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SOME RECURRENCE RELATIONS OF MULTIPLE ORTHOGONAL POLYNOMIALS

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Abstract. In this paper, we first find a necessary and sufficient condition for the existence of multiple orthogonal polynomials by the moments of a pair of measures $(d\mu, d\nu)$ and then give representations for multiple orthogonal polynomials. We also prove four term recurrence relations for multiple orthogonal polynomials of type II and several interesting relations for multiple orthogonal polynomials are given. A generalized recurrence relation for multiple orthogonal polynomials of type I is found and then four term recurrence relations are obtained as a special case.

1. Introduction

Multiple orthogonal polynomials were developed from the theory of simultaneous rational approximations, in particular, Hermite-Padé approximations whose root goes back to nineteenth century ([1, 12]). But only several examples of multiple orthogonal polynomials appeared in the literature until 1980's. Recently there has been a renewed interest in multiple orthogonal polynomials as a natural generalization of ordinary orthogonal polynomials so that many new results on multiple orthogonal polynomials are obtained. We refer to [2, 3, 12, 14].

There are two ways of extending the ordinary orthogonal polynomial system (OPS) to multiple OPS's as follows:

(a) A sequence $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ of polynomial vectors is called a multiple OPS of type I relative to a pair of measures $(d\mu, d\nu)$ if

(i) $\deg(A_{(n_1,n_2)}) = n_1$ and $\deg(B_{(n_1,n_2)}) = n_2$;

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(ii) $A_{(n_1,n_2)}$ and $B_{(n_1,n_2)}$ satisfy

$$\int_{-\infty}^{\infty} x^k (A_{(n_1,n_2)} d\mu + B_{(n_1,n_2)} d\nu) = 0, \ k = 0, 1, \cdots, n_1 + n_2. \tag{1.1}$$

(b) A sequence $\{P_{n_1 \oplus n_2}\}_{n_1,n_2=0}^{\infty}$ of polynomials is called a multiple OPS of type II relative to a pair of measures $(d\mu, d\nu)$ if

(i) $\deg(P_{n_1 \oplus n_2}) = n_1 + n_2$;
(ii) $P_{n_1 \oplus n_2}$ satisfies

$$\int_{-\infty}^{\infty} x^k P_{n_1 \oplus n_2} d\mu = 0, \ k = 0, 1, 2, \cdots, n_1 - 1 \tag{1.2}$$

and

$$\int_{-\infty}^{\infty} x^k P_{n_1 \oplus n_2} d\nu = 0, \ k = 0, 1, 2, \cdots, n_2 - 1. \tag{1.3}$$

In this paper, we first find a necessary and sufficient condition for the existence of multiple OPS by the moments of measures $d\mu$ and $d\nu$ and then give representations for multiple OPS (see Proposition 2.1 for type I and Proposition 2.3 for type II). We also prove four term recurrence relations for multiple OPS of type II (see Theorem 2.5) and several interesting relations for multiple OPS are given (see Theorem 2.6 and Theorem 2.7). A generalized recurrence relation for multiple OPS of type I is found (see Theorem 2.8 and Theorem 2.9) and then four term recurrence relations are obtained as a special case.

2. Main results

Let $d\mu$ and $d\nu$ be two signed Borel measures on the real line $\mathbb{R}$ whose moments $\mu_n = \int_{-\infty}^{\infty} x^n d\mu(x)$ and $\nu_n = \int_{-\infty}^{\infty} x^n d\nu(x)$ $(n \geq 0)$ are finite. A measure $d\mu$ is said to be quasi-definite if its Hankel matrix

$$\Delta_n(d\mu) := \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}, \quad n \geq 0,$$

is nonsingular. Throughout the paper, $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ and $\{P_{n_1 \oplus n_2}\}_{n_1,n_1=0}^{\infty}$ are always assumed to be the multiple OPS of type I and type II relative to $(d\mu, d\nu)$, respectively. We also use the following notations

$$\Delta^A_{(n_1,n_2)} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} \\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n_1-1} & \mu_{n_1} & \cdots & \mu_{2n_1+n_2-1} \\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} \end{pmatrix}$$

and

$$\Delta^B_{(n_1,n_2)} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} \\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} \\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n_2-1} & \nu_{n_2} & \cdots & \nu_{n_1+n_2-1} \end{pmatrix}.$$

Here, we first find a necessary and sufficient condition for multiple OPS of type I to exist ([9]), and then give its representation.

PROPOSITION 2.1. *Let $d\mu$ and $d\nu$ be quasi-definite. Then*

(a) $|\Delta^A_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$, *if and only if there exists a unique* (up to a constant multiple) *multiple OPS* $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ *with* $\deg(A_{(n_1,n_2)}) = n_1$.

(b) $|\Delta^B_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$, *if and only if there exists a unique* (up to a constant multiple) *multiple OPS* $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ *with* $\deg(B_{(n_1,n_2)}) = n_2$.

In each case, the vector $(A_{(n_1,n_2)}, B_{(n_1,n_2)})$ *is uniquely determined* (up to a constant multiple) *by*

$$
(2.1)\qquad A_{(n_1,n_2)} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} & x^{n_1} \\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} & 0 \end{vmatrix}
$$
$$
= (-1)^{n_2+1}\Delta^A_{(n_1,n_2)} x^{n_1} + \text{lower degree terms}
$$

and

$$
(2.2)\qquad B_{(n_1,n_2)} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} & 0 \\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} & 0 \\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} & x^{n_2} \end{vmatrix}
$$
$$
= \Delta^B_{(n_1,n_2)} x^{n_2} + \text{lower degree terms.}
$$

Moreover,

$$
(2.3)\qquad \int_{-\infty}^{\infty} x^{n_1+n_2+1}(A_{(n_1,n_2)}d\mu + B_{(n_1,n_2)}d\nu) = |\Delta^A_{(n_1+1,n_2)}| = |\Delta^B_{(n_1,n_2+1)}|.
$$

Proof. Let's write

$$
A_{(n_1,n_2)} = a_{n_1}x^{n_1} + a_{n_1-1}x^{n_1-1} + \cdots + a_0
$$

and

$$
B_{(n_1,n_2)} = b_{n_2}x^{n_2} + b_{n_2-1}x^{n_2-1} + \cdots + b_0.
$$

By (1.1), we have

$$\text{(2.4)}\quad (a_0, a_1, \cdots, a_{n_1}, b_0, \cdots, b_{n_2}) \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} \\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} \\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} \end{pmatrix} = 0,$$

which can be written as

$$\text{(2.5)}\quad (a_0, a_1, \cdots, a_{n_1-1}, b_0, \cdots, b_{n_2})\Delta^A_{(n_1,n_2)} = -a_{n_1}(\mu_{n_1}, \mu_{n_1+1}, \cdots, \mu_{2n_1+n_2})$$

and

$$(a_0, a_1, \cdots, a_{n_1}, b_0, \cdots, b_{n_2-1})\Delta^B_{(n_1,n_2)} = -b_{n_2}(\nu_{n_2}, \nu_{n_1+1}, \cdots, \nu_{n_1+2n_2}).$$

Since $d\mu$ is quasi-definite, the Hankel matrix $\Delta_{n_1+n_2}(d\mu)$ has its full rank $n_1 + n_2 + 1$, that is, all row vectors of the matrix are linearly independent. Hence, the row $(\mu_{n_1}, \mu_{n_1+1}, \cdots, \mu_{2n_1+n_2})$ is independent of $(\mu_k, \mu_{k+1}, \cdots, \mu_{k+n_1+n_2})$ for $k = 0, 1, 2, \cdots, n_1 - 1$. Similarly, the vector $(\nu_{n_2}, \nu_{n_2+1}, \cdots, \nu_{n_1+2n_2})$ is independent of $(\nu_k, \nu_{k+1}, \cdots, \nu_{k+n_1+n_2})$ for $k = 0, 1, \cdots, n_2 - 1$.

(a) Let $|\Delta^A_{(n_1,n_2)}| \neq 0$. Then there exist solutions $a_0, a_1, \cdots, a_{n_1-1}, b_0, \cdots, b_{n_2}$ which are unique up to a constant multiple. Hence, the solution $(A_{(n_1,n_2)}, B_{(n_1,n_2)})$ with $\deg(A_{(n_1,n_2)}) = n_1$ exists, which is unique up to a constant multiple. Conversely, if there exist solutions $A_{(n_1,n_2)}$ and $B_{(n_1,n_2)}$ satisfying the condition (ii) in the definition, then the coefficients $a_0, a_1, \cdots, a_{n_1}$, $b_0, \cdots, b_{n_2}$ satisfy the equation (2.4). Since μ is quasi-definite, the moments vector $(\mu_{n_1}, \mu_{n_1+1}, \cdots, \mu_{(2n_1+n_2)})$ in the right hand side of (2.5) is not zero. Hence, it follows $|\Delta^A_{(n_1,n_2)}| \neq 0$ from the uniqueness of $(A_{(n_1,n_2)}, B_{(n_1,n_2)})$ with $\deg(A_{(n_1,n_2)}) = n_1$ or equivalently $a_{n_1} \neq 0$.

The proof of (b) is similar to that of (a). The representations for $A_{(n_1,n_2)}$ and $B_{(n_1,n_2)}$ in (2.1) and (2.2) can be easily proved by a direct calculation. Moreover, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} x^k (A_{(n_1,n_2)} d\mu + B_{(n_1,n_2)} d\nu) \\
&= \int_{-\infty}^{\infty} x^k A_{(n_1,n_2)} d\mu + \int_{-\infty}^{\infty} x^k B_{(n_1,n_2)} d\nu \\
&= \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} & \mu_k \\
\mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} & \mu_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} & \mu_{n_1+k} \\
\nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} & 0
\end{vmatrix}
+ \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} & 0 \\
\mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} & 0 \\
\nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} & \nu_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} & \nu_{k+n_2}
\end{vmatrix} \\
&= \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2} & \mu_k \\
\mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1} & \mu_{k+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mu_{n_1} & \mu_{n_1+1} & \cdots & \mu_{2n_1+n_2} & \mu_{n_1+k} \\
\nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2} & \nu_k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\nu_{n_2} & \nu_{n_2+1} & \cdots & \nu_{n_1+2n_2} & \nu_{k+n_2}
\end{vmatrix} \\
&= \begin{cases} 0 & \text{for} \quad k = 0, 1, 2, \cdots, n_1+n_2 \\ |\Delta^A_{(n_1+1,n_2)}| = |\Delta^B_{(n_1,n_2+1)}| & \text{for} \quad k = n_1+n_2+1, \end{cases}
\end{aligned}
$$

which is the equation (2.3). □

In the proof of Proposition 2.1, the vector $(\hat{A}_{(n_1,n_2)}, B_{(n_1,n_2)})((A_{(n_1,n_2)}, \hat{B}_{(n_1,n_2)})$, respectively) is uniquely determined, where $\hat{A}_{(n_1,n_2)}$ ($\hat{B}_{(n_1,n_2)}$, respectively) is the monic polynomial. Even in this case, we can not say that $\deg(B_{(n_1,n_2)}) = n_2$ ($\deg(A_{(n_1,n_2)}) = n_1$, respectively). The next lemma proves that it has to be so.

LEMMA 2.2. *Let $d\mu$ and $d\nu$ be quasi-definite. Then $|\Delta^A_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$, if and only if $|\Delta^B_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$.*

Proof. First note that $\Delta^A_{(n_1+1,n_2)} = \Delta^B_{(n_1,n_2+1)}$ from the definition. Hence, if $|\Delta^A_{(n_1,n_2)}| \neq 0$ for $n_1 \geq 0$ and $n_2 \geq 0$, then $|\Delta^B_{(n_1,n_2)}| \neq 0$ for $n_1 \geq 0$ and $n_2 \geq 1$. Since the matrix $\Delta_{(n_1,0)}$ is the Hankel matrix of a quasi-definite measure $d\mu$, which is nonsingular, we have $|\Delta^B_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$. The converse can be proved by the same method. □

If $|\Delta^A_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$, then $(\hat{A}_{(n_1,n_2)}, B_{(n_1,n_2)})$ is a multiple OPS relative to $(d\mu, d\nu)$ by Proposition 2.1. By Lemma 2.2, we have $|\Delta^B_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$, and so $(A_{(n_1,n_2)}, \hat{B}_{(n_1,n_2)})$ is also a multiple OPS relative to $(d\mu, d\nu)$ by Proposition 2.1 again. Even in this case, we can not take the monic polynomials $\hat{A}_{(n_1,n_2)}$ and $\hat{B}_{(n_1,n_2)}$ simultaneously, that is, $(\hat{A}_{(n_1,n_2)}, \hat{B}_{(n_1,n_2)})$ need not be a multiple OPS relative to $(d\mu, d\nu)$.

Now, consider a multiple OPS $\{P_{n_1 \oplus n_2}\}_{n_1,n_2=0}^{\infty}$ of type II relative to $(d\mu, d\nu)$. Since there are n_1+n_2+1 unknowns but only n_1+n_2 equations, there may be infinitely many solutions of (ii) in the definition of multiple OPS of type II. Let $\Delta_{0\oplus 0} = 1$ and

$$\Delta_{n_1 \oplus n_2} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2-1} \\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n_1-1} & \mu_{n_1} & \cdots & \mu_{2n_1+n_2-2} \\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n_2-1} & \nu_{n_2} & \cdots & \nu_{n_1+2n_2-2} \end{pmatrix}.$$

Then we obtain a necessary and sufficient condition for the existence of the multiple OPS of type II ([9]). More precisely, we have

PROPOSITION 2.3. *Let $d\mu$ and $d\nu$ be quasi-definite. Then $|\Delta_{n_1 \oplus n_2}| \neq 0$, $n_1, n_2 \geq 0$, if and only if there exists a unique* (up to a constant multiple) *multiple OPS $\{P_{n_1 \oplus n_2}\}_{n_1,n_2=0}^{\infty}$ of type II relative to $(d\mu, d\nu)$.*

In this case, we have

$$P_{n_1\oplus n_2}=\begin{vmatrix}\mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2}\\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2+1}\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{n_1-1} & \mu_{n_1} & \cdots & \mu_{2n_1+n_2-1}\\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2}\\ \vdots & \vdots & \ddots & \vdots\\ \nu_{n_2-1} & \nu_{n_2} & \cdots & \nu_{n_1+2n_2-1}\\ 1 & x & \cdots & x^{n_1+n_2}\end{vmatrix}$$

$$= \Delta_{n_1\oplus n_2}x^{n_1+n_2} + \text{lower degree terms}$$

and moreover,

$$\text{(2.6)}\qquad \begin{aligned}\int_{-\infty}^{\infty} x^{n_1}P_{n_1\oplus n_2}d\mu &= (-1)^{n_2}|\Delta_{n_1+1\oplus n_2}|;\\ \int_{-\infty}^{\infty} x^{n_2}P_{n_1\oplus n_2}d\nu &= |\Delta_{n_1\oplus n_2+1}|.\end{aligned}$$

Proof. Let $P_{n_1\oplus n_2}=a_{n_1+n_2}x^{n_1+n_2}+a_{n_1+n_2-1}x^{n_1+n_2-1}+\cdots+a_0$. By (1.2) and (1.3), we have

$$\begin{pmatrix}\mu_0 & \mu_1 & \cdots & \mu_{n_1+n_2-1}\\ \mu_1 & \mu_2 & \cdots & \mu_{n_1+n_2}\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{n_1-1} & \mu_{n_1} & \cdots & \mu_{2n_1+n_2-2}\\ \nu_0 & \nu_1 & \cdots & \nu_{n_1+n_2-1}\\ \vdots & \vdots & \ddots & \vdots\\ \nu_{n_2-1} & \nu_{n_2} & \cdots & \nu_{n_1+2n_2-2}\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ \vdots\\ \vdots\\ a_{n_1+n_2-1}\end{pmatrix}=-a_{n_1+n_2}\begin{pmatrix}\mu_{n_1+n_2}\\ \mu_{n_1+n_2-1}\\ \vdots\\ \mu_{2n_1+n_2}\\ \nu_{n_1+n_2}\\ \vdots\\ \nu_{n_1+2n_2}\end{pmatrix}$$

If $|\Delta_{n_1\oplus n_2}|\neq 0$, then there exists a unique (up to a constant multiple) solution, which implies the unique existence of $P_{n_1\oplus n_2}$. Conversely, if there exists a unique (up to a constant multiple) multiple OPS, then $a_{n_1+n_2}\neq 0$ and $\Delta_{n_1\oplus n_2}$ must be nonsingular. The representation of $P_{n_1\oplus n_2}$ and the estimates of (2.6) can be easily proved by a direct calculation. □

Proposition 2.1 and Proposition 2.3 are already obtained in [5, 6, 9, 12]. In order to investigate the relation between multiple OPS of type I and multiple OPS of type II, we note:

$$\text{(2.7)}\quad \Delta_{n_1\oplus n_2+1}=\Delta^A_{(n_1,n_2)} \quad\text{and}\quad \Delta_{n_1+1\oplus n_2}=\Delta^B_{(n_1,n_2)},\quad n_1,n_2\geq 0.$$

From Proposition 2.1 and Proposition 2.3, we can obtain relations between multiple OPS of type I and type II ([9]).

THEOREM 2.4. *Let $d\mu$ and $d\nu$ be quasi-definite. Then the followings are all equivalent.*

(a) $|\Delta^A_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$.
(b) $|\Delta^B_{(n_1,n_2)}| \neq 0$, $n_1, n_2 \geq 0$.
(c) *there exists a unique* (up to a constant multiple) *multiple OPS of type I relative to* $(d\mu, d\nu)$.
(d) $|\Delta_{n_1 \oplus n_2}| \neq 0$, $n_1, n_2 \geq 0$.
(e) *there exists a unique* (up to a constant multiple) *multiple OPS of type II relative to* $(d\mu, d\nu)$.

Proof. The equivalences of (a), (b), and (c) are proved in Proposition 2.1 and Lemma 2.2, and the equivalence of (d) and (e) is proved in Proposition 2.3. By (2.7), (d) implies (a). Conversely, to show that (a) implies (d), by relation (2.7) again, it's enough to show that $|\Delta_{n_1\oplus 0}| \neq 0$ for $n_1 \geq 1$. Since $\Delta_{n_1\oplus 0}$ is the Hankel matrix for the quasi-definite measure $d\nu$, it must be nonsingular. □

Let $\hat{P}_{n_1\oplus n_2} = \frac{P_{n_1\oplus n_2}}{|\Delta_{n_1\oplus n_2}|}$, $\hat{A}_{(n_1,n_2)} = \frac{(-1)^{n_2+1}A_{(n_1,n_2)}}{|\Delta^A_{(n_1,n_2)}|}$, and $\hat{B}_{(n_1,n_2)} = \frac{B_{(n_1,n_2)}}{|\Delta^B_{(n_1,n_2)}|}$ are monic polynomials. Then, by the orthogonality, we have

(2.8)
$$\int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2} A_{(k,\ell)} d\mu = \begin{cases} (-1)^{n_2+\ell+1} \frac{|\Delta^A_{(k,\ell)}|}{|\Delta_{n_1\oplus n_2}|} |\Delta_{n_1+1\oplus n_2}| & \text{if} \quad k = n_1 \\ 0 & \text{if} \quad k < n_1 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2} B_{(k,\ell)} d\nu = \begin{cases} 0 & \text{if} \quad \ell < n_2 \\ (-1)^{n_1} \frac{|\Delta^B_{(k,\ell)}|}{|\Delta_{n_1\oplus n_2}|} |\Delta_{n_1\oplus n_2+1}| & \text{if} \quad \ell = n_2. \end{cases} \tag{2.9}$$

By the relations (2.8) and (2.9), we can easily see that

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2} A_{(n_1,n_2)} d\mu &= -\frac{|\Delta_{n_1\oplus n_2+1} \Delta_{n_1+1\oplus n_2}|}{|\Delta_{n_1\oplus n_2}|} \\ &= -\int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2} B_{(n_1,n_2)} d\nu. \end{aligned}$$

Moreover, by the orthogonality of multiple OPS of type I and type II and (2.8), (2.9), we have

$$\int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}(A_{(n_1,n_2)}d\mu + B_{(n_1,n_2)}d\nu)$$

$$= \begin{cases} 0 & \text{if } k+\ell < n_1+n_2 \\ 0 & \text{if } k > n_1 \text{ and } \ell > n_2 \\ \frac{(-1)^{\ell+n_2+1}|\Delta^A_{(n_1,n_2)}||\Delta_{k+1\oplus\ell}|}{|\Delta_{k\oplus\ell}|} & \text{if } k = n_1 \text{ and } \ell > n_2 \\ \frac{|\Delta^B_{(n_1,n_2)}|}{|\Delta_{k\oplus\ell}|}|\Delta_{k\oplus\ell+1}| & \text{if } k > n_1 \text{ and } \ell = n_2 \\ \int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}B_{(n_1,n_2)}d\nu & \text{if } k > n_1 \text{ and } \ell < n_2 \\ \int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}A_{(n_1,n_2)}d\mu & \text{if } k < n_1 \text{ and } \ell > n_2 \end{cases}$$

from which we have:

(i) if $k > n_1$ and $k+\ell < n_1+n_2$, then

$$\int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}B_{(n_1,n_2)}d\nu = \int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}(A_{(n_1,n_2)}d\mu + B_{(n_1,n_2)}d\nu) = 0;$$

(ii) if $\ell > n_2$ and $k+\ell < n_1+n_2$, then

$$\int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}A_{(n_1,n_2)}d\mu = \int_{-\infty}^{\infty} \hat{P}_{k\oplus\ell}(A_{(n_1,n_2)}d\mu + B_{(n_1,n_2)}d\nu) = 0;$$

(iii) $\displaystyle\int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2+1}A_{(n_1,n_2)}d\mu = |\Delta_{n_1+1\oplus n_2+1}| = \int_{-\infty}^{\infty} \hat{P}_{n_1+1\oplus n_2}B_{(n_1,n_2)}d\nu.$

Using these relations, we can obtain a four term recurrence relation of multiple OPS of type II, which was already known for proper multi-indices ([4, 7, 8, 10, 11, 13, 15]).

THEOREM 2.5. *For the multiple OPS* $\{\hat{P}_{n_1\oplus n_2}\}_{n_1,n_2=0}^{\infty}$ *of type II relative to a pair of quasi-definite measures* $(d\mu, d\nu)$, *we have for* $n_1, n_2 \geq 1$,

$$\begin{aligned} x\hat{P}_{n_1\oplus n_2} = {} & \hat{P}_{n_1\oplus n_2+1} + b_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} \\ & + c_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2} + d_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1}, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} b_{(n_1,n_2)} &= \frac{\int_{-\infty}^{\infty} x\hat{P}_{n_1\oplus n_2}(A_{(n_1-1,n_2)}d\mu + B_{(n_1-1,n_2)}d\nu)}{|\Delta_{n_1\oplus n_2+1}|}; \\ c_{(n_1,n_2)} &= \frac{|\Delta_{n_1-1\oplus n_2}\Delta_{n_1+1\oplus n_2}| + |\Delta_{n_1\oplus n_2-1}\Delta_{n_1\oplus n_2+1}|}{|\Delta_{n_1\oplus n_2}|^2}; \\ d_{(n_1,n_2)} &= \left|\frac{\Delta_{n_1-1\oplus n_2-1}}{\Delta_{n_1\oplus n_2}}\right|\left|\frac{\Delta_{n_1\oplus n_2+1}}{\Delta_{n_1-1\oplus n_2}}\right| \neq 0, \qquad n_1, n_2 \geq 1. \end{aligned} \tag{2.11}$$

Proof. We may assume that $n_1 \leq n_2$ because the other case can be proved similarly. Since $\hat{P}_{n_1 \oplus n_2}$ is of degree $n_1 + n_2$, we can write

$$x\hat{P}_{n_1 \oplus n_2} = \sum_{j=0}^{n_1} a_{n_1-j} \hat{P}_{n_1-j \oplus n_2-j+1} + \sum_{j=0}^{n_1} b_{n_2-j} \hat{P}_{n_1-j \oplus n_2-j} + \sum_{j=0}^{n_2-n_1} b_{n_2-n_1-j} \hat{P}_{0 \oplus n_2-n_1-j}.$$

Multiplying by $(A_{(n_1-k,n_2-k)}d\mu + B_{(n_1-k,n_2-k)}d\nu)$, $k = 2, 3, \cdots, n_1$, and then integrating both sides, we have by the orthogonality of type I and type II,

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} x\hat{P}_{n_1 \oplus n_2}(A_{(n_1-k,n_2-k)}d\mu + B_{(n_1-k,n_2-k)}d\nu) \\
&= \sum_{j=0}^{n_1} a_{n_1-j} \int_{-\infty}^{\infty} \hat{P}_{n_1-j \oplus n_2-j+1}(A_{(n_1-k,n_2-k)}d\mu + B_{(n_1-k,n_2-k)}d\nu) \\
&+ \sum_{j=0}^{n_1} b_{n_2-j} \int_{-\infty}^{\infty} \hat{P}_{n_1-j \oplus n_2-j}(A_{(n_1-k,n_2-k)}d\mu + B_{(n_1-k,n_2-k)}d\nu) \\
&+ \sum_{j=1}^{n_2-n_1} b_{n_2-n_1-j} \int_{-\infty}^{\infty} \hat{P}_{0 \oplus n_2-n_1-j}(A_{(n_1-k,n_2-k)}d\mu + B_{(n_1-k,n_2-k)}d\nu) \\
&= a_{n_1-k} \int_{-\infty}^{\infty} \hat{P}_{n_1-k \oplus n_2-k+1} A_{(n_1-k,n_2-k)}d\mu \\
&= a_{n_1-k} |\Delta_{n_1-k+1 \oplus n_2-k+1}|, \quad k = 2, 3, \cdots, n_1.
\end{aligned}$$

By Proposition 2.1, we have $|\Delta_{n_1-k+1 \oplus n_2-k+1}| \neq 0$ so that $a_{n_1-k} = 0$, $k = 2, 3, \cdots, n_1$. Hence, $a_k = 0$ for $k = 0, 1, 2, \cdots, n_1 - 2$. Now, multiplying by $(A_{(n_1-k,n_2-k-1)}d\mu + B_{(n_1-k,n_2-k-1)}d\nu)$, $k = 2, 3, \cdots, n_1$, and then integrating both sides, we have by the same process as above, $b_k = 0$ for $k = n_2 - n_1, n_2 - n_1 + 1, \cdots, n_2 - 2$. Finally, multiplying by $B_{(0,n_2-n_1-k)}d\nu$ and then integrating both sides, we obtain by the orthogonality of type I and type II,

$$\begin{aligned}
0 &= \int_{-\infty} x\hat{P}_{n_1 \oplus n_2} B_{(0,n_2-n_1-k)}d\nu \\
&= \int_{-\infty}^{\infty} (a_{n_1}\hat{P}_{n_1 \oplus n_2+1} + a_{n_1-1}\hat{P}_{n_1-1 \oplus n_2}) B_{(0,n_2-n_1-k)}d\nu \\
&+ \int_{-\infty}^{\infty} (b_{n_2}\hat{P}_{n_1 \oplus n_2} + b_{n_2-1}\hat{P}_{n_1-1 \oplus n_2-1}) B_{(0,n_2-n_1-k)}d\nu
\end{aligned}$$

$$+\sum_{j=1}^{n_2-n_1} b_{n_2-n_1-j}\int_{-\infty}^{\infty}\hat{P}_{0\oplus n_2-n_1-j}B_{(0,n_2-n_1-k)}d\nu$$
$$=\sum_{j=1}^{n_2-n_1} b_{n_2-n_1-j}\int_{-\infty}^{\infty}\hat{P}_{0\oplus n_2-n_1-j}B_{(0,n_2-n_1-k)}d\nu,$$
$$\text{for } k=1,2,\cdots,n_2-n_1.$$

First, put $k=n_2-n_1$. Then $\int_{-\infty}^{\infty}\hat{P}_{0\oplus n_2-n_1-j}B_{(0,0)}d\nu=0$ for $j=1,2,\cdots,n_2-n_1-1$ so that

$$0=b_0\int_{-\infty}^{\infty}\hat{P}_{0\oplus 0}B_{(0,0)}d\nu=\frac{|\Delta^B_{(0,0)}|}{|\Delta_{0\oplus 0}|}|\Delta_{0\oplus 1}|b_0.$$

Since $\frac{|\Delta^B_{(0,0)}|}{|\Delta_{0\oplus 0}|}|\Delta_{0\oplus 1}|\neq 0$, we obtain $b_0=0$. If we put $k=n_2-n_1-1$, then by the same process, we have

$$0=b_0\int_{-\infty}^{\infty}\hat{P}_{0\oplus 0}B_{(0,1)}d\nu+b_1\int_{-\infty}^{\infty}\hat{P}_{0\oplus 1}B_{(0,1)}d\nu$$
$$=b_1\int_{-\infty}^{\infty}\hat{P}_{0\oplus 1}B_{(0,1)}d\nu$$

from which $b_1=0$ follows. Continuing the same processes for $k=n_2-n_1-2,n_2-n_1-3,\cdots,1$, successively, we obtain for $m=2,3,\cdots,n_2-n_1-1$,

$$0=b_0\int_{-\infty}^{\infty}\hat{P}_{0\oplus 0}B_{(0,m)}d\nu$$
$$+b_1\int_{-\infty}^{\infty}\hat{P}_{0\oplus 1}B_{0,m}d\nu+\cdots+b_m\int_{-\infty}^{\infty}\hat{P}_{0\oplus m}B_{(0,m)}d\nu.$$

Using the fact

$$\int_{-\infty}^{\infty}\hat{P}_{0\oplus m}B_{(0,m)}d\nu=\frac{|\Delta^B_{(0,m)}|}{|\Delta_{0\oplus m}|}|\Delta_{0\oplus m+1}|\neq 0,\ m=0,1,\cdots,n_2-n_1-1,$$

we obtain $b_j=0$ for $j=0,1,\cdots,n_2-n_1-1$ successively. Hence, $\{\hat{P}_{n_1\oplus n_2}\}_{n_1,n_2=0}^{\infty}$ satisfies a four term recurrence relation (2.10).

By multiplying (2.10) by $A_{(n_1-1,n_2)}d\mu+B_{(n_1-1,n_2)}d\nu$, $A_{(n_1-1,n_2-1)}d\mu+B_{(n_1-1,n_2-1)}d\nu$, and $A_{(n_1-2,n_2-1)}d\mu+B_{(n_1-2,n_2-1)}d\nu$, and then integrating both sides, we can easily obtain (2.11) by the orthogonality of type I and type II. □

Note that if we multiply (2.10) by $A_{(n_1,n_2)}d\mu + B_{(n_1,n_2)}d\nu$ and then integrate both sides,

$$\int_{-\infty}^{\infty} x\hat{P}_{n_1\oplus n_2}(A_{(n_1,n_2)}d\mu + B_{(n_1,n_2)})d\nu = |\Delta_{n_1+1\oplus n_2+1}|.$$

By the same process as in Theorem 2.5, we have another four term recurrence relations such as

(2.12)

$$\begin{aligned} &x\hat{P}_{n_1\oplus n_2}\\ &= \hat{P}_{n_1\oplus n_2+1} + b_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + c_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2} + d_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1}\\ &= \hat{P}_{n_1\oplus n_2+1} + \tilde{b}_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + \tilde{c}_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2-1} + \tilde{d}_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1}\\ &= \hat{P}_{n_1+1\oplus n_2} + \beta_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + \gamma_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2} + \delta_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1}\\ &= \hat{P}_{n_1+1\oplus n_2} + \tilde{\beta}_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + \tilde{\gamma}_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2-1} + \tilde{\delta}_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1}. \end{aligned}$$

Comparing the leading coefficients, we can easily see that $b_{(n_1,n_2)} = \tilde{b}_{(n_1,n_2)}$ and $\beta_{(n_1,n_2)} = \tilde{\beta}_{(n_1,n_2)}$. Since $c_{(n_1,n_2)} = \tilde{c}_{(n_1,n_2)}$ by comparing the coefficient for $x^{n_1+n_2-1}$ in the first and second equation of (2.12), we have $c_{(n_1,n_2)} = \tilde{c}_{(n_1,n_2)}$ and similarly $\gamma_{(n_1,n_2)} = \tilde{\gamma}_{(n_1,n_2)}$. Multiplying the third equation of (2.12) by $A_{(n_1-1,n_2-1)}d\mu + B_{(n_1-1,n_2-1)}d\nu$ and then integrating, we obtain

$$\gamma_{(n_1,n_2)} = \frac{|\Delta_{n_1-1\oplus n_2}\Delta_{n_1+1\oplus n_2}| + |\Delta_{n_1\oplus n_2-1}\Delta_{n_1\oplus n_2+1}|}{|\Delta_{n_1\oplus n_2}|^2} = c_{(n_1,n_2)}.$$

Moreover, multiplying by $A_{(n_1-2,n_2-1)}d\mu + B_{(n_1-2,n_2-1)}d\nu$ and integrating gives

$$d_{(n_1,n_2)} = \left|\frac{\Delta_{n_1-1\oplus n_2-1}\Delta_{n_1\oplus n_2+1}}{\Delta_{n_1\oplus n_2}\Delta_{n_1-1\oplus n_2}}\right|$$

and multiplying by $A_{(n_1-1,n_2-2)}d\mu + B_{(n_1-1,n_2-2)}d\nu$ and integrating gives

$$\tilde{d}_{(n_1,n_2)} = -\left|\frac{\Delta_{n_1-1\oplus n_2-1}\Delta_{n_1+1\oplus n_2}}{\Delta_{n_1\oplus n_2}\Delta_{n_1\oplus n_2-1}}\right|.$$

By the same process, we can easily see that $d_{(n_1,n_2)} = \delta_{(n_1,n_2)}$ and $\tilde{d}_{(n_1,n_2)} = \tilde{\delta}_{(n_1,n_2)}$. Hence, the four term recurrence relation (2.12) becomes

(2.13)

$$\begin{aligned} &x\hat{P}_{n_1\oplus n_2} \\ &= \hat{P}_{n_1\oplus n_2+1} + b_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + c_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2} + d_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1} \\ &= \hat{P}_{n_1\oplus n_2+1} + b_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + c_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2-1} + \tilde{d}_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1} \\ &= \hat{P}_{n_1+1\oplus n_2} + \beta_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + c_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2} + d_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1} \\ &= \hat{P}_{n_1+1\oplus n_2} + \beta_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2} + c_{(n_1,n_2)}\hat{P}_{n_1\oplus n_2-1} + \tilde{d}_{(n_1,n_2)}\hat{P}_{n_1-1\oplus n_2-1}. \end{aligned}$$

Hence, from the first and second equations of (2.13),

$$(2.14)\quad c_{(n_1,n_2)}[\hat{P}_{n_1-1\oplus n_2} - \hat{P}_{n_1\oplus n_2-1}] = (d_{(n_1,n_2)} - \tilde{d}_{(n_1,n_2)})\hat{P}_{n_1-1\oplus n_2-1}$$

and from the first and third equations,

$$(2.15)\qquad \hat{P}_{n_1\oplus n_2+1} - \hat{P}_{n_1+1\oplus n_2} = (b_{(n_1,n_2)} - \beta_{(n_1,n_2)})\hat{P}_{n_1\oplus n_2}.$$

THEOREM 2.6. *Let a multiple OPS* $\{P_{n_1\oplus n_2}\}_{n_1,n_2=0}^{\infty}$ *of type II satisfy four term recurrence relations* (2.13). *If* $c_{(n_1,n_2)} \neq 0$, *then*

$$\hat{P}_{n_1-1\oplus n_2} - \hat{P}_{n_1\oplus n_2-1} = \left|\frac{\Delta_{n_1\oplus n_2}\Delta_{n_1-1\oplus n_2-1}}{\Delta_{n_1-1\oplus n_2}\Delta_{n_1\oplus n_2-1}}\right|\hat{P}_{n_1-1\oplus n_2-1}$$

and if $c_{(n_1,n_2)} = 0$, *then*

$$\hat{P}_{n_1\oplus n_2+1} - \hat{P}_{n_1+1\oplus n_2} = (b_{(n_1,n_2)} - \beta_{(n_1,n_2)})\hat{P}_{n_1\oplus n_2}.$$

Proof. It follows from (2.14), (2.15), and

$$\frac{d_{(n_1,n_2)} - \tilde{d}_{(n_1,n_2)}}{c_{(n_1,n_2)}} = \left|\frac{\Delta_{n_1\oplus n_2}\Delta_{n_1-1\oplus n_2-1}}{\Delta_{n_1-1\oplus n_2}\Delta_{n_1\oplus n_2-1}}\right|.$$

THEOREM 2.7. *Let* $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ *be a multiple OPS of type I relative to a pair of quasi-definite measures* $(d\mu, d\nu)$. *Then*

$$(2.16)\qquad A_{(n_1+1,n_2+1)} = \alpha^A_{(n_1,n_2)}A_{(n_1+1,n_2)} + \beta^A_{(n_1,n_2)}A_{(n_1,n_2+1)}$$

and

$$(2.17)\qquad B_{(n_1+1,n_2+1)} = \alpha^B_{(n_1,n_2)}B_{(n_1,n_2+1)} + \beta^B_{(n_1,n_2)}B_{(n_1+1,n_2)}$$

where

(2.18)

$$\alpha^A_{(n_1,n_2)} = -\left|\frac{\Delta_{n_1+1\oplus n_2+2}}{\Delta_{n_1+1\oplus n_2+1}}\right|;\ \beta^A_{(n_1,n_2)} = \frac{\int_{-\infty}^{\infty}\hat{P}_{n_1\oplus n_2+1}A_{(n_1+1,n_2+1)}d\mu}{\int_{-\infty}^{\infty}\hat{P}_{n_1\oplus n_2+1}A_{(n_1,n_2+1)}d\mu}$$

and
(2.19)

$$\alpha^B_{(n_1,n_2)} = -\Big|\frac{\Delta_{n_1+2\oplus n_2+1}}{\Delta_{n_1+1\oplus n_2+1}}\Big|;\ \beta^B_{(n_1,n_2)} = \frac{\int_{-\infty}^{\infty} \hat{P}_{n_1+1\oplus n_2} B_{(n_1+1,n_2+1)} d\nu}{\int_{-\infty}^{\infty} \hat{P}_{n_1+1\oplus n_2} B_{(n_1+1,n_2)} d\nu}.$$

Proof. We only prove the case $A_{(n_1,n_2)}$ since the other case can be proved by the same way. Let's write $A_{(n_1+1,n_2+1)} = a_{n+1}A_{(n_1+1,n_2)} + a_{n_1}A_{(n_1,n_2+1)} + \sum_{j=0}^{n_1-1} a_j A_{(j,n_2+1)}$. Multiplying by $P_{k\oplus n_2+2}$ and integrating both sides, we have by the orthogonality of type I and type II for $k = 0, 1, \cdots, n_1$,

$$\begin{aligned}&\int_{-\infty}^{\infty} P_{k\oplus n_2+2} A_{(n_1+1,n_2+1)} d\mu \\ &= \int_{-\infty}^{\infty} P_{k\oplus n_2+2}(A_{(n_1+1,n_2+1)} d\mu + B_{(n_1+1,n_2+1)} d\nu) \\ &= 0,\end{aligned}$$

and for $k = 0, 1, \cdots, n_1 - 1$,

$$\begin{aligned}\int_{-\infty}^{\infty} P_{k\oplus n_2+2}(a_{n_1+1}A_{(n_1+1,n_2)} + a_{n_1}A_{(n_1+1,n_2+1)} + \sum_{j=0}^{n_1-1} a_j A_{(j,n_2+1)}) d\mu \\ = a_{n_1+1}\int_{-\infty}^{\infty} P_{k\oplus n_2+2}(A_{(n_1+1,n_2)} d\mu + B_{(n_1+1,n_2)} d\nu) \\ + \sum_{j=0}^{n_1-1} a_j \int_{-\infty}^{\infty} P_{k\oplus n_2+2}(A_{(j,n_2+1)} d\mu + B_{(j,n_2+1)} d\nu) \\ = a_k \int_{-\infty}^{\infty} P_{k\oplus n_2+2} A_{(k,n_2+1)} d\mu.\end{aligned}$$

Hence, we have $a_k = 0$ for $k = 0, 1, \cdots, n_1 - 1$. The relations (2.18) ((2.19), respectively) can be obtained by multiplying by $P_{n_1+1\oplus n_2}$ and $P_{n_1\oplus n_2+1}$ ($P_{n_1\oplus n_2+1}$ and $P_{n_1+1\oplus n_2}$, respectively) and then integrating both sides. □

Since $A_{(n_1,n_2)} = (-1)^{n_2+1}|\Delta_{n_1\oplus n_2+1}|\hat{A}_{(n_1,n_2)}$, the equations (2.16) and (2.17) can be written as

$$\hat{A}_{(n_1+1,n_2+1)} = \hat{A}_{(n_1+1,n_2)} + \frac{|\Delta_{n_1\oplus n_2+2}|}{|\Delta_{n_1+1\oplus n_2+2}|}\beta^A_{(n_1,n_2)}\hat{A}_{(n_1,n_2+1)}$$

and

$$\hat{B}_{(n_1+1,n_2+1)} = \hat{B}_{(n_1,n_2+1)} + \frac{|\Delta_{n_1+2\oplus n_2}|}{|\Delta_{n_1+2\oplus n_2+1}|}\beta^B_{(n_1,n_2)}\hat{B}_{(n_1+1,n_2)}.$$

THEOREM 2.8. *Let $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ be a multiple OPS of type I relative to a pair of quasi-definite measures $(d\mu, d\nu)$ and*

$$xA_{(n_1,n_2)} = \sum_{j=0}^{n_1+1} a_j A_{(j,n_2+m_j)}, \quad a_j = a_j(n_1, n_2, m_j). \tag{2.20}$$

If there exists a sequence $\{s_j\}_{i=0}^{k}$ of integers such that for $i = 0, 1, \cdots, k$,

$$\begin{aligned} &s_i > m_j, \quad j = i, i+1, \cdots, n_1+1 \\ &s_i \le m_j + j - i, \quad j = i+1, i+2, \cdots, n_1+1 \\ &1 < s_i \le n_1 - i - 1, \end{aligned} \tag{2.21}$$

then $a_j = 0$ for $j = 0, 1, \cdots, k$.

Proof. Let $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ be a multiple OPS of type I relative to a pair of quasi-definite measures $(d\mu, d\nu)$. Since $i+s_i \le n_1-1$, for $i = 0, 1, \cdots, k$, we have by the orthogonality of type I and type II,

$$\begin{aligned} &\int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} x A_{(n_1,n_2)} d\mu \\ &= \int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} (xA_{(n_1,n_2)} d\mu + xB_{(n_1,n_2)} d\nu) = 0. \end{aligned} \tag{2.22}$$

On the other hand, by the relation (2.20) and the condition (2.21),

$$\begin{aligned} &\int_{-\infty}^{\infty} \sum_{j=0}^{n_1+1} a_j A_{(j,n_2+m_j)} \hat{P}_{i\oplus n_2+s_i} d\mu \\ &= \sum_{j=0}^{n_1+1} a_j \int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} A_{(j,n_2+m_j)} d\mu \\ &= \sum_{j=i}^{n_1+1} a_j \int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} A_{(j,n_2+m_j)} d\mu \\ &= \sum_{j=i}^{n_1+1} a_j \int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} (A_{(j,n_2+m_j)} d\mu + B_{(j,n_2+m_j)} d\nu) \\ &= a_i \int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} (A_{(i,n_2+m_i)} d\mu + B_{(i,n_2+m_i)} d\nu) \\ &= a_i \int_{-\infty}^{\infty} \hat{P}_{i\oplus n_2+s_i} A_{(i,n_2+m_i)} d\mu \\ &= (-1)^{n_1+n_2+m_i+s_i+1} \left| \frac{\Delta^A_{(i,n_2+m_i)}}{\Delta_{i\oplus n_2+s_i}} \right| |\Delta_{i+1\oplus n_2+s_i}| a_i. \end{aligned} \tag{2.23}$$

Hence, by (2.22) and (2.23), we have $a_i = 0$ for $i = 0, 1, \cdots, k$. □

By the same process, we can obtain the following.

THEOREM 2.9. *Let $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ be a multiple OPS of type I relative to a pair of quasi-definite measures $(d\mu, d\nu)$ and*

$$xB_{(n_1,n_2)} = \sum_{j=0}^{n_2+1} b_j B_{(n_1+m_j,j)}, \quad b_j = b_j(n_1, n_2, m_j).$$

If there exists a sequence $\{s_j\}_{i=0}^{k}$ of integers such that for $i = 0, 1, \cdots, k$,

$$\begin{aligned} &s_i > m_j, \quad j = i, i+1, \cdots, n_2+1 \\ &s_i \le m_j + j - i, \quad j = i+1, i+2, \cdots, n_1+1 \\ &1 < s_i \le n_2 - i - 1, \end{aligned} \tag{2.24}$$

then $b_j = 0$ for $j = 0, 1, \cdots, k$.

Proof. It can be proved by the same method as in the proof of Theorem 2.8. □

From the third condition of (2.21) in Theorem 2.8, we have $i < n_1 - 2$, i.e., $a_i = 0$ for $i = 0, 1, \cdots, n_1 - 3$, so that the recurrence relation is at least four term recurrence relation. Moreover if we take $m_j \le 0$ for some j, then

$$s_i \le m_j + j - i \le j - i, \quad j = i+1, i+2, \cdots, n_1+1$$

so that $s_i \le 1$, which is impossible by the third condition. By Theorem 2.8 and Theorem 2.9, we can find many recurrence relations for $A_{(n_1,n_2)}$ and $B_{(n_1,n_2)}$. For example, if we take $m_j = 2$, $k = n_1 - 4$, and $s_i = 3$, then $A_{(n_1,n_2)}$ satisfies a five term recurrence relation

$$xA_{(n_1,n_2)} = \sum_{j=n_1-3}^{n_1+1} a_j A_{(j,n_2+2)}.$$

Actually we are interested in a short term recurrence relation. In particular, if $m_j = 1$, we obtain a four term recurrence relation, which can be found for proper multi-indices ([9]).

COROLLARY 2.10. *Let $\{(A_{(n_1,n_2)}, B_{(n_1,n_2)})\}_{n_1,n_2=0}^{\infty}$ be a multiple OPS of type I relative to a pair of quasi-definite measures $(d\mu, d\nu)$. Then we have for $n_1, n_2 \ge 0$,*

$$\begin{aligned} xA_{(n_1,n_2)} = {} & a^A_{(n_1,n_2)} A_{(n_1+1,n_2+1)} + b^A_{(n_1,n_2)} A_{(n_1,n_2+1)} \\ & + c^A_{(n_1,n_2)} A_{(n_1-1,n_2+1)} + d^A_{(n_1,n_2)} A_{(n_1-2,n_2+1)}, \end{aligned} \tag{2.25}$$

and

$$xB_{(n_1,n_2)} = a^B_{(n_1,n_2)}B_{(n_1+1,n_2+1)} + b^B_{(n_1,n_2)}B_{(n_1+1,n_2)} + c^B_{(n_1,n_2)}B_{(n_1+1,n_2-1)} + d^B_{(n_1,n_2)}B_{(n_1+1,n_2-2)},$$

where $A_{(-1,n_2)} = A_{(-2,n_2)} = 0$, $B_{(n_1,-1)} = B_{(n_1,-2)} = 0$, $A_{(0,n_2)}$ *and* $B_{(n_1,0)}$ *are constants. In this case,*

$$a^A_{(n_1,n_2)} = -\left|\frac{\Delta_{n_1\oplus n_2+1}}{\Delta_{n_1+1\oplus n_2+2}}\right|;\ a^B_{(n_1,n_2)} = -\left|\frac{\Delta_{n_1+1\oplus n_2}}{\Delta_{n_1+2\oplus n_2+1}}\right|, \tag{2.26}$$

$$\begin{aligned} b^A_{(n_1,n_2)} &= \frac{\int_{-\infty}^{\infty} x\hat{P}_{n_1\oplus n_2+2}A_{(n_1,n_2)}d\mu}{|\Delta_{n_1+1\oplus n_2+2}|}; \\ b^B_{(n_1,n_2)} &= \frac{\int_{-\infty}^{\infty} x\hat{P}_{n_1+1\oplus n_2}B_{(n_1,n_2)}d\nu}{|\Delta_{n_1+2\oplus n_2+1}|}, \end{aligned} \tag{2.27}$$

$$\begin{aligned} c^A_{(n_1,n_2)} &= \frac{\int_{-\infty}^{\infty} x\hat{P}_{n_1-1\oplus n_2+2}A_{(n_1,n_2)}d\mu}{|\Delta_{n_1\oplus n_2+2}|}; \\ c^B_{(n_1,n_2)} &= \frac{\int_{-\infty}^{\infty} x\hat{P}_{n_1+2\oplus n_2-1}B_{(n_1,n_2)}d\nu}{|\Delta_{n_1+2\oplus n_2}|}, \end{aligned} \tag{2.28}$$

and

$$d^A_{(n_1,n_2)} = \frac{|\Delta_{n_1+1\oplus n_2+1}|}{|\Delta_{n_1-1\oplus n_2+2}|} \neq 0;\ d^B_{(n_1,n_2)} = \frac{\Delta_{n_1+1\oplus n_2+1}}{|\Delta_{n_1+2\oplus n_2-1}|} \neq 0. \tag{2.29}$$

Proof. We only prove the case $A_{(n_1,n_2)}$ since the other case can be proved by the same way. Take $s_i = 2$ and $m_j = 1$ for $i = 0, 1, \cdots, k$ and $j = 0, 1, \cdots, n_1 + 1$. Then the conditions of Theorem 2.8 are satisfied for $k = n_1 - 3$ so that $a_i = 0$ for $i = 0, 1, \cdots, n_1 - 3$. Hence, the recurrence relation (2.25) is obtained. Now, by multiplying by $\hat{P}_{n_1+1\oplus k}$ and then integrating, we have by the orthogonality of type II,

$$\int_{-\infty}^{\infty} x\hat{P}_{n_1+1\oplus k}A_{(n_1,n_2)}d\mu = a^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1+1\oplus k}A_{(n_1+1,n_2+1)}d\mu,$$

from which (2.26) follows. Multiplying (2.25) by $\hat{P}_{n_1 \oplus n_2+2}$ and then integrating gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} x\hat{P}_{n_1\oplus n_2+2} A_{(n_1,n_2)} d\mu \\
&= \int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2+2}(a^A_{(n_1,n_2)} A_{(n_1+1,n_2+1)} + b^A_{(n_1,n_2)} A_{(n_1,n_2+1)}) d\mu \\
&= a^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2+2}(A_{(n_1+1,n_2+1)} d\mu + B_{(n_1+1,n_2+1)} d\nu) \\
&\quad + b^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2+2} A_{(n_1,n_2+1)} d\mu \\
&= b^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2+2} A_{(n_1,n_2+1)} d\mu,
\end{aligned}$$

which is (2.27). Multiplying (2.25) by $\hat{P}_{n_1-1\oplus n_2+2}$ and then integrating gives the equation (2.28) for $c^A_{(n_1,n_2)}$. Lastly, multiplying by $\hat{P}_{n_1-2\oplus n_2+2}$ and then integrating both sides, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} x\hat{P}_{n_1-2\oplus n_2+2} A_{(n_1,n_2)} d\mu \\
&= a^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1-2\oplus n_2+2} A_{(n_1+1,n_2+1)} d\mu \\
&\quad + b^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1-2\oplus n_2+2} A_{(n_1,n_2+1)} d\mu \\
&\quad + c^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1-2\oplus n_2+2} A_{(n_1-1,n_2+1)} d\mu \\
&\quad + d^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1-2\oplus n_2+2} A_{(n_1-2,n_2+1)} d\mu \\
&= d^A_{(n_1,n_2)} \int_{-\infty}^{\infty} \hat{P}_{n_1-2\oplus n_2+2} A_{(n_1-2,n_2+1)} d\mu \\
&= d^A_{(n_1,n_2)} |\Delta_{n_1-1\oplus n_2+2}|.
\end{aligned} \tag{2.30}$$

On the other hand, if we let $h_n = \left|\frac{\Delta_{n_1\oplus n_2}\Delta_{n_1-1\oplus n_2-1}}{\Delta_{n_1-1\oplus n_2}\Delta_{n_1\oplus n_2-1}}\right|$, then

$$\begin{aligned}
&\int_{-\infty}^{\infty} x\hat{P}_{n_1-2\oplus n_2+2} A_{(n_1,n_2)} d\mu \\
&= \int_{-\infty}^{\infty} \hat{P}_{n_1-1\oplus n_2+2} A_{(n_1,n_2)} d\mu
\end{aligned} \tag{2.31}$$

$$= \int_{-\infty}^{\infty} (\hat{P}_{n_1\oplus n_2+1} + h_n\hat{P}_{n_1-1\oplus n_2+1})A_{(n_1,n_2)}d\mu$$
$$= \int_{-\infty}^{\infty} \hat{P}_{n_1\oplus n_2+1}A_{(n_1,n_2)}d\mu$$
$$= |\Delta_{n_1+1\oplus n_2+1}|$$

Hence, (2.29) is obtained by (2.30) and (2.31).

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