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QR-SUBMANIFOLDS OF MAXIMAL QR-DIMENSION IN QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. The purpose of this paper is to study n-dimensional QR-submanifolds of maximal QR-dimension isometrically immersed in a quaternionic projective space and to give sufficient conditions in order for such a submanifold to be a tube over a quaternionic invariant submanifold.

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ such that

$$F\nu_x \subset \nu_x, \ G\nu_x \subset \nu_x, \ H\nu_x \subset \nu_x,$$
$$F\nu_x^{\perp} \subset T_xM, \ G\nu_x^{\perp} \subset T_xM, \ H\nu_x^{\perp} \subset T_xM$$

at each point x in M, then M is called a *QR-submanifold of r QR-dimension*, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in $TM^{\perp}$ ([1, 10]). Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated by many authors ([2, 9, 10, 11, 12, 14]) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [7]).

In this paper we shall study QR-submanifolds of maximal QR-dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$ and prove the following theorem which is an extension of theorem proved in [12, Theorem 10] to the case of QR-submanifolds with maximal QR-dimension :

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THEOREM 1.1. *Let M be an n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the equalities appeared in (3.4) hold on M, then M is locally isometric to*

$$\pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)) \quad (r_1^2 + r_2^2 = 1)$$

for some integers n_1, n_2 with $4n_1 + 4n_2 = n - 3$, where π is the Hopf fibration $S^{n+4}(1) \to QP^{(n+1)/4}$.

Next, under the same assumptions as in Theorem 1.1, we bring into use an integral formula ([15]) which leads to an inequality among the Ricci curvature, the scalar curvature and the mean curvature of M. Using this inequality, we provide the following theorem as quaternionic analogue to theorem given in [4, Theorem 4.2]. Theorem 1.2 is also a generalization of Lawson's result ([11, Theorem 4]) for higher codimension, but avoiding the condition of minimality:

THEOREM 1.2. *Let M be an n-dimensional compact QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$, If the distinguished normal vector field ξ is parallel with respect to the normal connection and the inequality*

$$\frac{1}{3}\{Ric(U,U) + Ric(V,V) + Ric(W,W)\} + \rho - n^2\|\mu\|^2 \geq (n^2 + 8n - 1),$$

then M is isometric to

$$\pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}))$$

for some integers n_1, n_2 with $4n_1 + 4n_2 = n - 3$, where π is the Hopf fibration $S^{n+4} \to QP^{(n+1)/4}$.

2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting with tensor fields of type (1,1) over $\overline{M}$ satisfying the following conditions (a), (b), and (c) :

(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{cases} F^2 = -I, \ G^2 = -I, \ H^2 = -I, \\ FG = -GF = H, \ GH = -HG = F, \ HF = -FH = G. \end{cases} \tag{2.1}$$

(b) There is a Riemannian metric g which is hermite with respect to all of F, G and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to g

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix} \tag{2.2}$$

where p, q and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$ ([5, 6]).

For canonical local bases $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and $'\overline{\mathcal{U}}$, it follows that in $\overline{\mathcal{U}} \cap {}'\overline{\mathcal{U}}$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad (x,\ y = 1,\ 2,\ 3)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (2.1). As is well known ([5, 6]), every quaternionic Kähler manifold is orientable.

Now let M be an n-dimensional QR-submanifold of maximal QR-dimension, that is, of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition there is a unit normal vector field ξ such that $\nu_x^{\perp} = \text{Span}\{\xi\}$ at each point x in M. We set

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi. \tag{2.3}$$

Denoting by $\mathcal{D}_x$ the maximal quaternionic invariant subspace

$$T_xM \cap FT_xM \cap GT_xM \cap HT_xM$$

of T_xM, we have $\mathcal{D}_x^{\perp} \supset \text{Span}\{U, V, W\}$, where $\mathcal{D}_x^{\perp}$ means the complementary orthogonal subspace to $\mathcal{D}_x$ in T_xM. But, using (2.1), we can prove that $\mathcal{D}_x^{\perp} = \text{Span}\{U, V, W\}$ ([1, 10]). Thus we have

$$T_xM = \mathcal{D}_x \oplus \text{Span}\{U, V, W\}, \ {}^{\forall}x \in M,$$

which together with (2.1) and (2.3) implies

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus \text{Span}\{\xi\}.$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\}_{\alpha=1,\dots,p}$ ($\xi_1 := \xi$) of normal vectors to M, we have

$$FX = \phi X + u(X)\xi, \quad GX = \psi X + v(X)\xi, \\ HX = \theta X + w(X)\xi, \tag{2.4}$$

$$F\xi_\alpha = -U_\alpha + P_1\xi_\alpha, \quad G\xi_\alpha = -V_\alpha + P_2\xi_\alpha, \\ H\xi_\alpha = -W_\alpha + P_3\xi_\alpha \tag{2.5}$$

($\alpha = 1, \dots, p$). Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^\perp$, respectively. Moreover, the hermitian property of $\{F, G, H\}$ implies

$$\begin{aligned} g(X, \phi U_\alpha) &= -u(X)g(N_1, P_1\xi_\alpha), \\ g(X, \psi V_\alpha) &= -v(X)g(N_1, P_2\xi_\alpha), \\ g(X, \theta W_\alpha) &= -w(X)g(N_1, P_3\xi_\alpha), \quad \alpha = 1, \dots, p, \end{aligned} \tag{2.6}$$

$$\begin{aligned} g(U_\alpha, U_\beta) &= \delta_{\alpha\beta} - g(P_1\xi_\alpha, P_1\xi_\beta), \\ g(V_\alpha, V_\beta) &= \delta_{\alpha\beta} - g(P_2\xi_\alpha, P_2\xi_\beta), \\ g(W_\alpha, W_\beta) &= \delta_{\alpha\beta} - g(P_3\xi_\alpha, P_3\xi_\beta), \quad \alpha, \beta = 1, \dots, p. \end{aligned} \tag{2.7}$$

Also, from the hermitian properties

$$g(FX, \xi_\alpha) = -g(X, F\xi_\alpha), \quad g(GX, \xi_\alpha) = -g(X, G\xi_\alpha), \\ g(HX, \xi_\alpha) = -g(X, H\xi_\alpha),$$

it follows that

$$g(X, U_\alpha) = u(X)\delta_{1\alpha}, \quad g(X, V_\alpha) = v(X)\delta_{1\alpha}, \quad g(X, W_\alpha) = w(X)\delta_{1\alpha}$$

and hence

$$\begin{aligned} &g(U_1, X) = u(X), \ g(V_1, X) = v(X), \ g(W_1, X) = w(X), \\ &U_\alpha = 0, \ V_\alpha = 0, \ W_\alpha = 0, \ \alpha = 2, \dots, p. \end{aligned} \tag{2.8}$$

On the other hand, comparing (2.3) and (2.5) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, $W_1 = W$, which together with (2.3) and (2.8) implies

$$\begin{aligned} &g(U, X) = u(X), \quad g(V, X) = v(X), \quad g(W, X) = w(X), \\ &u(U) = 1, \quad v(V) = 1, \quad w(W) = 1. \end{aligned} \tag{2.9}$$

Here and in the sequel we use the notations U, V, W instead of U_1, V_1, W_1.

Next, applying F to the first equation of (2.4) and using (2.5), (2.8), and (2.9), we have

$$\phi^2 X = -X + u(X)U, \quad u(X)P_1\xi = -u(\phi X)\xi.$$

Similarly we have

$$\begin{gathered}\phi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \\ \theta^2 X = -X + w(X)W,\end{gathered} \tag{2.10}$$

$$\begin{gathered}u(X)P_1\xi = -\, u(\phi X)\xi, \quad v(X)P_2\xi = -v(\psi X)\xi, \\ w(X)P_3\xi = -w(\theta X)\xi,\end{gathered} \tag{2.11}$$

from which, taking account of the skew-symmetry of P_1, P_2, and P_3 and using (2.6) with $\alpha = 1$, we also have

$$\begin{aligned}&u(\phi X) = 0, \quad v(\psi X) = 0, \quad w(\theta X) = 0, \\ &\phi U = 0, \quad \psi V = 0, \quad \theta W = 0, \\ &P_1\xi = 0, \quad P_2\xi = 0, \quad P_3\xi = 0.\end{aligned} \tag{2.12}$$

So (2.5) can be rewritten of the form

$$\begin{aligned}&F\xi = -U, \quad G\xi = -V, \quad H\xi = -W, \\ &F\xi_\alpha = P_1\xi_\alpha = \sum_{\beta=2}^{p} P_{1\alpha\beta}\xi_\beta, \quad G\xi_\alpha = P_2\xi_\alpha = \sum_{\beta=2}^{p} P_{2\alpha\beta}\xi_\beta, \\ &HN_\alpha = P_3N_\alpha = \sum_{\beta=2}^{p} P_{3\alpha\beta}\xi_\beta \quad (\alpha = 2, \ldots, p).\end{aligned} \tag{2.13}$$

Applying G and H to the first equation of (2.4) and using (2.1), (2.4), and (2.13), we have

$$\begin{aligned}&\theta X + w(X)\xi = -\psi(\phi X) - v(\phi X)\xi + u(X)V, \\ &\psi X + v(X)\xi = \theta(\phi X) + w(\phi X)\xi - u(X)W,\end{aligned}$$

and consequently

$$\begin{aligned}&\psi(\phi X) = -\theta X + u(X)V, \quad v(\phi X) = -w(X), \\ &\theta(\phi X) = \psi X + u(X)W, \quad w(\phi X) = v(X).\end{aligned} \tag{2.14}$$

Similarly the other equations of (2.4) yield

$$\begin{aligned}\phi(\psi X) &= \theta X + v(X)U, \quad u(\psi X) = w(X),\\ \theta(\psi X) &= -\phi X + v(X)W, \quad w(\psi X) = -u(X),\end{aligned} \tag{2.15}$$

$$\begin{aligned}\phi(\theta X) &= -\psi X + w(X)U, \quad u(\theta X) = -v(X),\\ \psi(\theta X) &= \phi X + w(X)V, \quad v(\theta X) = u(X).\end{aligned} \tag{2.16}$$

From the first three equations of (2.13), we can also easily obtain

$$\begin{aligned}&\psi U = -W, \quad v(U) = 0, \quad \theta U = V, \quad w(U) = 0,\\ &\phi V = W, \quad u(V) = 0, \quad \theta V = -U, \quad w(V) = 0,\\ &\phi W = -V, \quad u(W) = 0, \quad \psi W = U, \quad v(W) = 0.\end{aligned} \tag{2.17}$$

The equations (2.8)–(2.10), (2.12), and (2.14)–(2.17) tell us that M admits the so-called almost contact 3-structure (for definition, see [7]) and consequently $n = 4m + 3$ for some integer m.

Now let ∇ be the Levi-Civita connection on M and let $\nabla^\perp$ the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.18}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha, \quad \alpha = 1, \ldots, p \tag{2.19}$$

for vector fields X, Y tangent to M. Here h denotes the second fundamental form and A_α the shape operator corresponding to ξ_α. They are related by

$$h(X, Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha$$

Furthermore, we put

$$\nabla^\perp_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta, \tag{2.20}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Differentiating the first equation of (2.4) covariantly and using (2.2), (2.4), (2.18), and (2.19), we have

$$\begin{aligned}(\nabla_Y \phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)A_1 Y - g(A_1 Y, X)U,\\ (\nabla_Y u)X &= r(Y)v(X) - q(Y)w(X) + g(\phi A_1 Y, X).\end{aligned} \tag{2.21}$$

From the other equations of (2.4) we also have

$$(2.22)\quad \begin{aligned}(\nabla_Y\psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)A_1Y - g(A_1Y,X)V,\\ (\nabla_Y v)X &= -r(Y)u(X) + p(Y)w(X) + g(\psi A_1Y, X),\end{aligned}$$

$$(2.23)\quad \begin{aligned}(\nabla_Y\theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)A_1Y - g(A_1Y,X)W,\\ (\nabla_Y w)X &= q(Y)u(X) - p(Y)v(X) + g(\theta A_1Y, X).\end{aligned}$$

Next, differentiating the first equation of (2.13) covariantly and using (2.2), (2.13), (2.18), and (2.19), we have

$$(2.24)\quad \begin{aligned}&\nabla_Y U = r(Y)V - q(Y)W + \phi A_1 Y,\\ &g(A_\alpha U, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{1\beta\alpha}, \quad \alpha = 2,\dots,p.\end{aligned}$$

From the other equations of (2.13), we have similarly

$$(2.25)\quad \begin{aligned}&\nabla_Y V = -r(Y)U + p(Y)W + \psi A_1 Y,\\ &g(A_\alpha V, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{2\beta\alpha}, \quad \alpha = 2,\dots,p,\end{aligned}$$

$$(2.26)\quad \begin{aligned}&\nabla_Y W = q(Y)U - p(Y)V + \theta A_1 Y,\\ &g(A_\alpha W, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{3\beta\alpha}, \quad \alpha = 2,\dots,p.\end{aligned}$$

Finally if the ambient manifold $\vee M$ is of constant Q-sectional curvature c, the equations of Gauss and Codazzi are given by

$$(2.27)\quad \begin{aligned}R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y\\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\\ &+ g(\psi Y,Z)\psi X - g(\psi X,Z)\psi Y - 2g(\psi X,Y)\psi Z\\ &+ g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y - 2g(\theta X,Y)\theta Z\}\\ &+ \sum_\alpha g(A_\alpha Y,Z)A_\alpha X - \sum_\alpha g(A_\alpha X,Z)A_\alpha Y,\end{aligned}$$

$$
\begin{aligned}
(2.28)\qquad & g((\nabla_X A_1)Y - (\nabla_Y A_1)X, Z) \\
&= \frac{c}{4}\{g(\phi Y, Z)u(X) - g(\phi X, Z)u(Y) - 2g(\phi X, Y)u(Z) \\
&+ g(\psi Y, Z)v(X) - g(\psi X, Z)v(Y) - 2g(\psi X, Y)v(Z) \\
&+ g(\theta Y, Z)w(X) - g(\theta X, Z)w(Y) - 2g(\theta X, Y)w(Z)\} \\
&+ \sum_\beta \{g(A_\beta X, Z)s_{\beta 1}(Y) - g(A_\beta Y, Z)s_{\beta 1}(X)\},
\end{aligned}
$$

respectively. Moreover, (2.9), (2.10), and (2.27) yield

(2.29)
$$
\begin{aligned}
&\mathrm{Ric}(X, Y) \\
&= \frac{c}{4}\{(n+8)g(X, Y) - 3(u(X)u(Y) + v(X)v(Y) + w(X)w(Y))\} \\
&+ \sum_\alpha \{(\mathrm{tr}A_\alpha)g(A_\alpha X, Y) - g(A_\alpha^2 X, Y)\},
\end{aligned}
$$

$$
(2.30)\qquad \rho = \frac{c}{4}(n+9)(n-1) + n^2\|\mu\|^2 - \sum_\alpha \mathrm{tr}A_\alpha{}^2,
$$

where Ric and ρ denote the Ricci tensor and the scalar curvature, respectively, and

$$
(2.31)\qquad \mu = \frac{1}{n}\sum_\alpha (trA_\alpha)\xi_\alpha
$$

is the mean curvature vector ([3]).

3. Some properties of the shape operator A_1

In this section, let M be an n-dimensional QR-submanifold of maximal QR dimension in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$ with constant Q-sectional curvature c. In what follows, *we assume that the distinguished normal vector field* $\xi_1 := \xi$ *is parallel with respect to the normal connection* $\nabla^\perp$, that is,

$$
\nabla^\perp_X \xi = 0
$$

for any vector field X tangent to M. Then it follows from (2.20) that $s_{1\beta} = 0$ and consequently (2.24)–(2.26) imply

$$A_\alpha U = 0, \quad A_\alpha V = 0, \quad A_\alpha W = 0, \quad \alpha = 2, \dots, p. \tag{3.1}$$

Moreover, since $s_{1\beta} = 0$, (2.20) yields

$$\nabla^{\perp}_X \xi_\alpha = \sum_{\beta=2}^{p} s_{\alpha\beta}(X)\xi_\beta, \quad \alpha = 2, \dots, p. \tag{3.2}$$

From now on we denote by A the shape operator A_1 corresponding to the normal vector $\xi = \xi_1$ and prepare a lemma for later use.

LEMMA 3.1. *Let M be an n-dimensional QR-submanifolds of maximal QR-dimension in a quaternionic space form $\overline{M}^{(n+p)/4}(c)$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and if the equalities*

$$\begin{gathered} h(X, \phi Y) = h(\phi X, Y), \; h(X, \psi Y) = h(\psi X, Y), \\ h(X, \theta Y) = h(\theta X, Y), \end{gathered} \tag{3.4}$$

hold for any vector fields X, Y tangent to M, then

$$A\phi = \phi A, \quad A\psi = \psi A, \quad A\theta = \theta A, \tag{3.5}$$

and $A_\alpha = 0$ for $\alpha = 2, \cdots p$.

Proof. Sine $n = 4m + 3$ and $p = 4t + 1$ for some integers m and t, and since the subspace ν is quaternionic invariant (see also (2.13)), we can take a local orthonormal basis $\{\xi, \xi_a, \xi_{a^*}, \xi_{a^{**}}, \xi_{a^{***}}\}_{a=1,\dots,t}$ of normal vectors M such that

$$\begin{gathered} \xi_{1^*} := F\xi_1, \; \dots \; , \xi_{t^*} := F\xi_t, \quad \xi_{1^{**}} := G\xi_1, \; \dots \; , \xi_{t^{**}} := G\xi_t, \\ \xi_{1^{***}} := H\xi_1, \; \dots \; , \xi_{t^{***}} := H\xi_t. \end{gathered}$$

Then we can express the second fundamental form h as

$$\begin{aligned} h(X, Y) = g(AX, Y)\xi + \sum_{a=1}^{t} \{ & g(A_a X, Y)\xi_a + g(A_{a^*} X, Y)\xi_{a^*} \\ & + g(A_{a^{**}} X, Y)\xi_{a^{**}} + g(A_{a^{***}} X, Y)\xi_{a^{***}} \}. \end{aligned}$$

Hence the assumption (3.4) implies

$$\begin{aligned}
&A\phi = \phi A, \quad A\psi = \psi A, \quad A\theta = \theta A,\\
&A_{a^*}\phi = \phi A_{a^*}, \quad A_{a^*}\psi = \psi A_{a^*}, \quad A_{a^*}\theta = \theta A_{a^*},\\
&A_{a^{**}}\phi = \phi A_{a^{**}}, \quad A_{a^{**}}\psi = \psi A_{a^{**}}, \quad A_{a^{**}}\theta = \theta A_{a^{**}}\\
&A_{a^{***}}\phi = \phi A_{a^{***}}, \quad A_{a^{***}}\psi = \psi A_{a^{***}}, \quad A_{a^{***}}\theta = \theta A_{a^{***}}.
\end{aligned} \tag{3.6}$$

On the other hand,

$$\xi_{a^*} = F\xi_a, \quad \xi_{a^{**}} = G\xi_a, \quad \xi_{a^{***}} = H\xi_a \quad (a = 1, \ldots, t)$$

give, respectively,

$$\begin{aligned}
&\overline{\nabla}_X F\xi_a = -A_{a^*}X + \nabla^{\perp}_X \xi_{a^*},\\
&\overline{\nabla}_X G\xi_a = -A_{a^{**}}X + \nabla^{\perp}_X \xi_{a^{**}},\\
&\overline{\nabla}_X H\xi_a = -A_{a^{***}}X + \nabla^{\perp}_X \xi_{a^{***}}, \qquad a = 1, \ldots, t.
\end{aligned} \tag{3.7}$$

Hence, using (2.2), (2.13), (2.19), and (3.2), it follows from the first equation of (3.7) that

$$\begin{aligned}
&-A_{a^*}X + \nabla^{\perp}_X \xi_{a^*}\\
&= r(X)G(\xi_a) - q(X)H(\xi_a)\\
&\quad + F(-A_aX + \sum_{b=1}^{t}\{s_{ab^*}(X)\xi_{b^*} + s_{ab^{**}}(X)\xi_{b^{**}} + s_{ab^{***}}(X)\xi_{b^{***}}\}),
\end{aligned}$$

from which, taking the tangential part, we can easily obtain

$$\phi A_a = A_{a^*}, \quad a = 1, \ldots, t. \tag{3.8}$$

Similarly, from the other equations of (3.7), we have

$$\psi A_a = A_{a^{**}}, \quad \theta A_a = A_{a^{***}}, \quad a = 1, \ldots, t. \tag{3.9}$$

Therefore, for any vectors X, Y tangent to M, it is clear from (3.8) that

$$g(A_{a^*}\phi X, Y) = -g(A_a\phi X, \phi Y)$$

and consequently

$$g(A_{a^*}\phi X, Y) = g(A_{a^*}\phi Y, X) = -g(\phi A_{a^*}X, Y),$$

that is,

$$A_{a^*}\phi = -\phi A_{a^*},$$

which and (3.6) imply $\phi A_{a^*} = 0$. Thus we have from (3.8) that $\phi^2 A_a = 0$, which together with (2.10) and (3.1) yields $A_a = 0$. Hence (3.9) yields

$$A_a = 0, \quad A_{a^*} = 0, \quad A_{a^{**}} = 0, \quad A_{a^{***}} = 0. \qquad \square$$

4. Codimension reduction and proof of Theorem 1.1

In this section we assume that the ambient manifold is a quaternionic projective space Q of constant Q-sectional curvature 4. Let

$$N_0(x) = \{\eta \in T_xM^\perp : A_\eta = 0\}$$

and let $H_0(x)$ be the maximal quaternionic invariant subspace of $N_0(x)$, that is,

$$H_0(x) = N_0(x) \cap FN_0(x) \cap GN_0(x) \cap HN_0(x).$$

Then Kwon and the first author of this paper [8] have proved the following theorem :

LEMMA 4.1. *Let M be an n-dimensional real submanifold of an $(n+p)$-dimensional quaternionic projective space $QP^{(n+p)/4}$. If the orthogonal complement $H_1(x)$ of $H_0(x)$ in $TM^\perp$ is invariant under the parallel translation with respect to the normal connection and q is the constant dimension of $H_1(x)$, then there exists a real $(n+q)$-dimensional totally geodesic quaternionic projective space $QP^{(n+p)/4}$ such that $M \subset QP^{(n+p)/4}$.*

In our cases, $N_0(x) = \mathrm{Span}\{\xi_2(x), \ldots, \xi_p(x)\}$. In fact, as a consequence of Lemma 3.1, $A_\alpha = 0$ for $\alpha = 2, \ldots, p$. Hence

$$\mathrm{Span}\{\xi_2(x), \ldots, \xi_p(x)\} \subset N_0(x).$$

On the other hand, for any η in $N_0(x)$, we can put $\eta = \sum_{\alpha=1}^{p} \lambda^\alpha \xi_\alpha$. But

$$A_\eta = \sum_{\alpha=1}^{p} \lambda^\alpha A_\alpha = \lambda^1 A_1 = 0$$

since $A_\alpha = 0$ for $\alpha = 2, \ldots, p$. Hence $\lambda^1 = 0$ and consequently

$$\eta \in \mathrm{Span}\{\xi_2(x), \ldots, \xi_p(x)\}.$$

Hence we have

$$N_0(x) = H_0(x) = \mathrm{Span}\{\xi_2(x), \ldots, \xi_p(x)\}.$$

Thus $H_1(x) = \mathrm{Span}\{\xi(x)\}$ and so our assumption yields that $H_1(x)$ is invariant under parallel translation with respect to the normal connection. Therefore we can apply Lemma 4.1 and obtain the following theorem.

THEOREM 4.2. *Let M be an n-dimensional QR-submanifold of maximal QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the equalities appeared in (3.4) hold on M, then there exists a real $(n+1)$-dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$.*

Proof of Theorem 1.1. From now on we shall give the proof of the theorem stated in Section 1. By means of Theorem 4.2 the submanifold M can be regarded as a real hypersurface of $QP^{(n+1)/4}$ which is totally geodesic in $QP^{(n+p)/4}$. Tentatively we denote $QP^{(n+1)/4}$ by M' and by i_1 the immersion of M into M' and by i_2 the totally geodesic immersion of M' into $QP^{(n+p)/4}$. Then it is clear from (2.18) that

$$\nabla'_{i_1X} i_1 Y = i_1 \nabla_X Y + h'(X,Y) = i_1 \nabla_X Y + g(A'X,Y)\xi', \tag{4.1}$$

where ∇' is the induced connection on M' from that of $QP^{(n+p)/4}$, h' the second fundamental form of M in M' and A' the corresponding shape operator to a unit normal vector field ξ' to M in M'. Since $i = i_2 \circ i_1$ and M' is totally geodesic in $QP^{(n+p)/4}$, we can easily see that (2.18) and (4.1) imply

$$\xi = i_2\xi', \quad A_1 = A'. \tag{4.2}$$

Since M' is a quaternionic invariant submanifold of $QP^{(n+p)/4}$, for any vector field X tangent to M

$$Fi_2X = i_2F'X, \ Gi_2X = i_2G'X, \ Hi_2X = i_2H'X \tag{4.3}$$

are valid, where $\{F', G', H'\}$ is the induced quaternionic Kähler structure on M'. Thus it follows from (4.2), (4.3), and the first equation of (2.4) that

$$FiX = Fi_2 \circ i_1 X = i_2F'i_1X = i_2(i_1\phi'X + u'(X)\xi') = i\phi'X + u'(X)\xi$$

for any vector field X tangent to M. Comparing this equation with the first equation of (2.4), we have $\phi = \phi'$ and $u = u'$. Similarly we have

$$\phi = \phi', \ \psi = \psi', \ \theta = \theta', \ u = u', \ v = v', \ w = w',$$

which and Lemma 3.1 imply

$$A'\phi' = \phi'A', \quad A'\psi' = \psi'A', \quad A'\theta' = \theta'A'.$$

Now applying the theorem proved in [12, Theorem 10, pp. 57] by the first author of this paper, we may conclude that M is locally isometric to

$$\pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)) \quad (r_1^2 + r_2^2 = 1)$$

for some n_1, n_2 with $4n_1 + 4n_2 = n - 3$, where π is the Hopf fibration $S^{n+4} \to QP^{(n+1)/4}$. □

5. An integral formula for the model space $\pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}))$

Let M be an n-dimensional QR-submanifold of maximal QR dimension in a quaternionic space form $\vee M^{(n+p)/4}(c)$ with constant Q-sectional curvature c. We put

$$T := \nabla_U U + \nabla_V V + \nabla_W W - (\mathrm{div} U)U - (\mathrm{div} V)V - (\mathrm{div} W)W$$

and take an orthonormal basis $\{U, V, W, e_a, e_{a^*}, e_{a^{**}}, e_{a^{***}}\}_{a=1,\dots,m}$ of tangent vectors to M such that

$$e_{a^*} := \phi e_a, \quad e_{a^{**}} := \psi e_a, \quad e_{a^{***}} = \theta e_a.$$

Then it follows from (2.8), (2.9), (2.12), and (2.24)–(2.26) that

$$T = \phi AU + \psi AV + \theta AW, \tag{5.1}$$

$$g(T,U) = g(T,V) = g(T,W) = 0. \tag{5.2}$$

Here and in the sequel we also denote by A the shape operator A_1 corresponding to the distinguished normal vector $\xi = \xi_1$. We note that T is a global vector field defined on M. For later use we compute

$$\mathrm{div} T = \sum_{i=1}^{n} g(e_i, \nabla_{e_i} T).$$

Differentiating (5.1) covariantly and using (2.21)–(2.26), we have

$$\begin{aligned}\nabla_X T = &\{u(AU) + v(AV) + w(AW)\}AX \\ &- g(A^2U,X)U - g(A^2V,X)V - g(A^2W,X)W \\ &+ \phi A\phi AX + \psi A\psi AX + \theta A\theta AX \\ &+ \phi(\nabla_X A)U + \psi(\nabla_X A)V + \theta(\nabla_X A)W,\end{aligned}$$

from which, taking account of (2.8)–(2.10), (2.12), and (2.14)–(2.17),

$$\begin{aligned}
&\operatorname{div}T\\
&= \{u(AU) + v(AV) + w(AW)\}\operatorname{tr}A - u(A^2U) - v(A^2V) - w(A^2W)\\
&\quad + \operatorname{tr}(\phi A\phi A) + \operatorname{tr}(\psi A\psi A) + \operatorname{tr}(\theta A\theta A) - g((\nabla_V A)W - (\nabla_W A)V, U)\\
&\quad - g((\nabla_W A)U - (\nabla_U A)W, V) - g((\nabla_U A)V - (\nabla_V A)U, W)\\
&\quad - \sum_{a=1}^{m}\{g((\nabla_{e_a}A)e_{a^*} - (\nabla_{e_{a^*}}A)e_a + (\nabla_{e_{a^{**}}}A)e_{a^{***}}\\
&\quad - (\nabla_{e_{a^{***}}}A)e_{a^{**}}, U) + g((\nabla_{e_a}A)e_{a^{**}} - (\nabla_{e_{a^{**}}}A)e_a + (\nabla_{e_{a^{***}}}A)e_{a^*}\\
&\quad - (\nabla_{e_{a^*}}A)e_{a^{***}}, V) + g((\nabla_{e_a}A)e_{a^{***}} - (\nabla_{e_{a^{***}}}A)e_i + (\nabla_{e_{a^*}}A)e_{a^{**}}\\
&\quad - (\nabla_{e_{a^{**}}}A)e_{a^*}, W)\},
\end{aligned}$$

or equivalently

$$\begin{aligned}
\text{(5.3)}\qquad \operatorname{div}T = {}& \{u(AU) + v(AV) + w(AW)\}\operatorname{tr}A\\
& - u(A^2U) - v(A^2V) - w(A^2W) + \frac{3(n-3)}{4}c\\
& + \operatorname{tr}(\phi A\phi A) + \operatorname{tr}(\psi A\psi A) + \operatorname{tr}(\theta A\theta A)
\end{aligned}$$

because of (2.28) with $s_{\beta 1} = 0$. On the other hand, using (2.8)–(2.10) and (2.12), we can easily obtain that

$$\begin{aligned}
&\|\phi A - A\phi\|^2 + \|\psi A - A\psi\|^2 + \|\theta A - A\theta\|^2\\
&= 6\operatorname{tr}A^2 - 2\{u(A^2U) + v(A^2V) + w(A^2W)\}\\
&\quad + 2\{\operatorname{tr}(\phi A\phi A) + \operatorname{tr}(\psi A\psi A) + \operatorname{tr}(\theta A\theta A)\},
\end{aligned}$$

which together with (5.3) yields

$$\begin{aligned}
\text{(5.4)}\qquad \operatorname{div}T = {}& \frac{1}{2}\{\|\phi A - A\phi\|^2 + \|\psi A - A\psi\|^2 + \|\theta A - A\theta\|^2\}\\
& + \frac{3(n-3)}{4}c - 3\operatorname{tr}A^2 + \operatorname{tr}A\{u(AU) + v(AV) + w(AW)\}.
\end{aligned}$$

On the other hand, it follows from (2.29)–(2.31) and (3.1) that

$$\begin{aligned}
\operatorname{Ric}(U,U) &= \frac{c}{4}(n+5) + (\operatorname{tr}A)u(AU) - u(A^2U),\\
\operatorname{Ric}(V,V) &= \frac{c}{4}(n+5) + (\operatorname{tr}A)v(AV) - v(A^2V),\\
\operatorname{Ric}(W,W) &= \frac{c}{4}(n+5) + (\operatorname{tr}A)w(AW) - w(A^2W),\\
\operatorname{tr}A^2 &= -\rho + \frac{c}{4}(n+9)(n-1) + n^2\|\mu\|^2 - \sum_{\alpha=2}^{p}\operatorname{tr}{A_\alpha}^2,
\end{aligned}$$

from which and (5.4), we have

$$(5.4)' \quad \begin{aligned} \mathrm{div}T = & \frac{1}{2}\{\|\phi A - A\phi\|^2 + \|\psi A - A\psi\|^2 + \|\theta A - A\theta\|^2\} \\ & + \mathrm{Ric}(U,U) + \mathrm{Ric}(V,V) + \mathrm{Ric}(W,W) \\ & + 3\{\rho - \frac{c}{4}(n^2 + 8n - 1) - n^2\|\mu\|^2\} + 3\sum_{\alpha=2}^{p} \mathrm{tr}A_\alpha^2 \\ & + \|AU\|^2 + \|AV\|^2 + \|AW\|^2. \end{aligned}$$

Thus we have

LEMMA 5.1. *Let M be an n-dimansional compact QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the inequality*

$$\frac{1}{3}\{Ric(U,U) + Ric(V,V) + Ric(W,W)\} + \rho - n^2\|\mu\|^2 \geq n^2 + 8n - 1,$$

holds on M, then

$$A\phi = \phi A \quad A\psi = \psi A, \quad A\theta = \theta A,$$

$$A_\alpha = 0, \quad \alpha = 2, \dots, p$$

and $AU = AV = AW = 0$.

Proof. Applying Stokes's theorem to (5.4)$'$, we can easily obtain the conclusions. □

Combining Lemma 4.1 and Lemma 5.1, we have

THEOREM 5.2. *Let M be an n-dimensional compact QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$. If the distinguished normal vector field ξ is parallel with respect to the normal connection and the inequality*

$$\frac{1}{3}\{Ric(U,U) + Ric(V,V) + Ric(W,W)\} + \rho - n^2\|\mu\|^2 \geq n^2 + 8n - 1,$$

holds on M, then there exists an $(n+1)$-dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$

Proof. By means of Lemma 5.1

$$A_\alpha = 0, \quad \alpha = 2, \dots, p.$$

As shown in the proof of Theorem 4.2, applying Lemma 4.1, we may conclude that there exists an $(n+1)$-dimensional totally geodesic quaternionic projective space $QP^{(n+1)/4}$ such that $M \subset QP^{(n+1)/4}$. □

Proof of Theorem 1.2. From now on we shall give the proof of Theorem 1.2 stated in Section 1. By means of Theorem 5.2 the submanifold M can be regarded as a real hypersurface of $QP^{(n+1)/4}$ which is totally geodesic in $QP^{(n+p)/4}$. By the same method as in the proof of Theorem 1.1, we can easily see that

$$A'\phi' = \phi' A', \quad A'\psi' = \psi' A', \quad A'\theta' = \theta' A',$$

where A' is the shape operator to a unit normal vector field ξ' to M in $QP^{(n+1)/4}$ and $\{\phi', \psi', \theta'\}$ denote the almost contact 3-structure induced on M from the quaternionic Kähler structure on $QP^{(n+1)/4}$. Now, applying the theorem proved in [12, Theorem 10, pp. 57] by the first author of this paper, we may conclude that M is isometric to

$$\pi(S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)) \quad (r_1^2 + r_2^2 = 1)$$

for some n_1, n_2 with $4n_1 + 4n_2 = n - 3$, that is, M is a tube over a totally geodesic submanifold QP^{n_1}. Moreover, $A'U = A'V = A'W = 0$ implies that the radius of the tube is $\frac{\pi}{4}$ (for details, see [2]). Thus M is isometric to

$$\pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2})) \quad (4n_1 + 4n_2 = n - 3). \qquad \square$$

REMARK. We consider special generalized Clifford tori in

$$S^{n+4} := \{(x_1, \dots, x_{n+5}) \in \mathcal{R}^{(n+5)} \mid \sum_{i=1}^{n+5} x_i^2 = 1\}$$

defined by

$$\begin{aligned} &S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}) \\ &= \{(x_1, \dots, x_{n+5}) \in S^{n+4} \mid \sum_{i=1}^{4n_1+4} x_i^2 = \frac{1}{2}, \sum_{i=4n_1+5}^{n+5} x_i^2 = \frac{1}{2}\}, \end{aligned}$$

where $4n_1 + 4n_2 = n - 3$ and $n = 4s + 3$ for some integer s. Then, since $S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2})$ is a real hypersurface of S^{n+4}, its shape operator $\tilde{A}$ is of the form

$$\tilde{A} = \mathrm{diag}(1, -1)$$

for suitable orthonormal basis. The multiplicities of 1 and -1 are $4n_1+3$ and $4n_2 + 3$, respectively ([13]). By choosing the spheres so that they lie in quaternionic subspace, we have fibrations

$$S^3 \to S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2}) \to M^Q_{n_1,n_2}$$

compatible with the Hopf fibration $\pi : S^{n+4} \to QP^{(n+1)/4}$, where we put

$$M^Q_{n_1,n_2} = \pi(S^{4n_1+3}(1/\sqrt{2}) \times S^{4n_2+3}(1/\sqrt{2})).$$

In this case we can easily see that the geodesic distance from QP^{n_1} to $M^Q_{n_1,n_2}$ is $\frac{\pi}{4}$ and that its principal curvatures are 1, -1, and 0 with multiplicities $n - 3 - 4n_1, 4n_1, 3$, respectively (for details, see [2]). Furthermore, let ξ be a unit normal vector field of $M^Q_{n_1,n_2}$ and let $\{F, G, H\}$ be the canonical quaternionic Kähler structure of $QP^{(n+1)/4}$. Then

$$U = -F\xi, \quad V = -G\xi, \quad W = -H\xi$$

are principal vectors corresponding to the principal curvature 0, that is

$$AU = 0, \quad AV = 0, \quad AW = 0,$$

where A denote the shape operator of $M^Q_{n_1,n_2}$ in $QP^{(n+1)/4}$. Applying (2.29)–(2.31) to the real hypersurface $M^Q_{n_1,n_2}$, we obtain

$$\begin{aligned}
&\mathrm{tr}A = n - 3 - 8n_1, \quad \mathrm{tr}A^2 = n - 3, \\
&\mathrm{Ric}(U, U) + \mathrm{Ric}(V, V) + \mathrm{Ric}(W, W) = 3(n + 5), \\
&\rho = n^2 + 7n - 6 + (n - 3 - 8n_1)^2.
\end{aligned}$$

Hence, for $M^Q_{n_1,n_2}$, we have

$$\frac{1}{3}\{\mathrm{Ric}(U, U) + \mathrm{Ric}(V, V) + \mathrm{Ric}(W, W)\} + \rho - n^2\|\mu\|^2 = n^2 + 8n - 1.$$

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