = n^*, N_1f = n', Ng = m^*, N_1g = m'$ . If  $n' = n^*$ , then (iii) is equivalent to (i). If  $m' = m^*$ , then (ii) is equivalent to (i). We may assume that  $n' \neq n^*, m' \neq m^*$ , so  $f(n^*), f(n'), g(m^*), g(m')$  all  $\neq 0$ . Assume that  $f$  and  $g$  are algebraically dependent over  $\mathbb{C}$ . Let  $p, q$  be distinct primes and  $F, G$  be the corresponding Dirichlet series of  $f, g$ , respectively. By Theorem 1,  $J(f, g/p, q) = 0$ , and so

$$\begin{aligned} 0 &= J(F, G/p, q) = \begin{vmatrix} d_p F & d_p G \\ d_q F & d_q G \end{vmatrix} \\ &= \left| \begin{array}{cc} \left( \frac{f(n^*)v_p(n^*)}{(n^*/p)^s} + \frac{f(n')v_p(n')}{(n'/p)^s} + \dots \right) & \left( \frac{g(m^*)v_p(m^*)}{(m^*/p)^s} + \frac{f(m')v_p(m')}{(m'/p)^s} + \dots \right) \\ \left( \frac{f(n^*)v_q(n^*)}{(n^*/q)^s} + \frac{f(n')v_q(n')}{(n'/q)^s} + \dots \right) & \left( \frac{g(m^*)v_q(m^*)}{(m^*/q)^s} + \frac{f(m')v_q(N_1g)}{(m'/q)^s} + \dots \right) \end{array} \right| \\ &= \left| \frac{f(n^*)v_p(n^*)}{(n^*/p)^s} \frac{g(m^*)v_p(m^*)}{(m^*/p)^s} \right| + \left| \frac{f(n^*)v_p(n^*)}{(n^*/q)^s} \frac{g(m')v_p(m')}{(m'/p)^s} \right| \\ &\quad + \left| \frac{f(n')v_p(n')}{(n'/p)^s} \frac{g(m^*)v_p(m^*)}{(m^*/q)^s} \right| + R \end{aligned}$$

$$\begin{aligned}
&= \frac{f(n^*)g(m^*)(pq)^s}{(n^*m^*)^s} \left| \begin{array}{cc} v_p(n^*) & v_p(m^*) \\ v_q(n^*) & v_q(m^*) \end{array} \right| \\
&+ \frac{f(n^*)g(m')(pq)^s}{(n^*m')^s} \left| \begin{array}{cc} v_p(n^*) & v_p(m') \\ v_q(n^*) & v_q(m') \end{array} \right| \\
&+ \frac{f(n')g(n^*)(pq)^s}{(n'm^*)^s} \left| \begin{array}{cc} v_p(n') & v_p(m^*) \\ v_q(n') & v_q(m^*) \end{array} \right| + R,
\end{aligned}$$

where  $R$  is the sum of remaining terms all of whose denominators are greater than  $(\frac{n'm^*}{pq})^s$  and  $(\frac{n^*m'}{pq})^s$ . Since  $f(n^*), f(n'), g(m^*), g(m')$  are all  $\neq 0$  and  $n^*m' \neq n'm^*$ , then

$$\left| \begin{array}{cc} v_p(n^*) & v_p(m^*) \\ v_q(n^*) & v_q(m^*) \end{array} \right| = \left| \begin{array}{cc} v_p(n^*) & v_p(m') \\ v_q(n^*) & v_q(m') \end{array} \right| = \left| \begin{array}{cc} v_p(n') & v_p(m^*) \\ v_q(n') & v_q(m^*) \end{array} \right| = 0.$$

From  $\left| \begin{array}{cc} v_p(n^*) & v_p(m^*) \\ v_q(n^*) & v_q(m^*) \end{array} \right| = 0$ , we deduce that there exist  $x_1, x_2 \in \mathbb{Z}$ , not all zero, such that for all primes  $r$ ,  $x_1 v_r(n^*) + x_2 v_r(m^*) = 0$ , i.e.,  $v_r((n^*)^{x_1} (m^*)^{x_2}) = 0$ , which renders  $(Nf)^{x_1} (Ng)^{x_2} = (n^*)^{x_1} (m^*)^{x_2} = 1$ . The remaining assertions follow analogously using the other two determinantal values.  $\square$

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