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THE TOPOLOGY OF S^2-FIBER BUNDLES

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Abstract. Let $P \xrightarrow{\pi} M$ be an oriented S^2-fiber bundle over a closed manifold M and let Q be its associated $SO(3)$-bundle, then we investigate the ring structure of the cohomology of the total space P by constructing the coupling form τ_A induced from an $SO(3)$ connection A. We show that the cohomology ring of total space splits into those of the base space and the fiber space if and only if the Pontrjagin class $p_1(Q) \in H^4(M;\mathbb{Z})$ vanishes. We apply this result to the twistor spaces of 4-manifolds.

1. Introduction

(1.1) In this article, we are going to investigate the cohomology ring structure of the total space of an S^2-fiber bundle P over a closed manifold M. Many of such examples can be constructed by the projectivization of rank 2 complex vector bundle E over M, i.e., $\pi : P(E) \to M$. In this case, the cohomology ring $H^*(P(E);\mathbb{R})$ is already known by the Leray-Hirsch theorem as a free $H^*(M;\mathbb{R})$-module generated by 1, $c_1(\xi)$ with a relation such as $c_1^2(\xi) - \pi^*(c_1(E)) \cdot c_1(\xi) + \pi^*(c_2(E)) = 0$ where ξ is the tautological line bundle over $P(E)$. This ring structure of the total space can be recovered by constructing a closed 2-form τ on P which is called a coupling form. By the result of this paper, we can identify the cohomology class of the coupling 2-form $[\tau] = -c_1(\xi) + \frac{1}{2}c_1(\pi^*(E))$ for the case $P \cong P(E)$. Then we have $[\tau^2] = \frac{1}{4}\pi^*((c_1^2(E) - 4c_2(E))) \in \pi^*(H^4(M;\mathbb{R})) \subset H^4(P(E);\mathbb{R})$ which completely determines the cohomology ring structure of the total space $P(E)$ of the S^2-fiber bundle. In turn, we can conclude that $c_1^2(E) = 4c_2(E)$ if and only if the cohomology ring $H^*(P(E))$ splits. This kind of characterization of the

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cohomology ring structure of S^2-fiber bundle P over M will be studied in terms of coupling form τ_A which is induced by a symplectic connection A which comes from an $SO(3)$ connection. Let us start with some basic preliminaries about S^2-fibration.

(1.2) Suppose $P \xrightarrow{\pi} M$ is an S^2-fiber bundle over M and the system of local coefficients on M induced by the fiber is simple. Then there exists an exact sequence, so called Gysin short exact sequence, such as

$$0 \to H^k(M;\mathbb{R}) \xrightarrow{\pi^*} H^k(P;\mathbb{R}) \xrightarrow{\pi_*} H^{k-2}(M;\mathbb{R}) \to 0,$$

where $\pi_* = PD \circ \pi_{\#} \circ PD^{-1}$, $\pi_{\#} = H_{n-k}(P) \to H_{n-k}(M)$ is the cohomology map induced by π and PD is the Poincare dual map [1]. Moreover π_* is called the map of integration along the fiber which it will be defined in the Section 3 via a given $SO(3)$-connection on P. Let $\tau \in H^2(P;\mathbb{R})$ be an element such that $\pi_*(\tau) = 1 \in H^0(M;\mathbb{R})$. It leads the splitting of the Gysin sequence by defining $s(\alpha) = \tau \cup \pi^*(\alpha) \in H^k(P;\mathbb{R})$ where $\alpha \in H^{k-2}(M;\mathbb{R})$. The splitting induced by the map s is followed by the projection formula [1], i.e., $\pi_*(\tau \cup \pi^*(\alpha)) = \pi_*(\tau) \cup \alpha = \alpha$. Then it completely determine the linear structure of the cohomology of the total space P as the tensor product of those of the base M and the fiber S^2. It says that

$$H^*(P;\mathbb{R}) \cong H^*(M;\mathbb{R}) \otimes H^*(S^2;\mathbb{R}),$$

where the isomorphism is induced by the splitting map s_τ as above. With a given cohomology class τ, the ring structure of $H^*(P;\mathbb{R})$ is determined by the square $\tau^2 \in H^4(P;\mathbb{R})$. Suppose we have $\tau^2 = \pi^*(\alpha) \cup \tau + \pi^*(\beta)$, by changing τ to $\tau - \frac{1}{2}\pi^*(\alpha)$, we may assume that the square of the cohomology class τ is the pull-back of some cohomology class β, i.e., $\tau^2 = \pi^*(\beta)$. We will show that the square of the τ is equal to the pull-back of $\frac{1}{4}p_1(Q) \in H^4(M;\mathbb{R})$ where Q is the $SO(3)$-bundle over M associated the P. In the next subsection, it will be discussed the way of getting the principal $SO(3)$-bundle Q from the S^2-fiber bundle P.

2. Reduction of structure group

(2.1) For a given oriented S^2-fiber bundle $\pi : P \to M$, the bundle P admits the structure of symplectic fibration since the $\mathrm{Diff}^+(S^2)/\mathrm{Symp}(S^2, \omega_{S^2})$ can be identified with the space of the symplectic forms on S^2, which is the contractible space of positive volume form. Hence the structure group $Diff^+(S^2)$ can be reduced to the group of symplectomorphisms, $\mathrm{Symp}(S^2, \omega_{S^2})$. This reduction always holds for the case when the fiber F is a compact Riemann surface [8]. Moreover since the group $Diff^+(S^2)$ defromation retract onto its linear part $SO(3)$ we can associate the principal $SO(3)$-bundle Q such that $P \cong Q \times_{SO(3)} S^2$, where $SO(3)$ acts S^2 as the symplectomorphism of the standard symplectic form ω_{S^2}. Note that linear subgroup $SO(3)$ is naturally isomorphic to the isometry group of the Kähler metric on $\mathbb{C}P^1 = S^2$. Suppose the dimension of the base space M is less than or equal to 4 then the principal $SO(3)$-bundles Q over M are completely classified by the pair of characteristic classes $(\omega_2(Q), p_1(Q))$ such that $p_1(Q) \equiv \omega_2(Q)^2(\mathrm{mod}2)$ where $\omega_2(Q) \in H^2(M;\mathbb{Z}/2)$ is the 2nd Stiefel-Whitney class and $p_1 \in H^4(M;\mathbb{Z})$ is the first Pontrjagin class. This classification result is due to the theorem of Dold and Whitney[3]. The diffeomorphism class of the principal $SO(3)$-bundles over M is unique up to homotopy class of maps from $M \rightarrow BSO(3)$ where $BSO(3)$ is the classifying space of $SO(3)$. We now discuss the extensions of the linear structure group $SO(3)$ to Spin(3) or Spin^c(3) structure. Recall that Spin(3) $\cong SU(2)$ and Spin^c(3) $\cong$ Spin(3) $\times_{\mathbb{Z}_2} U(1) \cong U(2)$. The following results can be found in [5].

(2.1.1) Spin(3) = $SU(2)$ case. The obstruction for the extension of the structure group from $SO(3)$ to $SU(2)$ is completely determined by the second Stiefel-Whitney class $\omega_2(Q) \in H^2(M;\mathbb{Z}_2)$. It implies that the vanishing of $\omega_2(Q) \in H^2(M;\mathbb{Z}_2)$ gives an equivalent condition of the extension to Spin(3). In this case $p_1(Q) = -4c_2(E) \in H^4(M;\mathbb{Z})$ where E is the complex $SU(2)$-bundle associated the extension.

(2.1.2) Spin^c(3) = $U(2)$ case. The obstruction is that there is a complex line bundle L whose first Chern class $c_1(L) \in H^2(M;\mathbb{Z})$ is the integral lift of $\omega_2(Q) \in H^2(M;\mathbb{Z}_2)$ i.e., $c_1(L) \equiv \omega_2(Q)$ mod2. And we have $p_1(Q) = c_1^2(E) - 4c_2(E) \in H^4(M;\mathbb{Z})$ where E is the complex vector bundle associated to the extension.

(2.2) Existence of section of $\pi : P \rightarrow M$. Since the action of $SO(3)$ on S^2 is transitive, we can view S^2 as a homogeneous space as $SO(3)/SO(2) = SO(3)/S^1$. Hence the existence of a section of $\pi : P \cong Q \times_{SO(3)} S^2 = Q \times_{SO(3)} SO(3)/S^1 \rightarrow M$ gives an existence condition such that there is an S^1 reduction of the principal $SO(3)$-bundle, i.e., $Q \cong \tilde{Q}_{S^1} \times_{S^1} SO(3)$. It also gives an equivalent condition such that there exist a line bundle L whose first Chern class $c_1(L)$ is an integral lift of $\omega_2(Q)$ and $c_1(L)^2 = p_1(Q)$.

(2.2.1) Note that the condition for the existence of the section of an S^2-fiber bundle is exactly the same as that of existence of an almost complex structure on oriented 4-manifold by the Wu's theorem. This is just because an almost complex structure on 4-manifold can be realized as a section of the twistor space $\tau(X)$ which is an S^2-fiber bundle over M[5].

(2.2.2) Note that even though we know all the characteristic classes $p_1(Q)$, $\omega_2(Q)$ associated to the $SO(3)$-bundle Q, it does not determine all the homotopy clases of maps from M to $BSO(3)$ for dim$M \geq 5$. However, the cohomology ring structure of the associated S^2-fiber bundle $P \cong Q \times_{SO(3)} S^2$ is completely determined by the characteristic classes of $SO(3)$-bundle Q, which will be discussed in the Section 3. Before getting into that, we need to discuss the Hamiltonian group action of $SO(3)$ on S^2 which induces an invariant positive definite pairing on the Lie algebra of $SO(3)$ in terms of the Hamiltonian functions.

3. Hamiltonian group action and semi-simple Lie algebra

(3.1) In this section, we discuss the Hamiltonian group action of a semi-simple Lie group on a symplectic manifold and the local isometry between its Lie algebra and the Hamiltonian functions induced by a moment map. Let us recall some basic facts from the Hamiltonian group action. Let G be a compact Lie group with its Lie algebra $\mathcal{G} =$ Lie(G) which acts covariantly on a symplectic manifold (X, ω) b symplectomorphisms. This implies that there is a group homomorphism G $\to$ Symp(X, ω) : $g \mapsto \psi_g$. The infinitesimal action determines a Lie algebra homomorphism $\mathcal{G} \to \chi(X, \omega) : \xi \mapsto X_\xi$ defined by

$$X_\xi = \frac{d}{dt}\bigg|_{t=0} \psi_{\exp(t\xi)}$$

for every $\xi \in \mathcal{G}$.

Since ψ_g is a symplectomorphism for every $g \in G$ it follows that each X_ξ is a symplectic vector field. This means that the 1-form $\iota(X_\xi)\omega$ is closed for every ξ. Suppose the 1-form $\iota(X_\xi)\omega$ is exact, $dH_\xi = \iota(X_\xi)\omega$, for every $\xi \in \mathcal{G}$, we call the action of G on X weakly Hamiltonian. Moreover the action is called Hamiltonian if the map

$$\mathcal{G} \to C^\infty(X) : \xi \mapsto H_\xi$$

can be chosen to be a Lie algebra homomorphism with respect to be the Lie algebra structure on $\mathcal{G}$ and Poisson structure on $C_\infty(X)$. Note that in general, a weakly Hamiltonian action need not be Hamiltonian. The obstruction takes the from of a Lie algebra cocycle in $H^2(\mathcal{G};\mathbb{R})$. For details, see chapter 5 in [8]. However suppose (X,ω) is a compact symplectic manifold then there is a way of normalizing the Hamiltonian function so that $\int_X H\omega^n = 0$. Since $\int_X H_{[\xi,\eta]}\omega^n = 0$ and $\int_X \{H_\xi, H_\eta\}\omega^n = \int_X dH_\xi \wedge dH_\eta \wedge \omega^{n-1} = 0$, we have $H_{[\xi,\eta]} = \{H_\xi, H_\eta\}$. Hence with this normalization, we can show that every weakly Hamiltonian action is Hamiltonian. Assume that the action of G on X is Hamiltonian and G is connected. Then it follows by straightforward calculation that

$$H_{g^{-1}\xi g} = H_\xi \circ \psi_g$$

for $g \in G$ and $\xi \in \mathcal{G}$.

(3.2) Consider a bilinear symplectic paring on the Lie Algebra $\mathcal{G}$ with a Hamiltonian action of G on a compact symplectic manifold (X,ω) :

$$\ll \xi,\eta \gg := \int H_\xi \cdot H_\eta \; \omega^n,$$

where $\omega^n = \omega \wedge \cdots \wedge \omega$. By the equation(1) and $\psi_g^*\omega = \omega$, we have

$$\begin{aligned} \ll (Adg)\xi, (Adg)\eta \gg &= \int_X H_{g^{-1}\xi g}(x) \cdot H_{g^{-1}\eta g}(x) \; \omega^n \\ &= \int_X H_\xi(\psi_g(x)) \cdot H_\eta(\psi_g(x)) \; \psi_g^*(\omega^n) \\ &= \int_X H_\xi \cdot H_\eta \; \omega^n \\ &= \ll \xi,\eta \gg . \end{aligned}$$

Thus we can prove the following proposition.

PROPOSITION 3.2.1. *Let G be a connected Lie group. Suppose the action of G on a compact symplectic manifold, (X,ω) is Hamiltonian. Then the paring*

$$\ll \xi,\eta \gg = \int_X H_\xi \cdot H_\eta \; \omega^n$$

defines an adjoint invariant semi-positive definite form on the Lie algebra $\mathcal{G}$.

Note that we have chosen the canonical orientation induced by the form $\omega^n \in \Omega^{2n}(X)$. To make a the form $\ll \cdot,\cdot \gg$ being positive definite, it only needs to have $H_\xi \neq 0$ for all $0 \neq \xi \in \mathcal{G}$. It leads to the following definition.

DEFINITION 3.2.2. The symplectic group action of G on (X,ω) is *effective* if the induced Lie algebra homomorphism $\mathcal{G} \to \chi(X,\omega)$ is injective.

The definition of effectiveness is equivalent to that it has only discrete stabilizers in G. For instance, consider the Hamiltonian action of $G = U(n)$ on $(\mathbb{C}P^{n-1}, \tau_0)$ induced by the obvious action on $\mathbb{C}^n$ where τ_0 the standard symplectic form on $\mathbb{C}P^{n-1}$. Then this action is not effective since the diagonal matrices $(e^{it}E)$ act trivially on $\mathbb{C}P^{n-1}$ where E is the identity matrix. However if the action is restricted to the subgroup $SU(n) \subset U(n)$ then it becomes effective. In that case, one can compare the positive definite form $\ll \cdot,\cdot \gg$ defined above and canonical inner product $<\xi,\eta>= \mathrm{trace}(\xi^*\eta)$, where ξ^* denotes the conjugate transpose of ξ. By the uniqueness of the invariant definite form on the semi-simple Lie algebra $su(n)$, one can compute the constant c such that $<\cdot,\cdot>= c \ll \cdot,\cdot \gg$. For the sake of this exposition, we are going to compute this constant for the effective Hamiltonian action of $SU(2)$ on $\mathbb{C}P^1 \cong S^2$.

(3.3) The effective Hamiltonian action of $SU(2)$ on $(\mathbb{C}P^1 = S^2, \omega_{S^2})$. Let us define the symplectic form ω_{S^2} on S^2 as follows. Let $pr : \mathbb{C}^2 - 0 \to \mathbb{C}P^1$ denote the obvious projection and define $pr^*\omega_{S^2} = \frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2$ where $\|z\|^2 = z_0\bar{z}_0 + z_1\bar{z}_1$. Then it is easily checked that ω_{S^2} is a well defined, $U(2)$ invariant symplectic 2-form and $\int_{S^2}\omega_{S^2} = 1$. Now let $\{\omega_1 = z_1/z_0\}$ be the coordinates on the open set $U_0 \equiv \{(z_0 \neq 0)\}$ in $\mathbb{C}P^1$ and use the lifting $z = (1,\omega_1)$ on U_0 ; we have

$$\omega_{S^2} = \frac{i}{2\pi}\frac{d\omega_1 \wedge d\bar{\omega}_1}{(1+|\omega_1|^2)^2}.$$

Let $\omega_1 = \omega$ be the complex coordinate on $U_0 = (z_0 \neq 0)$ and let

$$\xi = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in su(2).$$

Then in the polar coordinate system, $X_\xi = \frac{d}{dt}\big|_{t=0} e^{2it}\cdot\omega (= re^{i\theta}) = 2(\frac{d}{d\theta})_\omega$ where $\frac{d}{d\theta}$ is the angular tangent vector such that $d\theta(\frac{d}{d\theta}) = 1$ so we have $\omega = \frac{1}{\pi}\frac{r}{(1+r^2)^2}dr d\theta$ and $X_\xi \angle \omega = -\frac{1}{\pi}\frac{2r}{(1+r^2)^2}dr = d(-\frac{1}{\pi(1+r^2)})$. Thus we have $H_\xi = \frac{1}{\pi}(\frac{1}{1+r^2} - \frac{1}{2}) = -\frac{1}{2\pi}\frac{1-r^2}{1+r^2}$, here we take the normalization such that $\int_{S^2} H_\xi \quad \omega_{S^2} = 0$. Then by the direct integration we have

$$\int_{S^2} H_\xi^2\, \omega = \frac{1}{12\pi^2}.$$

Also $< \xi, \xi >= \text{trace}\xi^*\xi = -\text{trace}\xi^2 = 2$. Then we have $< \xi, \xi >= 24\pi^2 \ll \xi, \xi \gg$.By the invariance of the adjoint action of the both inner product, the constant is universal i.e.,$< \xi, \eta >= 24\pi^2 \ll \xi, \eta \gg_{su(2)}$ for all $\xi, \eta \in su(2)$. In particular, we have $\text{Tr}(\xi^2) = -24\pi^2 \ll \xi, \xi \gg$ where Tr is the trace map.

(3.4) Local Hamiltonian action of $SO(3)$ on $\mathbb{C}P^1$. As we already know, there is a local isometry between $su(2)$ and $so(3)$ which is induced from the double cover $SU(2) \to SO(3)$. Note that $SU(2)$ is naturally identified with $\text{Spin}(3) \cong \mathbf{S}^3$. By using this local isometry, any $\xi \in so(3)$ can be viewed as an element $\frac{1}{2}\xi \in su(2)$. Let $\xi \in so(3) = su(2)$ be a element of the Lie algebra of $SO(3)$. Let $\exp t\xi \in SO(3)$ be a local curve in $SO(3)$. Then $\exp t\frac{\xi}{2}$ becomes its local lifting to $SU(2)$. Since the group action of $SU(2)$ and $SO(3)$ on S^2 coincide for the lifting of $g \in SO(3)$ to $\tilde{g} \in SU(2)$, the symplectic vector field $X_{\tilde{\xi}}$ induced by $SO(3)$ action is the same as that of $\frac{1}{2}X_\xi$ by the $SU(2)$ action. Also the Hamiltonian function H_ξ from the $SO(3)$ action is the half of that from $SU(2)$. It follows that

$$\begin{aligned} < \xi, \eta >_{so(3)} &= \frac{1}{4} < \xi, \eta >_{su(2)} \\ &= \frac{1}{24\pi^2} \ll \frac{\xi}{2}, \frac{\eta}{2} \gg_{su(2)} \\ &= \frac{1}{24\pi^2} \ll \xi, \eta \gg_{so(3)} . \end{aligned}$$

Hence it can be summarized as the following lemma.

LEMMA 3.4.1. *Under the assumption of the Hamiltonian action of $SO(3)$ on S^2, we have*

$$\text{Tr}(\xi \cdot \eta) = -24\pi^2 \int_{S^2} H_\xi \cdot H_\eta \, \omega_{S^2},$$

where H_ξ is the normalized Hamiltonian function associated to $\xi, \eta \in su(2)$.

4. The ring structure of $H^*(P; \mathbb{R})$

(4.1) $SO(3)$ connection and coupling 2-form. As we discussed in Section 1, for every S^2 fiber bundle P over M there is an $SO(3)$ principal bundle Q over M such that $P \cong Q \times_{SO(3)} S^2$ by the reduction of structure group to the linear subgroup $SO(3)$ of $\text{Symp}(S^2, \omega)$. Note that we take the symplectic form ω to the canonical one defined as above. Then each fiber F_m of the symplectic fibration $\pi : P \to M$ carries a natural symplectic structure $\omega_m \in \Omega^2(F_m)$ defined by

$$\omega_m = \phi_\alpha(m)^*\omega$$

for some local trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \simeq U_\alpha \times S^2$ and $\phi_\alpha(m) = \phi_\alpha|_{F_m} : F_m \cong S^2$. Note that this form is independent of the choice of α. We call 2-form $\tau \in \Omega^2(P)$ is compatible with the symplectic fibration $\pi : P \to M$ if the restriction of τ to each fiber F_m is eaual to ω_m defined as above. Note that the symplectic fibration P is induced from the principal $SO(3)$-bundle Q, i.e., $P \cong Q \times_{SO(3)} S^2$ and $SO(3) \to \text{Symp}(S^2, \omega_{S^2})$ is a Hamiltonian action. We can apply the following theorem due to Weinstein [9].

THEOREM 4.1.1. *Let $G \to \text{Sypm}(F, \omega) : g \mapsto \psi_g$ be a Hamiltonian action. Then every connection A on a principal G -bundle $Q \to M$ gives rise to a closed 2-form τ_A on the associated fibration $Q \times_G F \to M$ which restricts to the forms ω_m on the fibers.*

Such a τ_A is called the coupling 2-form of the symplectic connection induced by the connection A. The above theorem is generalized by Guillemin-Lerman-Sternberg by constructing the coupling 2-form induced by the symplectic connection with a compact simply-connected fiber. This construction is extensively discussed in the book [4, 8]. Let us briefly explain how the construction goes. At each point $x \in P$ denote by $\text{Vert}_x = \ker d\pi(x) = T_x F_{\pi(x)}$ the vertical tangent space to the fiber. Let us define Γ to be the connection on the fibration $\pi : P \to M$, which defines a field of horizontal subspace $\text{Hor}_x \subset T_x P$ such that $TP_x = \text{Vert}_x \oplus \text{Hor}_x$. This leads to a splitting of the tangent bundle of P, i.e., $TP = \text{Vert} \oplus \text{Hor}$. Then every path $\gamma : [0,1] \to M$ determines a diffeomorphism $\Psi_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$. The diffeomorphism Ψ_γ is called the holonomy of the path γ. The connection Γ is called symplectic if the associated diffeomorphism Ψ_γ preserves the symplectic structure in the fiber, i.e.,$\Psi_\gamma^*\omega_{\gamma(1)} = \omega_{\gamma(0)}$ for every path γ. Let $\tilde{v} \in \text{Hor}_x$ be a horizontal vector then $\tau_\Delta = 0$ for all vertical vector $\omega \in \text{Vert}_x$. It now remains to define $\tau_\Gamma(\tilde{v_1}, \tilde{v_2})$. It is defined as follows, let v_1, v_2 be two vector fields on M then the vertical part of the communicator $[\widetilde{v_1}, \widetilde{v_2}]$ of the horizontal lifts $\tilde{v}_1, \tilde{v}_2$ respectively is a symplectic vector field on each fiber $F_{\pi(x)}$ and so, by the assumption of Hamiltonian action, is generated by a unique Hamiltonian function $H_(\tilde{v}_1, \tilde{v}_2)$of mean value zero, i.e., $\int_F H_(\tilde{v}_1, \tilde{v}_2)\omega = 0$. We therefore define

$$\tau_\Gamma(\tilde{v}_1, \tilde{v}_2) = H_{[\tilde{v}_1, \tilde{v}_2]^{\text{vert}}}(x).$$

The proof that τ_Γ is a well-defined closed 2-form reduces to some basic facts about connection, gauge transformation, and curvature on symplectic fibration [8].

Let A be a connection on $\pi : Q \to M$ then it induces a connection Γ_A on $\pi : P = Q \times_{SO(3)} S^2 \to M$. This connection becomes a symplectic connection because the holonomy is induced from the $SO(3) \subset \mathrm{Symp}(S^2, \omega_{S^2})$. Let τ_A be the 2-form on $P = Q \times_{SO(3)} S^2$ associated with an $SO(3)$-connection A on Q. In our case, we have

$$\tau_A(\tilde{v}_1(x), \tilde{v}_2(x)) = H_{A[\tilde{v}_1,\tilde{v}_2]}(x),$$

where $\xi(m) = [q, A[\tilde{v}_1, \tilde{v}_2](q)] \in \Gamma(M, \mathrm{ad}Q)$ and H_ξ the normalized Hamiltonian function on each fiber $F_m = S^2_{\pi(x)}$. Here we denote $\xi(m) = [q, \xi_q] = [q \cdot g, g^{-1}\xi g] \in \mathrm{ad}Q$ and note that $R_g^*((A[\tilde{v}_1, \tilde{v}_2])(q) = A[\tilde{v}_1 g, \tilde{v}_2 g] (q \cdot g) = g^{-1}A[\tilde{v}_1, \tilde{v}_2](q)g$.

It can be explained as follows. Let $x = [q, y] = [q \cdot g, g^{-1} \cdot y] \in P = Q \times_{SO(3)} S^2$ where we denote that $g^{-1} \cdot y = \psi_{g^{-1}}(y) \in S^2$ is the group action of $g \in SO(3)$ at $y \in S^2$. Let $\xi \in \Gamma(M, \mathrm{ad}Q)$ be a section of the adjoint vector bundle, $\mathrm{ad}Q$, associated by Q. Then it defines a vector field X_ξ on P which is vertical along the fiber as follows,

$$X_{\xi,x} = [q \cdot \xi, y] = \left[\frac{d}{dt}\bigg|_{t=0} q \cdot \exp\, t\xi, y\right] = \left[q, \frac{d}{dt}\bigg|_{t=0} \exp\, t\xi \cdot y\right].$$

It follows from the equivariance of ξ, i.e., $\xi_{q\cdot g} = g^{-1}\xi_q g$, this vector field is well defined and independent of the representative $x = [q, y] = [q \cdot g, g^{-1} \cdot y]$. By definition, it defines a symplectic vector field ξ which induces the unique Hamiltonian funtions H_ξ on each fiber $F_m \cong S^2$ of mean value zero. For any pair of vector fields v_1, v_2 on M, the vertical part of the commutator of the horizontal lifts $[\overline{v}_1, \overline{v}_2]$ is exactly defined by $\xi_{v_1,v_2} = A[\widetilde{v}_1, \widetilde{v}_2] \in \Gamma(M, \mathrm{ad}Q)$. Hence it defines the Hamiltonian function $H_{A[\widetilde{v}_1,\widetilde{v}_2]}(x) : P \to \mathbb{R}$. Moreover note that $A([\widetilde{v}_1, \widetilde{v}_2]) = F_A(\widetilde{v}_1, \widetilde{v}_2)$ where $F_A \in \Omega^2(M, \mathrm{ad}Q)$ is the curvature tensor induced by A.

(4.2) The ring structure of $H^*(P; \mathbb{R})$ and the Pontrjagin class of Q. From Section 1.2, we have the Gysin sequence of an S^2-fiber bundle P over M as follows,

$$0 \to H^k(M; \mathbb{R}) \xrightarrow{\pi^*} H^k(P; \mathbb{R}) \xrightarrow{\pi_*} H^{k-2}(M; \mathbb{R}) \to 0,$$

where π_* is the integration along the fiber.

We want to define this map $\pi_* : \Omega^k(P, \mathbb{R}) \to \Omega^{k-2}(M, \mathbb{R})$ as follows

$$\pi_*(\alpha)(v_1, v_2, \cdots, v_{k-2})(m) = \int_{F_m} \alpha(\widetilde{v}_1, \cdots, \widetilde{v}_{k-2})(x),$$

where $\widetilde{v}_i$ is the horizontal lift of v_i with respect to the connection A. Note that this map does not make any difference if $\widetilde{v}_i$ has been chosen to be any lifting of v_i.

Then we have $\pi_*(\tau_A) = 1$ and by the projection formula we have $\pi_*(\pi^*(\beta) \wedge \tau_A) = \beta$ for all $\beta \in \Omega^k(M, \mathbb{R})$ and the commutativity of π_* and d follows from the direct local computation, i.e., we have $\pi_* d\alpha = d\pi_*\alpha$ [1]. We need to verify the following identities to show the main result Theorem 4.2.2 below of this paper.

LEMMA 4.2.1. $[\tau_A^2] = \frac{1}{4}\pi^* p_1(Q)$ *where* $p_1(Q)$ *is the Pontrjagin class of* Q.

Proof. We are going to establish the following identities to prove Lemma 4.2.1.

(1) $\tau_A^2 = \pi^*(\beta) + da$.

From the Gysin sequence, we have

$$\tau_A^2 = \tau_A \wedge \tau_A = \pi^*(\beta) + \pi^*(\alpha) \wedge \tau_A + d\gamma$$

and we have $\pi_*(\tau_A^2) = \alpha + d(\pi_*(\gamma))$. By the normalization condition, $\pi_*(\tau_A^2)(\widetilde{v}_1, \widetilde{v}_2) = 2\int_F \tau_A(v_1, v_2)\tau_A = 0$, it implies that α is an exact form on M. This also shows that $\tau_A^2 = \pi^*(\beta) + da$ where $a = \pi^*(\pi_*(\gamma)) \wedge \tau_A + \gamma$.

(2) $\pi_*(\tau_A^3)(v_1, \cdots, v_4) = 3\int_F \tau_A^2(\widetilde{v}_1, \cdots, \widetilde{v}_4)\tau_A$.

For sake of brevity, we denote

$$\begin{aligned} \tau_A(\widetilde{v}_i, \widetilde{v}_j) &= H_{ij}, \ \tau_A(\widetilde{v}_i, \cdot) = \tau_i, \\ \tau_A^2(\widetilde{v}_1, \cdots, \widetilde{v}_4) &= 2(H_{12}H_{34} - H_{13}H_{24} + H_{14}H_{23}), \\ \tau_A^2(\widetilde{v}_i, \widetilde{v}_j, \cdot, \cdot) &= 2(H_{ij}\tau_A = \tau_i \wedge \tau_j), \end{aligned}$$

$$\begin{aligned} \iota(\widetilde{v}_4) \cdots \iota(\widetilde{v}_1)(\tau_A^3) &= \iota(\widetilde{v}_4) \cdots \iota(\widetilde{v}_1)\tau_A^2 \tau_A \\ &\quad + \sum_{i>j>k,l} (-1)^{l-1} \iota(\widetilde{v}_i)\iota(\widetilde{v}_j)\iota(\widetilde{v}_k)\tau_A^2 \wedge \iota(\widetilde{v}_l)\tau_A \\ &\quad + \sum_{\sigma(1)<\sigma(2), \sigma(3)<\sigma(4)} \operatorname{sign}(\sigma)\iota(\widetilde{v}_{\sigma(2)})\iota(\widetilde{v}_{\sigma(1)})\tau_A^2 \iota(\widetilde{v}_{\sigma(4)})\iota(\widetilde{v}_{\sigma(3)})\tau_A \\ &= \tau_A(\widetilde{v}_1, \cdots, \widetilde{v}_4)\tau_A + 4(H_{12}H_{34} - H_{13}H_{24} + H_{14}H_{23})\tau_A \\ &\quad + \sum a_{ij}\tau_i \wedge \tau_j \\ &= 3\tau_A^2(\widetilde{v}_1, \cdots, \widetilde{v}_4)\tau_A + \sum a_{ij}\tau_i \wedge \tau_j. \end{aligned}$$

Since each term $\tau_i \wedge \tau_j$ in the last sum vanishes by the integration along the fiber, it completes the equation.

(3) $[\beta] = \frac{1}{4}p_1(Q)$.

From above, we have

$$\tau_A^3 = \tau_A^2 \wedge \tau_A = \pi^*(\beta) \wedge \tau_A + d(a \wedge \tau_A),$$
$$\beta(v_1, \cdots, v_4) = \pi_*(\tau_A^3) - d(\pi_*(a \wedge \tau_A)),$$

$$\begin{aligned}\pi_*(\tau_A^3)(v_1, \cdots, v_4) &= \int_F \tau_A^2(\widetilde{v}_1, \cdots, \widetilde{v}_4)\tau_A \\ &= 3\int_F 2(H_{12}H_{34} - H_{13}H_{24} + H_{14}H_{23})\tau_A \\ &= 3\int_{S^2} 2(H_{F_{12}}H_{F_{34}} - H_{F_{13}}H_{F_{24}} + H_{F_{14}}H_{F_{23}})\omega_{S^2} \\ &= \frac{1}{4\pi^2}(< F_{12}, F_{34} > - < F_{13}, F_{24} > + < F_{14}, F_{23} >) \\ &= -\frac{1}{4\pi^2}\mathrm{Tr}(F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23}) \\ &= -\frac{1}{8\pi^2}\mathrm{Tr}((F_A^2)(v_1, \cdots, v_4)),\end{aligned}$$

where $F_{i,j} = F_A(v_1, v_2) \in so(3) \cong su(2)$, and $H_{F_{ij}}$ is the normalized Hamiltonian function induced by $F_{ij} \in so(3)$. Hence we have $[\pi^*(\beta)] = [\tau_A^2] = \frac{1}{4}\pi^* p_1(Q)$. □

THEOREM 4.2.2. *Let P be the S^2-fiber bundle over M then there is a closed two form $\tau \in \Omega^2(P, \mathbb{R})$ such that its cohomology class defines the linear isomorphism $H^*(P; \mathbb{R}) \cong H^*(M) \otimes H^*(S^2)$ and it also determine the ring structure $H^*(P; \mathbb{R})$ such that $[\tau]^2 = \frac{1}{4}\pi^* p_1 \in \pi^*(H^4(M; \mathbb{R})) \subset H^4(P; \mathbb{R}))$, where $p_1 = p_1(Q)$ is the first Pontrjagin class of the associated principal $SO(3)$-bundle Q.*

COROLLARY 4.2.3. *$H^*(P; \mathbb{R}) \cong H^*(M; \mathbb{R}) \otimes H^*(S^2; \mathbb{R})$ splits as a ring iff the Pontrjagin class of the associated $SO(3)$-bundle Q vanishes i.e., $p_1(Q) = 0 \in H^4(M; \mathbb{R})$. For the case $P = P(E)$ is a projectivization of rank 2 vector bundle, $p_1(Q) = p_1(adE) = c_1(E)^2 - 4c_2(E) \in H^4(M; \mathbb{R})$.*

Proof. Suppose the ring $H^*(P; \mathbb{R})$ splits as a ring $H^*(M; \mathbb{R}) \otimes H^*(S^2; \mathbb{R})$ then there is an element $\tau \in H^2(P; \mathbb{R})$ such that $\pi_*(\tau) = 1$ and $[\tau^2] = 0$. Comparing the coupling 2-form τ_A with τ, we know that $[\tau_A] - [\tau] = [\pi^* a]$ by the Gysin sequence. Therefore we have $[\tau_A]^2 = [\tau]^2 + 2[\tau][\pi^* a] + [\pi^* a]^2$ and $0 = \pi_*[\tau_A^2] = 2[a] \in H^2(M; \mathbb{R})$, i.e., $p_1(Q) = 4[\tau_A^2] = 0 \in H^4(M; \mathbb{R}) \subset H^4(P; \mathbb{R})$. This completes the proof. □

Note that the splitting condition of $P(E)$, the projectivization of rank 2 vector bundle E, is achieved if E is projectively flat which implies that $\mathrm{End}(E)$ is flat.

LEMMA 4.2.3. *We can prove that the cohomology class of the coupling two-form* $[\tau_A]$ *does not depend on any symplectic connection* Γ *on* P *by the same argument given in the proof of the Corollary 4.2.2, i.e.,* $[\tau_A] = [\tau_\Gamma] \in H^2(P;\mathbb{R})$. *See* [4].

In the S^2-fiber bundle case, the cohomology class of the coupling form is characterized uniquely as an element $\tau \in H^2(P;\mathbb{R})$ such that $\pi_*\tau = 1$ and $\tau^2 \in \pi^*(H^4(M;\mathbb{R}))$.

5. Twistor space of 4-manifold

(5.1) In this section, we are going to study the twistor space $\tau(M)$ of oriented 4-manifold M which is an S^2-fiber bundle over M. The twistor space $\tau(M)$ is naturally induced by the projectivization of positive pure spinors on M which is isomorphic to the space of orthogonal almost complex strucrures on M. Suppose the dimension of M is 4, nonzero positive spinor defines a pure spinor in turn, the twistor space $\tau(M)$ is canonically isomorphic to the projectivization of positive spinor bundle, i.e., $\tau(M) \cong P(S_C^+)$ where S_C^+ is the positive spinor bundle which is rank 2 complex vector bundle. Thus the twistor space $\tau(M)$ is an S^2-fiber bundle over M canonically associated to the Riemannian manifold M. Topological characterization of the existence of the positive spinor bundle is whether there exists an integral lifts of the second Stiefel-Whitney class of M, $\omega_2(M)$ which indicates the Spin^c structure of given manifold M. It is well known that there exists as Spin^c-structure on any oriented smooth 4-manifold. For more discussion on the twistor space and Spin^c structures, see [5]. Note that the twistor space is well-defined independent of Spin^c structure, S_C^+. As we discussed before, there is an associated principal $SO(3)$-bundle $Q_{\tau(M)}$ which isomorphic to the adjoint bundle of the unitary bundle S_C^+. We will show that $p_1(Q) = 3\sigma(M) + 2\chi(M)$ where $\chi(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature of M.

LEMMA 5.1.1. *Let* S^+ *be a positive spinor bundle of almost complex 4-manifold* M *then we have*

$$c_2(S^+) = \frac{1}{4}(c_1(S^+)^2 - 3\sigma - 2\chi),$$

where σ is the signature of M and χ is the Euler characteristic of M.

Before we prove the lemma, it might be good place to recall the basic facts about the spinor bundle over 4-manifold. This material can be found in [5] and [6]. Let $\omega_2(M) \in H^2(M;\mathbb{Z}/2)$ be the second Stiefel-Whitney class and then the Spin^c structures are naturally isomorphic to the cohomology class of the integral lift of $\omega_2(M)$ which is naturally isomorphic to the set of characteristic line bundles $\{L \mid \text{complex line bundle } c_1(L) \equiv \omega_2(M)\}$. It is a principal $H^2(M;\mathbb{Z})$ space since the difference between two characteristic line bundles contained in $2H^2(M;\mathbb{Z})$. We will abuse the notation for complex line bundle L as $l = c_1(L) \in H^2(M;\mathbb{Z})$, vice versa. This sums up to as follows:

$$\{\text{Spin}^c \text{ structures over } M\} \cong \{L_0 + 2l \mid l \in H^2(M;\mathbb{Z})\}$$

where L_0 is a characteristic line bundle. Suppose S^+ be the positive spinor bundle then the determinantal line bundle $\det S^+ = L$ defines the Spin^c structure. We denote $S^+ = S^+(L)$ for $L = \det S^+$. For any other spinor bundle, it can be written as tensor product of some line bundle l, i.e., $S^+(L) = S_0^+ \otimes l$ where $l = (L \otimes L_0)^{\frac{1}{2}}$. It induces $c_2(S^+(L)) = c_2(S_0^+) + c_1(S_0^+) \cdot c_1(l) + c_1^2(l)$ and $c_1(S^+(L))^2 = c_1^2(S_0^+) + 4c_1(S_0^+) \cdot c_1(l) + 4c_1^2(l)$ which proves the lemma. It suffices to show that there exists a positive spinor bundle S_0^+ such that $c_2(S_0^+) = \frac{1}{4}(c_1^2(S_0^+)) - 3\sigma - 2\chi)$.

Proof of Lemma 5.1.1. Suppose M has an almost complex structure then the $J : TM \to TM$ defines a canonical Spin^c structure and the induced positive spinor bundle is isomorphic to $S_J^+ \cong II \otimes K_J^{-1}$ with $K_J^{-1} = \det TM_J$ for TM_J being the complex tangent bundle induced by J and II being trivial line bundle ([6]). We have $c_2(S_J^+) = 0$ and $c_1^2(K_J^{-1}) = 2\chi + 3\sigma$ by the Hirzebruch signature theorem.

Note that the canonical negative spinor bundle S_J^- induced by an almost complex structure J is canonically isomorphic ro complex tangent bundle TM_J. Since $c_2(S_J^-) = c_2(M) = \chi(M)$ and $c_1^2(S_J^-) = c_1^2(M) = 2\chi(M) + 3\sigma(M)$, we have $c_2(S_C^-) = \frac{1}{4}(c_1^2(S_C^-) - 3\sigma(M) - 2\chi(M))$. Let $Q_{S_C^-}$ be the principal $SO(3)$-bundle associated to the the negative spinor bundle S_C^-. Then we have $p_1(Q_{S_C^-}) = 3\sigma(M) - 2\chi(M)$. □

COROLLARY 5.1.2. *The rational cohomology ring of $P(S_C^\pm)$, the projectivization of the positive* (resp. negative) *spinor bundle, splits if and only if $3\sigma(M) = \mp 2\chi(M)$ respectively.*

Example 5.1.3. We know that the complex projective space $\mathbb{C}P^3$ becomes the twistor space over S^4. Identity $\mathbb{C}^4$ with quaternionic plane

$\mathbb{H}^2$ then the obvious fibration $\pi\mathbb{C}P^3 \to \mathbb{H}P^1$ induces the S^2-fiber bundle structure. Canonically we can identify $\mathbb{H}P^1 \cong S^4$ and $\mathbb{C}P^3 = \tau(S^4)$. For details, see [5], [8]. In this case, the cohomology class of the form induces by the Fubini-Study metric, $\omega \in H^2(\mathbb{C}P^3;\mathbb{R})$ defines the same class of the coupling 2-form since $\int_{S^2}\omega = \deg_\omega S^2 = 1$ where S^2 is the fiber class.

Note that the above example explains that the square of the cohomology class of coupling form is the generator of $\mathbb{H}^4(\mathbb{C}P^3;\mathbb{Z})$ which equals to π^*(positive generator)$= \frac{1}{4}\pi^* p_1(Q) = \frac{1}{4}(3\sigma(S^4) + 2\chi(S^4)) = 1 \in \mathbb{Z} \cong \mathbb{H}^4(M;\mathbb{Z})$.

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