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The fuzzy linear maps

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Abstract

We investigate some situations in connection with two exact sequences of fuzzy linear maps.

Key words : Fuzzy R-module, Fuzzy linear maps, Quasi-monic maps.

1. Introduction

Fuzzy modules were introduced by Negoita and Ralescu[5]. Katsaras and Liu[2], and Lowen[4] have developed the theory of fuzzy vector spaces.

Fu-Zhen Pan[7] investigated fuzzy vector spaces for the following purposes; to establish a fundamental frame of fuzzy vector space by virtue of homological algebra and modular theory, and to stretch it out to study general fuzzy modules. In fact, fuzzy vector spaces are the simplest kind of fuzzy free modules. The theory of fuzzy modules has been a virgin field for a long time. Recently, many authors presented the same research on fuzzy modules, properties of fuzzy finitely generated modules and fuzzy quotient modules, etc. In particular, Kim [3] investigated the properties of the sequence of fuzzy linear maps and studied the situations in connection with two exact sequences of fuzzy linear maps.

In this paper, we investigate some situations in connection with two exact sequences of fuzzy linear maps.

2. Preliminaries

In this section, we review some definitions and some results which will be used in the later sections. Throughout this paper, we assume that all modules are equipped with the same underlying commutative ring R

Definition 2-1 [8] A R-module M together with a function χ from M into [0, 1] is called a *fuzzy R-module* if it satisfies the following conditions ;

(1) for any $a, b \in M$, $\chi(a+b) \geq \min\{\chi(a), \chi(b)\}$
(2) for any $a \in M$, $\chi(-a) = \chi(a)$
(3) for the identity of M, $\chi(0) = 1$
(4) for any $a \in M$ and $r \in R$, $\chi(ra) \geq \chi(a)$

denoted it by (M, χ) or χ_M

Definition 2-2 [8] Let χ_M, η_N be any two fuzzy R-modules, then $\mathcal{F} : \chi_M \to \eta_N$ is called a *fuzzy linear map* (or *fuzzy R-map*) if there exists a linear map $f: M \to N$ such that $\eta(f(a)) \geq \chi(a)$ for all $a \in M$.

Remark. Let χ_M, η_N be any two fuzzy R-modules. Then $\mathcal{F} : \chi_M \to \eta_N$ is called a *fuzzy strong linear map* if there exists a linear map $f: M \to N$ such that $\eta f(a) = \chi(a)$ for all $a \in M$.

Definition 2-3 [8] For a fuzzy linear map , $\mathcal{F} : \chi_M \to \eta_N$ η_{Imf} is called the *image of* $\mathcal{F}$ denoted it by $\eta_{Im\mathcal{F}}$. Furthermore, χ_{M_0}, where $M_0 = \{m \in M \mid \eta(f(m)) = 1\}$ is called the *Kernel of* $\mathcal{F}$ denoted it by $\chi_{Ker\mathcal{F}}$.

Theorem 2-4 [8] *Let $\mathcal{F} : \chi_M \to \eta_N$ be a fuzzy linear map, then $\chi_{Ker\mathcal{F}}$ is a fuzzy subspace of χ_M and $\eta_{Im\mathcal{F}}$ is a fuzzy subspace of η_N.*

Remark. For any fuzzy linear map $\mathcal{F} : \chi_M \to \eta_N$ $\chi_{Kerf} \leq \chi_{Ker\mathcal{F}}$ and $\eta_{Imf} \leq \eta_{Im\mathcal{F}}$

Definition 2-5 [8] A fuzzy linear map $\mathcal{F} : \chi_M \to \eta_N$ is called *epic* (or *monic*) *iff* $f: M \to N$ is *epic* (or *monic*)

Definition 2-6 [8] A fuzzy linear map $\mathcal{F} : \chi_M \to \eta_N$ is called a *quasi-monic iff* $\chi_{Ker\mathcal{F}} = \chi_{M'}$, where $M' = \{m \in M \mid \chi(m) = 1\}$

Remark. Obviously, when $\chi_{Ker\mathcal{F}} = \{0\}$, *quasi-monic* is just *ordinary monic*

Definition 2-7 [8] Two fuzzy maps $\chi_M \xrightarrow{\mathcal{F}} \eta_N \xrightarrow{\mathcal{G}} \rho_V$ are exact at η_N iff $\eta_{Im\mathcal{F}} = \eta_{Ker\mathcal{G}}$

Remark. By the induction, from Definition 2-7, we can define an exact sequence of fuzzy linear maps.

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Definition 2-8 [8] A fuzzy R-module χ_M is called a *fuzzy singular R-module iff* $\chi(m)=1$ for all $m \in M$, denoted it by 1.

Definition 2-9 [8] An exact sequence

$$1 \xrightarrow{\tilde{i}} \chi_M \xrightarrow{\tilde{f}} \eta_N \xrightarrow{\tilde{g}} \rho_V \xrightarrow{\tilde{f}} 1,$$

where the two 1's are the appropriate singular fuzzy R-modules and $\tilde{i}$, $\tilde{f}$ are the fuzzy identity map and an epic map, respectively, is called a short exact sequence of fuzzy linear maps.

Theorem 2-10 [8] Given a short exact sequence of fuzzy linear maps, $1 \xrightarrow{\tilde{i}} \chi_M \xrightarrow{\tilde{f}} \eta_N \xrightarrow{\tilde{g}} \rho_V \xrightarrow{\tilde{f}} 1$

(1) $Im\,\tilde{i} = Ker\,\tilde{f} = \chi_{M'}$

(2) $Im\,\tilde{f} = Ker\,\tilde{g} \geq \eta_{N'}$

(3) $\tilde{g}$ is epic

where $M' = \{m \in M \mid \chi(m)=1\}$ and $N' = \{n \in N \mid \eta(n)=1\}$.

3. Connections with two exact sequences

In this section, we investigate some situations in connection with exact sequences of fuzzy linear maps of fuzzy R-modules.

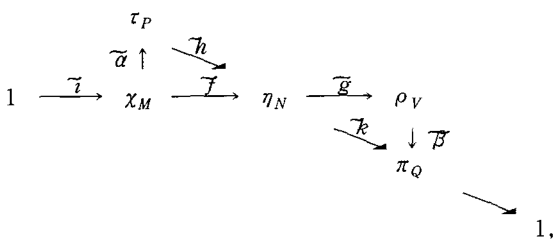
Lemma 3-1 [4] *For any fuzzy linear map* $\tilde{f}: \chi_M \to \eta_N$,

$$\chi_{M'} \subseteq \chi_{Ker\,\tilde{f}}\ ,$$

where $M' = \{m \in M \mid \chi(m)=1\}$.

Pan[8] investigated the situation (Corollary 3-5) and Kim[3] investigated the other situation (Theorem 3-2) in connection with two exact sequences of fuzzy linear maps of fuzzy R-modules.

Theorem 3-2 [3] *Consider the commutative diagram of two exact sequences of fuzzy linear maps*



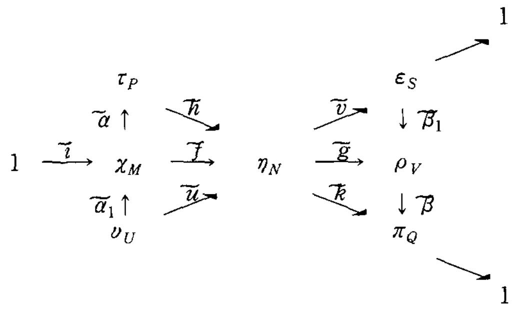
where i *is the identity map.*

Then (1) $\tilde{\alpha}$ *is quasi-monic and* $\tilde{\beta}$ *is epic*

(2) If $\tilde{\alpha}$ *is epic and* $Ker\,\tilde{\beta} \subset Im\,\tilde{g}$*, then* $\tilde{\beta}$ *is quasi-monic.*

We investigate the another situations in connection with exact sequences of fuzzy linear maps in the following theorems.

Theorem 3-3. *Consider the commutative diagram of exact sequences of fuzzy linear maps*



where i *is the identity map.*

Then (1) $\tilde{\beta}_1$ *is epic*

(2) If $\tilde{u}$ *is quasi-monic, then* $\tilde{\alpha}_1$ *is quasi-monic*

Proof. (1) Let $y \in V$. Then $\beta(y) \in Q$. Since $\tilde{k}$ is epic, there exists $x \in N$ such that $k(x) = \beta(y)$. Since $\beta_1(v(x)) = g(x) \in \rho_{Im\,\tilde{\beta}_1} = \rho_{Ker\,\tilde{\beta}}$, $\pi(\beta(g(x))) = 1$. Since $\beta(g(x)) = k(x) = \beta(y)$,

$$\pi(\beta(g(x)-y)) = 1.$$

So $g(x)-y \in \rho_{Ker\,\tilde{\beta}}$. Since $\rho_{Ker\,\tilde{\beta}} = \rho_{Im\,\tilde{\beta}_1}$, there exists $z \in S$ such that $\beta_1(z) = g(x)-y = \beta_1(v(x))-y$. Thus

$$y = \beta_1(v(x)) - \beta_1(z) = \beta_1(v(x)-z).$$

Hence $\tilde{\beta}_1$ is epic.

(2) To prove that $\tilde{\alpha}_1$ is quasi-monic, we must show that $\nu_{Ker\,\tilde{\alpha}} \subset \nu_{U'}$, where $U' = \{u \in U \mid \nu(u)=1\}$ by Lemma 3-1. Let $x \in \nu_{Ker\,\tilde{\alpha}_1}$. Then $\chi(\alpha_1(x)) = 1$. Thus $\alpha_1(x) \in \chi_{M'}$, where $M' = \{m \in M \mid \chi(m)=1\}$. Since $\eta(f(\alpha_1(x))) \geq \chi(\alpha_1(x)) = 1$,

$$f(\alpha_1(x)) \in \eta_{N'},$$

where $N' = \{n \in N \mid \eta(n)=1\}$.

By the commutativity of the diagram,

$$f(\alpha_1(x)) = u(x) \in \eta_{N'},$$

which implies that $\eta(u(x)) = 1$. So $x \in \nu_{Ker\,\tilde{u}}$. Since $\tilde{u}$ is quasi-monic, $\nu_{Ker\,\tilde{u}} = \nu_{U'}$. So $x \in \nu_{U'}$. This completes the proof of (2).

Theorem 3-4. *Let the following diagram of fuzzy linear maps of fuzzy R-modules be commutative and let the two rows be exact.*

$$
\begin{array}{ccccccc}
 & & \mu_A & \xrightarrow{\overline{f}} & \nu_B & \xrightarrow{\overline{g}} & \varepsilon_C \xrightarrow{\overline{f}} 1 \\
 & & \downarrow \overline{\alpha} & & \downarrow \overline{\beta} & & \downarrow \overline{\gamma} \\
1 & \xrightarrow{\overline{i}} & \chi_M & \xrightarrow{\overline{h}} & \eta_N & \xrightarrow{\overline{k}} & \rho_S
\end{array}
$$

Then (1) If $\overline{\alpha}$ and $\overline{\gamma}$ are epic, then $\overline{\beta}$ is epic

(2) If $\overline{\alpha}$ and $\overline{\gamma}$ are quasi-monic, then $\overline{\beta}$ is quasi-monic.

Proof. (1) Suppose that $\overline{\alpha}$ and $\overline{\gamma}$ are epic. Let $y \in N$. Then $k(x) \in S$. Since $\overline{\gamma}$ is epic, there exists $x \in C$ such that $\gamma(x) = k(y)$. Since $\overline{g}$ is epic, there exists $z \in B$ such that $g(z) = x$. Since $k(\beta(z)) = \gamma((g(z)) = \gamma(x) = k(y)$,

$$\rho(k(\beta(z)-y)) = \rho(k(\beta(z)) - k(y)) = 1.$$

So $\beta(z) - y \in \eta_{N'} = \eta_{Ker \overline{k}} = \eta_{Im \overline{h}}$. Thus there exists $m \in M$ such that $h(m) = \beta(z) - y$. Since $\overline{\alpha}$ is epic, there exists $a \in A$ such that $\alpha(a) = m$. Thus, by the commutativity of the diagram

$$
\begin{aligned}
y &= \beta(z) - h(m) \\
&= \beta(z) - h(\alpha(a)) \\
&= \beta(z) - \beta(f(a)) \\
&= \beta(z - f(a))
\end{aligned}
$$

Thus $\overline{\beta}$ is epic.

(2) Suppose that $\overline{\alpha}$ and $\overline{\gamma}$ are quasi-monic. To prove that $\overline{\beta}$ is quasi-monic, we must show that

$$\nu_{Ker \overline{\beta}} \subset \nu_{B'},$$

where $B' = \{b \in B \mid \nu(b) = 1\}$ by Lemma 3-1. Let $x \in \nu_{Ker \overline{\beta}}$. Then

$$\eta(\beta(x)) = 1.$$

So $\beta(x) \in \eta_{N'}$. Since $\rho(k(\beta(x))) \ge \eta(\beta(x)) = 1$ and $k\beta(x) = \gamma g(x)$,

$$g(x) \in \varepsilon_{Ker \overline{\gamma}}.$$

Since $\overline{\gamma}$ is quasi-monic,

$$\varepsilon_{Ker \overline{\gamma}} = \varepsilon_{C'},$$

where $C' = \{x \in C \mid \varepsilon(x) = 1\}$. Thus $\varepsilon(g(x)) = 1$.

Hence $x \in \nu_{Ker \overline{g}} = \nu_{Im \overline{f}}$. So there exists $y \in A$ such that $f(y) = x$.

Since $h(\alpha(y)) = \beta(f(y)) = \beta(x) \in \eta_{N'}$,

$$\eta(h(\alpha(y))) = 1$$

which implies that $\alpha(y) \in \chi_{Ker \overline{h}} = \chi_{Im \overline{i}} = \chi_M'$. Thus

$$\chi(\alpha(y)) = 1$$

and so $y \in \mu_{Ker \overline{\alpha}}$.

Since $\overline{\alpha}$ is quasi-monic, $\mu_{Ker \overline{\alpha}} = \mu_{A'}$. So $y \in \mu_{A'}$ which implies that $\mu(y) = 1$. Since $\nu(x) = \nu(f(y)) \ge \mu(y) = 1, x \in \nu_{B'}$.

This completes the proof of (2).

We have the following Corollary 3-5 which is the special case of Theorem 3-4.

Corollary 3-5 [8] *Consider the diagram with two exact sequences of fuzzy linear maps*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \chi_M & \xrightarrow{\overline{f}} & \eta_N & \xrightarrow{\overline{g}} & \rho_V & \xrightarrow{\overline{f}} & 1 \\
 & & \downarrow \overline{\alpha} & & \downarrow \overline{\beta} & & \downarrow \overline{\gamma} & & \\
1 & \longrightarrow & \mu_W & \xrightarrow{\overline{f}_1} & \nu_P & \xrightarrow{\overline{g}_1} & \varepsilon_S & \longrightarrow & 1
\end{array}
$$

where both $\overline{\alpha}$ and $\overline{\gamma}$ are epic and quasi-monic. Then $\overline{\beta}$ is epic and quasi-monic.

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