Construction Algorithm of Grassmann Space Parameters in Linear Output Feedback Systems

Su-Woon Kim

Abstract: A general construction algorithm of the Grassmann space parameters in linear systems — so-called, the Plücker matrix, "L" in m-input, p-output, n-th order static output feedback systems and the Plücker matrix, "L^{aug}" in augmented $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order static output feedback systems — is presented *for numerical checking* of necessary conditions of complete static and complete minimum d-th order dynamic output feedback pole-assignments, respectively, and also *for discernment of deterministic computation condition* of their pole-assignable real solutions. Through the construction of L, it is shown that certain generically pole-assignable strictly proper $mp > n$ system is *actually none pole-assignable* over any (real and complex) output feedbacks, by intrinsic rank deficiency of some submatrix of L. And it is also concretely illustrated that this none pole-assignable $mp > n$ system by static output feedback can be *arbitrary pole-assignable* system via minimum d-th order dynamic output feedback, which is constructed by deterministic computation under full-rank of some submatrix of L^{aug}.

Keywords: Grassmann space, Plücker matrix in static output feedback system, some submatrix of Plücker matrix, complete/generic output feedback pole-assignment, deterministic computation condition of real solutions.

1. INTRODUCTION

The static output feedback (simply, SOF) and minimum order dynamic output feedback (simply, m-DOF, or minimum order DOF) pole-assignments and their parametrizations in canonical forms for parametric solutions in linear systems shall be one of the fundamental open problems in linear system theory and control engineering [1-6]. However, previous major results regarding to the pole-assignment problem were stayed on the level of *generic* pole-assignability without reaching to final goal, *complete* (i.e., exact) pole-assignability that is naturally required for system design [3]. When a pure mathematical power, so-called, algebraic geometry method was firstly applied to this high nonlinear problem of SOF pole-assignment, it was thought that the genericity problem could be negligible in the sense that it lies in a union of algebraic subsets of lower dimension [7]. But in recent papers [8,9], it was shown that in the generic pole-assignable condition, some essential control engineering attributes (like sensitivity, stability, etc.) can be lost.

For this incomplete problem of the generic pole-assignability, the author systematically investigated in recent paper[10] *in what extend* the incomplete outcomes (of necessary and sufficient conditions of *generic* SOF and *generic* m-DOF pole-assignment on complex field $\mathbb{C}$, respectively) are valid and invalid, comparing with the complete outcomes (of necessary conditions of *complete* SOF and *complete* m-DOF pole-assignment on $\mathbb{C}$, respectively). Observe that if certain sufficient condition of generic pole-assignment does not satisfy a necessary condition of complete pole-assignment condition, then the sufficient condition of generic pole-assignment *is invalid*, i.e., the generically pole-assignable system is actually *none pole-assignable* over any (real and complex) SOF or m-DOF.

Through certain "lattice diagram analysis" (as a simplified signal flow graph analysis of OF gain loops; see Fig. 2) and "full-rank test in dynamic Grassmann invariant L^{aug}" (defined by the Plücker matrix of *augmented* $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order SOF system; see (20a)) — as a necessary condition of *complete* m-DOF pole-assignment [10,Theorem 1][11,Corollary 5.1.1], the validities and invalidities of generic m-DOF pole-assignability were totally configured in [10,Table 1].

Recall that the *Grassmann invariant* (so-called

Manuscript received August 12, 2004; revised April 18, 2005; accepted May 6, 2005. Recommended by Editorial Board member Jae Weon Choi under direction of Editor-in-Chief Myung Jin Chung.

Su-Woon Kim is with the Department of Electrical and Electronic Engineering, Cheju National University, 66 Jejudae-hakro, Jeju-si, Jeju-do 690-756, Korea (e-mail: swkim15@kornet.net).

Plücker matrix) L, as a SOF invariant in the canonical SOF vector equation for system poles,

$$Lk = a \tag{1}$$

was theoretically derived by Giannakopoulos and Karcanias in mid 1980s [11,12] (where $\boldsymbol{k}$ denotes extended OF gain vector whose elements constitute Plücker coordinates in $\boldsymbol{k} = [1, k_{11}, \ldots, k_{mp}, k_{i1}, \ldots, k_{ir}]^t \in \mathbb{P}^{\sigma}$ constrained with quadratic relations among their elements, and $\boldsymbol{a} = [1, a_1, \ldots, a_n]^t \in \mathbb{R}^{n+1}$ denotes arbitrary real coefficient vector of closed-loop characteristic polynomial in m-input, p-output, n-th order linear systems; $\sigma = \binom{m+p}{m} - 1$). But the concrete construction algorithm of L in $Lk = a$ for numerical full-rank test of some submatrix of L — as a necessary condition of *complete* SOF pole-assignment [11,Corollary 5.1.1] — has been not known.

And the *dynamic Grassmann invariant* (as the Grassmann invariant for minimum d-th order DOF m-input, p-output, n-th order system) was naturally induced and defined, through the lattice diagram analysis, by "the Grassmann invariant in *augmented* $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order SOF system" L^{aug} [10, Definition 4], within the canonical m-DOF vector equation for system poles,

$$L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug} \tag{2}$$

(where $\boldsymbol{k}^{aug}$ denotes extended SOF gain vector whose elements constitute Plücker coordinates in $\boldsymbol{k}^{aug} = [1, k_{11}, \ldots, k_{m+d,p+d}, k_{i1}, \ldots, k_{ir}]^t \in \mathbb{P}^{\sigma^*}$ constrained with QRs among their elements, and $\boldsymbol{a}^{aug} = [1, a_1, \ldots, a_{n+d}]^t \in \mathbb{R}^{n+d+1}$ denotes arbitrary real coefficient vector of closed-loop characteristic polynomial in *augmented* $(m+d)$-input, $(p+d)$-output, $(n + d)$-th order SOF systems; $\sigma^* = \binom{m+d+p+d}{m+d} - 1$).

In this paper, we shall call the real coefficient matrix $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ of $\boldsymbol{k}$ in the SOF vector equation $Lk = a$ and the real coefficient matrix $L^{aug} \in \mathbb{R}^{(n+d+1)\times(\sigma^*+1)}$ of $\boldsymbol{k}^{aug}$ in the m-DOF vector equation $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ by "Grassmann space parameters" of linear OF systems, because the elements of $\boldsymbol{k}$ and $\boldsymbol{k}^{aug}$ constitute the Plücker coordinates of the Grassmann spaces, $Grass(m, m+p)$ and $Grass(m+d, m+d+p+d)$, respectively (refer to Appendix A on the notions of Grassmann space and its Plücker embedding into projective space).

The goal of this paper is to present a general construction algorithm of the Grassmann space parameters L and L^{aug}, under the equations $Lk = a$ and $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$, *for numerical checking of the necessary conditions* of complete (SOF and m-DOF) pole-assignments and *for discernment of deterministic (nonsingular) computation conditions* of their real (SOF and m-DOF) compensators, respectively.

Through the construction algorithm of L, it is also concretely illustrated that a certain generically SOF pole-assignable system $mp > n$ is actually none pole-assignable over any real and complex SOF, as worried in [8,9]. Recall that the *complete* SOF pole-assignment and *generic* SOF pole-assignment in the linear system of transfer function matrix $G(s)$ are defined as follows [11,13,14].

Definition 1 (complete SOF pole-assignment): In the closed-loop characteristic polynomial $det\,[D_L(s) + N_L(s)K] = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$ of irreducible strictly proper (or proper) transfer function matrix $G(s) = D_L(s)^{-1}N_L(s)$, if there exist real OF gain matrices $K \in \mathbb{R}^{m\times p}$ for all arbitrary real coefficients $(a_1, \ldots, a_n) \in \mathbb{R}^n$ (or $(1, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$)), then it is called that the linear system $G(s)$ is *completely* pole-assignable by SOF (of real OF gain matrix).

Definition 2 (generic SOF pole-assignment): In the closed-loop characteristic polynomial $det\,[D_L(s) + N_L(s)K] = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$ of irreducible strictly proper (or proper) transfer function matrix $G(s) = D_L(s)^{-1}N_L(s)$, if there exist open dense sets of real coefficients $(a_1, \ldots, a_n) \in \mathbb{R}^n$ (or $(1, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$)) over all real OF gain matrices $K \in \mathbb{R}^{m\times p}$, then it is called that the linear system $G(s)$ is *generically* pole-assignable by SOF (of real OF gain matrix).

This paper is organized as follows. In Section 2, theoretical background is grossly explained. In Section 3, a construction algorithm of L in the SOF vector equation $Lk = a$ for system poles is presented. In Section 4, through the construction algorithm of L, an invalid case (i.e., none pole-assignable case) in the well-known sufficient condition of generic SOF pole-assignment $mp > n$ [15] is concretely revealed. And in Section 5, it is also illustrated that even in the valid case (i.e., pole-assignable case) of $mp > n$, the real solutions for construction of SOF compensator can not be always obtained if the numerical computation in the equation $Lk = a$ is carried under certain non-deterministic (singular) computational condition. In Section 6, it is demonstrated how the none pole-assignable system over any (real and complex) SOF in Section 4 can be changed into the arbitrary pole-assignable system via real m-DOF that is constructed under deterministic computation condition and full-rank of a submatrix of L^{aug} in the equation $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$. Concluded remarks are given in Section 7.

2. THEORETICAL BACKGROUND

2.1. Canonical SOF (vector) equation for system poles

In the SOF configuration of Fig. 1, the closed-loop transfer function matrix $G_{C\text{-}L}(s,K)$ is obtained by $G_{C\text{-}L}(s,K) = (I_p + G(s)K)^{-1}G(s)$ and the closed-loop characteristic polynomial $p(s,K)$ is obtained by

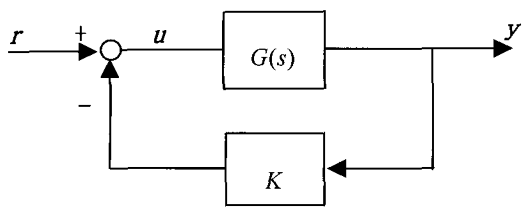


Fig. 1. SOF configuration.

$$\begin{aligned} p(s, K) &= f_0(k_{11}, \dots, k_{mp})s^n + f_1(k_{11}, \dots, k_{mp})s^{n-1} + \\ &\quad \dots + f_n(k_{11}, \dots, k_{mp}) \qquad (3a) \\ &= det[D_L(s) + N_L(s)K], \end{aligned}$$

where $G(s) = D_L(s)^{-1}N_L(s)$, and $k_{11}, \dots, k_{mp}$ indicate the entries of $K \in \mathbb{R}^{m\times p}$. In [11], using Grassmann algebra (or exterior algebra), it was shown that the closed-loop characteristic polynomial $p(s,K)$ can be represented by a linear vector function constrained with QRs

$$p(s, K) = \boldsymbol{e}_n(s)L\boldsymbol{k}, \qquad (3b)$$

(where $\boldsymbol{e}_n(s) = [s^n, s^{n-1}, \dots, s_1, 1]$, and $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ denotes Plücker matrix, and $\boldsymbol{k} \in \mathrm{P}^{\sigma}$ denotes extended OF gain vector whose elements constitute Plücker coordinates in projective space constrained with QRs).

From the equality (3a) = (3b), *a canonical SOF (vector) equation* for system poles,

$$L\boldsymbol{k} = \boldsymbol{a}$$

is derived where $\boldsymbol{a} = [1, a_1, \dots, a_n]^t \in \mathbb{R}^{n+1}$ indicates arbitrary real coefficient vector of closed-loop characteristic polynomial $p(s,K)$. It is notable that this vector equation is "canonical" by the fixed dimensions of L, $\boldsymbol{k}$ and $\boldsymbol{a}$, and is also "unique" over (minimal) transfer function matrix $G(s)$; see Remark 1.

2.2. Structural quantitative relationship (S.Q.R.) between SOF and m-DOF for pole-assignment

In recent author's paper [10,Theorem 1], through a lattice diagram analysis (as "a simplified signal flow graph analysis of all OF gain loops" in SOF linear systems), it is proved that there exists a *fixed* quantitative relationship between SOF compensation and m-DOF compensation for pole-assignment like:

"Minimum d-th order DOF compensator for pole-assignment in original m-input, p-output, n-th order linear systems *is decomposed into* SOF compensator and its associated d number of arbitrary 1st order strictly proper or proper dynamic (transfer function) elements in *augmented* $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order linear systems." (4)

See the Fig. 2. It shows $d!$ number of lattice diagrams representing "(SOF) gain-loop decomposition modes" of minimum d-th order DOF compensation in augmented $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order SOF system.

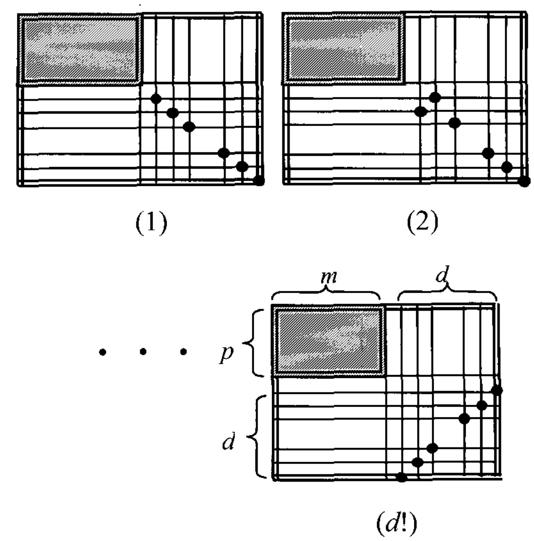


Fig. 2. Lattice diagrams of gain-loop decomposition modes of minimum d-th order DOF compensation in augmented SOF system.

The black dots(•) indicate the d number of 1st order dynamic (strictly proper or proper transfer function) elements locating in the crosses of (two different) horizontal lines and vertical lines, whose all combinative numbers amount to $d!$. And the slashed rectangular diagram indicates the original plant of m-input, p-output (degenerate or nondegenerate) transfer function matrix.

The fixed numerical-quantitative relationship in (3) was named by "structural quantitative relationship (simply, S.Q.R.)" in [10,Theorem 1]. The S.Q.R. (between SOF compensation and m-DOF compensation for pole-assignment) is significant one with following meaning:

"Any outcomes (like generic or complete necessary and/or sufficient condition, dynamic invariants, canonical forms, etc.) regarding to m-DOF pole-assignment can *be induced directly* from the pre-known outcomes (of generic or complete necessary and/or sufficient condition, static invariants, canonical forms, etc.) regarding to SOF pole-assignment, or *vice versa*." (5)

Thus through the S.Q.R., *a canonical m-DOF (vector) equation* for system poles is directly induced by

$$L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$$

(see the Example 2 in Section 5), and *dynamic Grassmann invariant* (as the Grassmann invariant for minimum d-th order DOF system) is directly induced by L^{aug} in *augmented* SOF system, and a new sufficient condition of generic m-DOF pole-assignment can be also directly induced from the well-known sufficient condition of generic SOF pole-assignment $mp > n$ (see the Theorem in Section 6).

2.3. General gain formula (G.G.F.) for computation of m-DOF compensator

From the S.Q.R. in (3) and (4), we can derive a *general gain formula* for computation of real m-DOF compensator, $H(s)^{min} \in \mathbb{R}(s)^{m\times p}$ for pole-assignment, through composition of the SOF compensator $K^{aug} \in \mathbb{R}^{(m+d)\times(p+d)}$ and its associated d number of arbitrary 1st order dynamic elements $e_1(s), \dots, e_d(s)$. Considering the decomposition mode with diagonally descending dynamic elements in Fig. 2(1), the general gain formula (simply, G.G.F.) for computation of each element $H_{ij}(s)^{min}$ of $H(s)^{min}$ is obtained by

$$-H_{ij}(s)^{min} = k_{ij} + \Big(k_{i,m+1}\frac{e_1(s)}{1-e_1(s)W_1^{ij}(s)}k_{p+1,\,j} + \dots + k_{i,m+d}\frac{e_d(s)}{1-e_d(s)W_d^{ij}(s)}\,k_{p+d,\,j}\ \Big), \tag{6}$$

where the $W_1^{ij}(s), \dots, W_d^{ij}(s)$ are outfitted by

$$W_1^{ij}(s) = k_{p+1,\ m+1} + \sum_{\lambda=2}^{d} k_{p+1,m+\lambda}\cdot\frac{e_\lambda(s)}{1-e_\lambda(s)k_{p+\lambda,\ m+\lambda}} \cdot k_{p+\lambda,\ m+1}$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$W_d^{ij}(s) = k_{p+d,\ m+d} + \sum_{\lambda=1}^{d-1} k_{p+d,m+\lambda}\cdot\frac{e_\lambda(s)}{1-e_\lambda(s)k_{p+\lambda,\ m+\lambda}} \cdot k_{p+\lambda,\ m+d}$$

for all $i = 1, \dots, m$ and $j = 1, \dots, p$. See (22) in Example 3 in Section 6.

3. CONSTRUCTION ALGORITHM OF A GRASSMANN SPACE PARAMETER, L

3.1. Polynomial system determinant

From Fig. 1, the closed-loop characteristic polynomial $p(s, K)$ is obtained by

$$p(s, K) = det\,[D_L(s)]\ det\,[I_p + G(s)K]$$

and since $det\,[D_L(s)]$ indicates open-loop characteristic polynomial (where $G(s) = D_L(s)^{-1}N_L(s)$), the system determinant (Δ) is obtained by

$$\Delta = det\,[I_p + G(s)K]$$

$$= det\left(\begin{bmatrix} 1 & \cdots & 0 & G_{11}(s) & \cdots & G_{1m}(s) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & G_{p1}(s) & \cdots & G_{pm}(s) \end{bmatrix} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ k_{11} & \cdots & k_{1p} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mp} \end{bmatrix} \right)$$

$$= det\,[T(s)\,F]. \tag{7}$$

Hence, from Binet-Cauchy theorem (refer to Appendix B),

$$\Delta = \sum_{p=1}^{\binom{m+p}{p}} (p\times p \text{ subdeterminant of } T(s)) \times (\text{correspondent } p\times p \text{ subdeterminant of } F)$$

$$= \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix} \times \begin{vmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} G_{11} & 0 & \cdots & 0 \\ G_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1} & 0 & \cdots & 1 \end{vmatrix} \times \begin{vmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} + \dots$$

$$+ \begin{vmatrix} 1 & \cdots & 0 & G_{1m} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & G_{p-1,m} \\ 0 & \cdots & 0 & G_{pm} \end{vmatrix} \times \begin{vmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ k_{m1} & \cdots & k_{m,p-1} & k_{mp} \end{vmatrix}$$

$$+ \begin{vmatrix} G_{11} & G_{12} & 0 & \cdots & 0 \\ G_{21} & G_{22} & 0 & \cdots & 0 \\ G_{31} & G_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{p1} & G_{p2} & 0 & \cdots & 1 \end{vmatrix} \times \begin{vmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1p} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2p} \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} + \dots$$

$$+ \begin{vmatrix} G_{1,m-p+1} & G_{1,m-p+2} & \cdots & G_{1m} \\ G_{2,m-p+1} & G_{2,m-p+2} & \cdots & G_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ G_{p,m-p+1} & G_{p,m-p+2} & \cdots & G_{pm} \end{vmatrix}$$

$$\times \begin{vmatrix} k_{m-p+1,1} & k_{m-p+1,2} & \cdots & k_{m-p+1,p} \\ k_{m-p+2,1} & k_{m-p+2,2} & \cdots & k_{m-p+2,p} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mp} \end{vmatrix}$$

$$= 1 + \sum_{(j,i)\,=\,(1,1)}^{(p,\,m)} G_{ji}(s)k_{ij} + \sum_{\ell=2}^{\min\{m,\,p\}} (\ell\times\ell \text{ subdeterminant of } G(s)) \times (\text{correspondent } \ell\times\ell \text{ subdeterminant of } K^t). \tag{8}$$

Lemma 1 (polynomial system determinant)**:** In SOF linear systems, the system determinant (Δ) of Mason's formula is described by following polynomial function formula,

$$\Delta = 1 + \sum_{(j,i)=(1,1)}^{(p,m)} G_{ji}(s)k_{ij} + \sum_{(u,v)=(1,1)}^{\left(\binom{p}{2},\binom{m}{2}\right)} \left|G_{(u,v)}^{2\times 2}\right| \left|[K^t]_{(u,v)}^{2\times 2}\right| + \cdots + \sum_{(u,v)=(1,1)}^{\left(\binom{p}{N},\binom{m}{N}\right)} \left|G_{(u,v)}^{N\times N}\right| \left|[K^t]_{(u,v)}^{N\times N}\right|$$

where $\sum_{(j,i)=(1,1)}^{(p,m)}$: the sum numbered in dictionary orders $(1,1), (1,2), \ldots, (p, m-1), (p,m)$,

$\sum_{(u,v)=(1,1)}^{\left(\binom{p}{\ell},\binom{m}{\ell}\right)}$: the sum numbered in dictionary orders $(\ell,(1,1))$, $(\ell, (1,2)), \ldots, (\ell, (\binom{p}{\ell}, \binom{m}{\ell}-1)), (\ell, (\binom{p}{\ell}, \binom{m}{\ell}))$; $\ell = 2, \ldots, \min\{m,p\}(=N)$,

$|G_{(u,v)}^{\ell\times\ell}|$: the $(\ell, (u,v))$-th $\ell\times\ell$ subdeterminants of $G(s) \in \mathbb{R}(s)^{p\times m}$, $|[K^t]_{(u,v)}^{\ell\times\ell}|$: the correspondent $(\ell, (u, v))$-th $\ell\times\ell$ subdeterminants of $K^t \in \mathbb{R}^{p\times m}$ (or $\mathbb{C}^{p\times m}$) over $|G_{(u,v)}^{\ell\times\ell}|$.

Proof: From (8), the proof is immediate. □

From the Lemma 1 and (3b), the closed-loop characteristic polynomial $p(s, K)$ can be re-written by

$$p(s, K) = f_0(k_{11}, \ldots, k_{mp})s^n + f_1(k_{11}, \ldots, k_{mp})s^{n-1} + \ldots + f_n(k_{11}, \ldots, k_{mp})$$
$$= \boldsymbol{e}_n(s)L\boldsymbol{k} \tag{9a}$$

$$= D_{O-L}(s) \begin{pmatrix} 1 + \sum_{(j,i)=(1,1)}^{(p,m)} G_{ji}(s)k_{ij} \\ + \sum_{(u,v)=(1,1)}^{\left(\binom{p}{2},\binom{m}{2}\right)} \left|G_{(u,v)}^{2\times 2}\right| \left|[K^t]_{(u,v)}^{2\times 2}\right| \\ + \ldots + \sum_{(u,v)=(1,1)}^{\left(\binom{p}{N},\binom{m}{N}\right)} \left|G_{(u,v)}^{N\times N}\right| \left|[K^t]_{(u,v)}^{N\times N}\right| \end{pmatrix}. \tag{9b}$$

Hence, the number of total interacting (nonlinear) terms in MIMO systems is immediately obtained as following Lemma 2 and Corollary.

Lemma 2 (number of square submatrices)**:** In a $p\times m$ matrix, the number r of all $N\times N$ submatrices is

$$r = \binom{m+p}{m} - mp - 1,$$

where $N = 2, \ldots, \min\{m, p\}$.

Proof: See the Appendix C. □

Corollary (number of nonlinear interacting terms)**:** In a $p\times m$ transfer function matrix of SOF linear systems, the number r of all $N\times N$ submatrices (as independent components of OF gain loops among all OF gain loops) is

$$r = \binom{m+p}{m} - mp - 1,$$

where $N = 2, \ldots, \min\{m, p\}$.

Proof: From Lemma 2, the proof is immediate. □

Thus from the equality (9a) = (9b) and the Corollary, the real coefficient Plücker matrix L of vector $\boldsymbol{k}$ can be constructed as following way:

"Every column of L in the SOF vector equation $L\boldsymbol{k} = \boldsymbol{a}$, is *one-to-one correspondently* constructed from *i*) the real coefficients of an open-loop characteristic polynomial $D_{O-L}(s)$ and *ii*) mp number of all normalized numerators over $D_{O-L}(s)$ and *iii*) $(\sigma-mp)$ number of the real coefficients of all kinds of normalized $\ell\times\ell$ sub-determinants of transfer function matrix $G(s)$ over $D_{O-L}(s)$ (where $\ell = 2, \ldots, \min\{m,p\}$)." (10)

From (10), we shall subdivide the $L\boldsymbol{k}$ into following three parts:

$$D_{O-L}(s) \cdot 1$$

as constant terms over an element '1' of $\boldsymbol{k}$, and

$$D_{O-L}(s) \cdot \sum_{(i,j)=(1,1)}^{(m,p)} G_{ji}(s)k_{ij}$$

as linear terms over mp number of OF gain variable elements $k_{11}, \ldots, k_{mp}$ of $\boldsymbol{k}$,

$$D_{O-L}(s) \begin{pmatrix} \sum_{(u,v)=(1,1)}^{\left(\binom{p}{2},\binom{m}{2}\right)} \left|G_{(u,v)}^{2\times 2}\right| \left|[K^t]_{(u,v)}^{2\times 2}\right| + \ldots \\ + \sum_{(u,v)=(1,1)}^{\left(\binom{p}{N},\binom{m}{N}\right)} \left|G_{(u,v)}^{2\times 2}\right| \left|[K^t]_{(u,v)}^{N\times N}\right| \end{pmatrix}$$

as nonlinear terms over $(\sigma - mp)$ number of interacting gain variable elements $k_{i1}, \ldots, k_{ir}$ of $\boldsymbol{k}$.

3.2. Construction algorithm of L in $L\boldsymbol{k} = \boldsymbol{a}$

From the 3 subdivision of the $L\boldsymbol{k}$, the construction procedure of Grassmann space parameter L in $L\boldsymbol{k} = \boldsymbol{a}$ can be presented by following 4 steps:

Step 1: *Normalized* transfer function matrix $G(s)$ (over the open-loop characteristic polynomial $D_{O-L}(s)$), then the first column $\mathbf{c}_0$ of $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ is constructed by the real coefficients of $D_{O-L}(s)$, in descending order $s^n, s^{n-1}, \ldots, 1$ coincident with the order of real coefficient vector $\boldsymbol{a}$ of $p(s, K)$.

Step 2: The next q $(= mp)$ columns of $\mathbf{c}_1, \ldots, \mathbf{c}_q$ of $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ are constructed by the real coefficients of *normalized* numerators. $N_{11}(s), N_{21}(s), \ldots, N_{pm}(s)$ over $D_{O-L}(s)$, in descending order $s^n, s^{n-1}, \ldots, 1$ coincident with the order of real coefficient vector $\boldsymbol{a}$

of $p(s, K)$.

Step 3: The next $\sigma - q$ $(= r)$ columns of $\mathbf{c}_{q+1}, \dots, \mathbf{c}_\sigma$ of $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ are constructed by the real coefficients of *normalized* subdeterminants, $I_{i1}(s)$, $I_{i2}(s)$, ... , $I_{ir}(s)$ over $D_{O\text{-}L}(s)$ in descending order s^n, s^{n-1}, ... , 1 coincident with the order of real coefficient vector $\boldsymbol{a}$ of $p(s, K)$
(where $N_{11}(s)$, $N_{21}(s)$, ... , $N_{pm}(s)$ indicate *normalized* numerators over denominator $D_{O\text{-}L}(s)$ in $G_{11}(s) = n_{11}(s)/d_{11}(s)$, $G_{21}(s) = n_{21}(s)/d_{21}(s)$, ... , $G_{pm}(s) = n_{pm}(s)/d_{pm}(s)$ and $I_{i1}(s)$, $I_{i2}(s)$, ... , $I_{ir}(s)$ indicate all kinds of *normalized* subdeterminants of $G(s)$ over $D_{O\text{-}L}(s)$).

Step 4: In the (extended) OF vector $\boldsymbol{k} = [1, k_{11}, \dots, k_{mp}, k_{i1}, \dots, k_{ir}]'$, the interacting gains (of MIMO system), $k_{i1}, \dots, k_{ih}, \dots, k_{ir}$ are constructed by numbering $h = 1, \dots, r$ in dictionary order from the first 2×2 sub-determinants to the last $N\times N$ subdeterminants like

$$k_{ih} := \left| [K^t]_{(u,v)}^{\ell\times\ell} \right|_{(2,\ (1,1))}^{(\ell,\ (u,v))}, \tag{11}$$

where $r = \sigma - mp$ and $h = 1, \dots, r$ denotes the h-th dictionary orders (2, (1,1)), (2, (1,2)), … , (2, (u,v-1)), (2, (u,v)), … , (N, (1,1)), … , (N, (u,v-1)), (N, (u,v)); $N = \min\{m, p\}$, $u = \binom{p}{2}$, $v = \binom{m}{2}$.

In (11), k_{ih} represents the interacting gains of arbitrary-order nonlinear relations of OF gains of $k_{11}, \dots, k_{mp}$. So we shall simply describe them by "NRs" for distinction with "QRs", the 2nd-order nonlinear relations of the OF gains.

Remark 1: A mathematical interesting issue, *construction algorithm of 2^{nd} order QRs* as the Grassmann variety of Grassmannian *Grass*(m, m+p) in Plücker coordinates $\mathbb{P}^\sigma$[16] can be developed as follows. From the arbitrary-order nonlinear relations (NRs) of OF gains in (11), the QRs of OF gains can be derived by consecutive $(\ell\text{-}1)\times(\ell\text{-}1)$ minor expansions of the $\ell\times\ell$ subdeterminants from 2×2 subdeterminants like

$$\begin{pmatrix} (\ell-1)\times(\ell-1) \ \ minor \ \ expansion \\ of \ \ \left| [K^t]_{(u,v)}^{\ell\times\ell} \right|_{(2,\ (1,1))}^{(\ell,\ (u,v))} \end{pmatrix} - k_{ih}\cdot k_0 = 0$$

[17,p.14][18,p.1077]; see Remark 2. Hence k_0 indicates the first coordinate in homogeneous projective space which is usually set by "1" in inhomogenized projective space $\mathbb{P}^\sigma(1, k_{11}, \dots, k_{mp}, k_{i1}, \dots, k_{ir})$, and which corresponds to the coefficients of open-loop characteristic polynomial $D_{O\text{-}L}(s)$ as the first column $\mathbf{c}_0$ of $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ in $L\boldsymbol{k} = \boldsymbol{a}$). Thus there are 2ℓ numbers of ways to make the QRs, through various $(\ell\text{-}1)\times(\ell\text{-}1)$ minor expansions of a $\ell\times\ell$ subdeterminant. Thus various "nonunique QRs" can be obtained from "unique NRs" of OF gains in a $\ell\times\ell$ subdeterminant (where $\ell \geq 3$).

4. NONE POLE-ASSIGNABLE CASE IN $mp > n$ STRICTLY PROPER SYSTEMS

We shall concretely illustrate an invalid (none pole-assignable) case in the generic outcome (i.e., sufficient condition of generic SOF pole-assignment $mp > n$ [15]), *through concrete construction algorithm of Grassmann space parameter* L in $L\boldsymbol{k} = \boldsymbol{a}$.

Example 1: In a strictly proper 3-input, 3-output, 8-th order unstable plant $G(s)$, but which satisfies generic SOF pole-assignment condition $mp > n$,

$$G(s) = \begin{bmatrix} \frac{(s-1)^2}{s^3} & 0 & 0 \\ 0 & \frac{(s+2)^2}{s^3+2} & 0 \\ 0 & 0 & \frac{s+3}{s^2+2} \end{bmatrix} \tag{12}$$

let's check whether this plant can be stabilized in arbitrary desired pole positions *by SOF*.

(1) Construction of Grassmann space parameter L and OF gain vector $\boldsymbol{k}$ in $L\boldsymbol{k} = \boldsymbol{a}$.

Step 1: From (12), the *normalize transfer function matrix $G(s)$ over $D_{O\text{-}L}(s)$* is constructed by

$$G(s) = \begin{bmatrix} \frac{(s-1)^2(s^3+2)(s^2+2)}{s^3(s^3+2)(s^2+2)} & 0 & 0 \\ 0 & \frac{(s+2)^2 s^3(s^2+2)}{s^3(s^3+2)(s^2+2)} & 0 \\ 0 & 0 & \frac{(s+3)s^3(s^3+2)}{s^3(s^3+2)(s^2+2)} \end{bmatrix}. \tag{12$'$}$$

Then the first column vector of L as real coefficients of $D_{O\text{-}L}(s)$ is obtained by

$\mathbf{c}_0 = [1, 0, 2, 2, 0, 4, 0, 0, 0]'$ over 1

(from $D_{O\text{-}L}(s) = s^3(s^3+2)(s^2+2)$) [19].

Step 2: From (12)$'$, the next q columns of $\mathbf{c}_1, \dots, \mathbf{c}_q$ of $L \in \mathbb{R}^{(n+1)\times(\sigma+1)}$ is constructed by

$\mathbf{c}_1 = [0, 1, -2, 3, -2, -2, 6, -8, 4]'$ over k_{11}
$\mathbf{c}_5 = [0, 1, 4, 6, 8, 8, 0, 0, 0]'$ over k_{22}
$\mathbf{c}_9 = [0, 1, 3, 0, 2, 6, 0, 0, 0]'$ over k_{33}

(from $N_{11}(s) = (s\text{-}1)^2(s^3+2)(s^2+2)$, $N_{22}(s) = (s+2)^2 s^3(s^2+2)$, $N_{33}(s) = (s+3)s^3(s^3+2)$), respectively, and $\mathbf{c}_2 = \mathbf{c}_3 = \mathbf{c}_4 = \mathbf{c}_6 = \mathbf{c}_7 = \mathbf{c}_8 = [0, 0, 0, 0, 0, 0, 0, 0, 0]'$ over $k_{21}, k_{31}, k_{12}, k_{32}, k_{13}, k_{23}$, respectively).

Step 3: From all the nonzero 2×2 subdeterminant and 3×3 determinant in $G(s)$, the nonlinear interacting terms (of MIMO system) are obtained as follows.

$D_{O\text{-}L}(s)\,|G_{(1,1)}^{2\times2}|\,|[K^t]_{(1,1)}^{2\times2}|$
$= D_{O\text{-}L}(s)\,(G_{11}G_{22} - G_{12}G_{21})(k_{11}k_{22} - k_{21}k_{12})$
$= (s^6 - 6s^5 + 15s^4 - 24s^3 + 30s^2 - 24s + 8)\, k_{i1}$

$D_{O\text{-}L}(s)\,|G_{(1,2)}^{2\times2}|\,|[K^t]_{(1,2)}^{2\times2}|$
$= D_{O\text{-}L}(G_{11}G_{32} - G_{12}G_{31})(k_{11}k_{23} - k_{13}k_{21}) = 0 \times k_{i2}$

$D_{O\text{-}L}(s)\,|G_{(1,3)}^{2\times2}|\,|[K^t]_{(1,3)}^{2\times2}|$
$= D_{O\text{-}L}(G_{21}G_{32} - G_{22}G_{31})(k_{12}k_{23} - k_{13}k_{22}) = 0 \times k_{i3}$

$D_{O\text{-}L}(s)\,|G_{(2,1)}^{2\times2}|\,|[K^t]_{(2,1)}^{2\times2}|$
$= D_{O\text{-}L}(G_{11}G_{23} - G_{13}G_{21})(k_{11}k_{32} - k_{12}k_{31}) = 0 \times k_{i4}$

$D_{O\text{-}L}(s)\,|G_{(2,2)}^{2\times2}|\,|[K^t]_{(2,2)}^{2\times2}|$
$= D_{O\text{-}L}(G_{11}G_{33} - G_{13}G_{31})(k_{11}k_{33} - k_{13}k_{31})$
$= (s^6 + s^5 - 5s^4 + 5s^3 + 2s^2 - 10s + 6)\, k_{i5}$

$D_{O\text{-}L}(s)\,|G_{(2,3)}^{2\times2}|\,|[K^t]_{(2,3)}^{2\times2}|$
$= D_{O\text{-}L}(G_{21}G_{33} - G_{23}G_{33})(k_{12}k_{33} - k_{13}k_{32}) = 0 \times k_{i6}$

$D_{O\text{-}L}(s)\,|G_{(3,1)}^{2\times2}|\,|[K^t]_{(3,1)}^{2\times2}|$
$= D_{O\text{-}L}(G_{12}G_{23} - G_{13}G_{22})(k_{21}k_{32} - k_{22}k_{31}) = 0 \times k_{i7}$

$D_{O\text{-}L}(s)\,|G_{(3,2)}^{2\times2}|\,|[K^t]_{(3,2)}^{2\times2}|$
$= D_{O\text{-}L}(G_{12}G_{33} - G_{13}G_{32})(k_{21}k_{33} - k_{23}k_{31}) = 0 \times k_{i8}$

$D_{O\text{-}L}(s)\,|G_{(3,3)}^{2\times2}|\,|[K^t]_{(3,3)}^{2\times2}|$
$= D_{O\text{-}L}(G_{22}G_{33} - G_{23}G_{32})\,(k_{22}k_{33} - k_{23}k_{32})$
$= (s^6 + 7s^5 + 16s^4 + 12s^3)\, k_{i9}$

$D_{O\text{-}L}(s)\,|G_{(1,1)}^{3\times3}|\,|[K^t]_{(1,1)}^{3\times3}|$
$= D_{O\text{-}L}(G_{11}G_{22}G_{33} + G_{12}G_{23}G_{31} + G_{13}G_{21}G_{32} - G_{11}G_{23}G_{32} - G_{12}G_{21}G_{33} - G_{13}G_{22}G_{31}) \times (k_{11}k_{22}k_{33} + k_{21}k_{32}k_{13} + k_{31}k_{12}k_{23} - k_{11}k_{32}k_{23} - k_{21}k_{12}k_{33} - k_{31}k_{22}k_{13})$
$= (s^5 + 5s^4 + 3s^3 - 13s^2 - 8s + 12)\, k_{i10}$ (13)

Thus, the next r (= σ - q) column vectors of L are constructed by

$$\mathbf{c}_{10} = [0, 0, 1, -6, 15, -24, 30, -24, 8]' \quad \text{over } k_{i1}$$
$$\mathbf{c}_{14} = [0, 0, 1, 1, -5, 5, 2, -10, 6]' \quad \text{over } k_{i5}$$
$$\mathbf{c}_{18} = [0, 0, 1, 7, 16, 12, 0, 0, 0]' \quad \text{over } k_{i9}$$
$$\mathbf{c}_{19} = [0, 0, 0, 1, 5, 3, -13, -8, 12]' \quad \text{over } k_{i10}$$

where $\mathbf{c}_{11} = \mathbf{c}_{12} = \mathbf{c}_{13} = \mathbf{c}_{15} = \mathbf{c}_{16} = \mathbf{c}_{17} = [0, 0, 0, 0, 0, 0, 0, 0]'$ over $k_{i2}, k_{i3}, k_{i4}, k_{i6}, k_{i7}, k_{i8}$, respectively.

Step 4: From Step 1 ~ Step 3, the SOF equation $L\boldsymbol{k} = \boldsymbol{a}$ excluding zero columns is constructed by

$$\left[\begin{array}{c|cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & -2 & 4 & 3 & 1 & 1 & 1 & 0 \\ 2 & 3 & 6 & 0 & -6 & 1 & 7 & 1 \\ 0 & -2 & 8 & 2 & 15 & -5 & 16 & 5 \\ 4 & -2 & 8 & 6 & -24 & 5 & 12 & 3 \\ 0 & 6 & 0 & 0 & 30 & 2 & 0 & -13 \\ 0 & -8 & 0 & 0 & -24 & -10 & 0 & -8 \\ 0 & \multicolumn{7}{c}{\underbrace{4 \quad 0 \quad 0 \quad 8 \quad 6 \quad 0 \quad 12}_{L^{sub}}} \end{array}\right] \begin{bmatrix} 1 \\ k_{11} \\ k_{22} \\ k_{33} \\ k_{i1} \\ k_{i5} \\ k_{i9} \\ k_{i10} \end{bmatrix} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}, \tag{14a}$$

where the interacting gains $k_{i1}, \ldots, k_{i10}$ are determined by

$$k_{i1} = k_{11}k_{22} - k_{21}k_{12},$$
$$k_{i5} = k_{11}k_{33} - k_{31}k_{13},$$
$$k_{i9} = k_{22}k_{33} - k_{32}k_{23},$$
$$k_{i10} = k_{11}k_{22}k_{33} + k_{21}k_{32}k_{13} + k_{31}k_{12}k_{23} - k_{11}k_{32}k_{23} - k_{21}k_{12}k_{33} - k_{31}k_{22}k_{13}. \tag{14b}$$

(2) Check a necessary condition of complete pole-assignability. From the SOF equation $L\boldsymbol{k} = \boldsymbol{a}$ in (14a), one can check a necessary condition of complete SOF pole-assignability. In the rank test of Plücker submatrix L^{sub} (where the 12 zero columns correspondent to the 12 degenerate variables $k_{21}, k_{31}, \ldots, k_{23}, k_{i2}, \ldots, k_{i8}$ are excluded),

$$rank\, L^{sub} < 8.$$

Thus, this generically pole-assignable 3-input, 3-output, 8th-order ($mp > n$) system [15,Theorem 4.1] is *intrinsically none pole-assignable* over any real and complex SOF.

This example clearly shows a case of *invalidity* of generic SOF pole-assignability as worried in [9,10]. □

Remark 2: As mentioned in Remark 1, a 3rd-order NR in (14b) can be re-written by a QR among following 6 various expressions:

$k_{i10} = k_{11}k_{i9} - k_{12}k_{i8} + k_{13}k_{i7},$
or $k_{i10} = - k_{21}k_{i6} + k_{22}k_{i5} - k_{23}k_{i4},$
or $k_{i10} = k_{31}k_{i3} - k_{32}k_{i2} + k_{33}k_{i1},$
or $k_{i10} = k_{11}k_{i9} - k_{21}k_{i6} + k_{31}k_{i3},$
or $k_{i10} = - k_{12}k_{i9} + k_{22}k_{i5} - k_{32}k_{i2},$
or $k_{i10} = k_{13}k_{i7} - k_{23}k_{i4} + k_{33}k_{i1},$
(where $k_{i2} = k_{11}k_{23} - k_{13}k_{21}$, $k_{i3} = k_{12}k_{23} - k_{13}k_{22}$, $k_{i4} = k_{11}k_{32} - k_{12}k_{31}$, $k_{i6} = k_{12}k_{33} - k_{13}k_{32}$, $k_{i7} = k_{21}k_{32} - k_{22}k_{31}$, $k_{i8} = k_{21}k_{33} - k_{23}k_{31}$).

5. NON-DETERMINISTIC COMPUTATION CASE IN *mp* > *n* SYSTEMS

We shall show another invalid (non-deterministic singular computation) case in the sufficient condition of generic SOF pole-assignment $mp > n$, which depends upon "the manner of selection" *of pre-assignable surplus variables* of the degree freedom, $mp - n$ in the equation $L\boldsymbol{k} = \boldsymbol{a}$.

Example 2: In a generically SOF pole-assignable system (of 3-input, 3-output, 8-th order, strictly proper transfer function matrix $G(s)$),

$$G(s) = \begin{bmatrix} \dfrac{1}{s} & \dfrac{s+1.5}{(s-2)^2} & 0 \\ \dfrac{s+0.5}{s^2-3} & \dfrac{2(s-1)}{s^2+2} & 0 \\ 0 & 0 & \dfrac{1}{s+1} \end{bmatrix}$$

we shall show that certain (bad) selection of a pre-assignable surplus variable in the degree freedom $mp - n = 9 - 8 = 1$ can yields non-deterministic (pathological) computation for real solutions !

(1) Construction of Grassmann space parameter L and OF gain vector $\boldsymbol{k}$ in $L\boldsymbol{k} = \boldsymbol{a}$. By the same way shown in Example 1, the SOF equation $L\boldsymbol{k}=\boldsymbol{a}$ is constructed by

$$\underbrace{\left[\begin{array}{c:ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 2.5 & -0.5 & -4 & -2 & 1 & 1 & 1 & 1 & 2 & 0 \\ 2 & -2 & 0.5 & 0.5 & -6 & 0 & -7 & -2 & 1.5 & -1.5 & -6 & 1 \\ -5 & 2 & -2.5 & -0.5 & 16 & 2 & -10.75 & 0 & -1 & 2 & 0 & -8 \\ 5 & -5 & -7.5 & -2.5 & -2 & -7 & 9.25 & 2 & -1.5 & -2.5 & 16 & -2.75 \\ 6 & 5 & -15 & 1 & -12 & 12 & -7.5 & -7 & -6 & 0 & -18 & 12 \\ -6 & 6 & -9 & 1 & 6 & -6 & -12.5 & 12 & -9 & 1 & 6 & -19.5 \\ 0 & -6 & 0 & 0 & 0 & 0 & 6 & -6 & 0 & 0 & 0 & 6 \end{array}\right]}_{L^{sub}} \begin{bmatrix} 1 \\ k_{11} \\ k_{21} \\ k_{12} \\ k_{22} \\ k_{33} \\ k_{i1} \\ k_{i5} \\ k_{i6} \\ k_{i8} \\ k_{i9} \\ k_{i10} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}, \qquad (15a)$$

where
$$\begin{aligned} &k_{i1} = k_{11}k_{22} - k_{12}k_{21}, \quad k_{i5} = k_{11}k_{33} - k_{13}k_{31}, \\ &k_{i6} = k_{12}k_{33} - k_{13}k_{32}, \quad k_{i8} = k_{21}k_{33} - k_{23}k_{31}, \\ &k_{i9} = k_{22}k_{33} - k_{23}k_{32}, \\ &k_{i10} = k_{11}k_{22}k_{33} + k_{12}k_{23}k_{31} + k_{13}k_{21}k_{32} \\ &\qquad - k_{11}k_{23}k_{32} - k_{12}k_{21}k_{33} - k_{13}k_{22}k_{31} \end{aligned} \qquad (15b)$$

(2) Check a necessary condition of complete pole-assignability. In rank test of the Plücker submatrix L^{sub},

$$rank\, L^{sub} = 8. \qquad (16)$$

Thus, a necessary condition of complete SOF pole-assignability is satisfied.

(3) Check deterministic computation condition. In (15a) and (15b), we should be careful that the 4 QRs,

$$\begin{aligned} &k_{i4} = k_{11}k_{33} - k_{13}k_{31}, \quad k_{i6} = k_{12}k_{33} - k_{13}k_{32}, \\ &k_{i8} = k_{21}k_{33} - k_{23}k_{31}, \quad k_{i9} = k_{22}k_{33} - k_{23}k_{32}, \end{aligned} \qquad (17a)\square$$

where the variables $k_{13}, k_{23}, k_{31}, k_{32}$ are not exposed in $L\boldsymbol{k}=\boldsymbol{a}$ in this degenerate system, can produce "non-deterministic (singular) computation problem" for obtaining real solutions. In other words, let

$$k_{13}k_{31}=\alpha_1,\quad k_{13}k_{32}=\alpha_2,\quad k_{23}k_{31}=\alpha_3,\quad k_{23}k_{32}=\alpha_4 \qquad (17b)$$

then these 4 equations provide (1-dimensional) infinite number of solutions if $\alpha_1/\alpha_2 = \alpha_3/\alpha_4$, but have no solutions if $\alpha_1/\alpha_2 \neq \alpha_3/\alpha_4$ over arbitrary values of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$; see Remark 4.

To avoid this non-deterministic computation case, we should be careful to the *selection manner of the pre-assignable surplus variables* (which can get arbitrary real values within the degree freedom $mp - n$ in the equation $L\boldsymbol{k}=\boldsymbol{a}$). From (17a) and (17b), in this degenerate 3-input, 3-output system (which has "1" surplus variables in the degree freedom 3×3 – 8 = 1), we can easily discern the good pre-assignable area and bad pre-assignable area for the selection of surplus variable. See the Fig. 3 where the slash-shaded areas indicate good pre-assignable areas for deterministic computation and the dot-shaded area indicates bad pre-assignable area of possible non-deterministic (singular) computation.

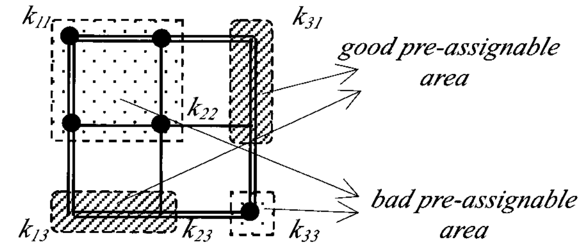


Fig. 3. Pre-assignable areas of a surplus gain-variable.

Table 1a. Number (#) of real solutions in deterministic good pre-assignable area

Selection of surplus gain	k_{13}	k_{32}	k_{13}	k_{23}
# Real solutions	**12**	**10**	**10**	**10**
# Complex solutions	12	11	4	12

Table 1b. Number (#) of real solutions in nondeterministic bad pre-assignable area

Selection of surplus gain	k_{11}	k_{21}	k_{12}	k_{22}	k_{33}
# Real solutions	**none**	**2**	**4**	**none**	**4**
# Complex solutions	12	8	6	10	4

(For the numerical iterative computation, "NSolve" of MATHEMATCA 5.0 is applied.)

Let the desired closed-loop system poles be set by

$$(-1, -1, -1, -1, -1, -1, -1, -1) \in (s+1)^8$$

for simplicity, and let any surplus gain-variable in the degree freedom be set by $k_{ij} = 1$, then the numbers of real solutions K in the 2 cases (of deterministic good pre-assignable area and non-deterministic bad pre-assignable area) are compared in Table 1a and Table 1b.

From the Tables, it is shown that over the SOF equations in (15a) and (15b), the selection of surplus variable in deterministic good pre-assignable area produces *sufficient 10 ~ 12 real solutions*, meanwhile the selection in non-deterministic bad pre-assignable area yields *only deficient 0 ~ 4 real solutions.*

Remark 3: From (16), by inverse function theorem [20], the *generic* pole-assignability in $mp > n$ and $rank\ L^{sub} = n$ implies *arbitrary* pole-assignability *in a weak sense* that the desired real gain matrices K are determined by arbitrary real coefficient vector $\boldsymbol{a}$, but which has only open small neighborhoods, *as open dense sets*, to guarantee real gain matrices K from the Definition 2).

Remark 4: We can easily check that the variety in non-deterministic or 1-dimensional deterministic computation case in (17b) is "singular" by Jacobian ($\boldsymbol{J}$) rank test over 4 eqns.(or generators), $g_1 = k_{13}k_{31} - \alpha_1$, $g_2 = k_{13}k_{32} - \alpha_2$, $g_3 = k_{23}k_{31} - \alpha_3$, $g_4 = k_{23}k_{32} - \alpha_4$. Hence it is obtained that $det\boldsymbol{J} = |\partial(g_1, g_2, g_3, g_4)/\partial(k_{13}, k_{23}, k_{31}, k_{32})| = 0$ [21, Theorem 12.2.20].

Remark 5: It is interesting to see in the iterative numerical computation algorithm of "NSolve" in MATHEMATCA 5.0 that in the non-deterministic (singular) computation case of Table 1b, *some (complex and real) solutions* still come out in place of that *no solutions* should be come out if algebraic computation can be carried out.

6. m-DOF COMPENSATION FOR SOF NONE POLE-ASSIGNABLE $mp > n$ SYSTEM

6.1. Induction of new sufficient condition of generic m-DOF pole-assignment

From the S.Q.R. in (3) and (4), a new sufficient condition of generic m-DOF pole-assignment is directly induced from the well-known sufficient condition of generic SOF pole-assignment $mp > n$ by X. Wang [15] as follows. Recall that previous best sufficient condition of generic m-DOF pole-assignment (which does not allow partial SOF) was

$$mp + d(p+m) - \min\{r_m(p-1),\ r_p(m-1)\} > n+d$$

by Rosenthal and Wang in 1996 (where $r_m = d - m[d/m]$ and $r_p = d - p[d/p]$ are the remainders of d divided by m and p, respectively)[22], and another best sufficient condition of generic m-DOF pole-assignment (which allows partial SOF) was

$$d \geq \lceil (n-\varphi)/\max\{m, p\} \rceil$$

by Söylemez and Munro in 2002 (where $\varphi = \max\{m, p\} + \lfloor \max\{m, p\}/2 \rfloor + \ldots + \lfloor \max\{m, p\}/\min\{m, p\} \rfloor$ and $\lfloor x \rfloor$ indicates the nearest integer lower than or equal to x, and $\lceil x \rceil$ indicates the nearest integer greater than or equal to x)[23]. For the definitions of complete DOF pole-assignment and generic DOF pole-assignment, refer to [13].

Theorem (induction of new sufficient condition of generic m-DOF pole-assignment)**:** In the m-input, p-output, n-th order linear strictly proper systems, new sufficient condition of generic pole-assignment by minimum d-th order DOF is induced by

$$(m+d)(p+d) > n+d \quad \text{and} \quad rank\, L^{aug.sub} = n+d \quad (18a)$$

when $d = 1$, or is induced by

$$(m+d)(p+d) > n+d+1 \text{ and } rank\, L^{aug.sub'} = n+d+1 \quad (18b)$$

when $d \geq 2$ (where $L^{aug.sub} \in \mathbb{R}^{(n+d)\times\sigma*}$ indicates first column, and first row curtailed Plücker submatrix of L^{aug} and $L^{aug.sub'} \in \mathbb{R}^{(n+d+1)\times\sigma*}$ indicates first column curtailed Plücker submatrix of $L^{aug} \in \mathbb{R}^{(n+d)\times(\sigma*+1)}$).

Proof: In m-input, p-output, n-th order linear strictly proper systems, $mp > n$ is sufficient condition of *generic* pole-assignment by SOF [15,Theorem 4.1]. Applying the S.Q.R. in (3) and (4) [10,Theorem 1] to $mp > n$, new sufficient condition of *generic* pole-assignment by minimum d-th order DOF is induced by $(m+d)(p+d) > n+d$ or $(m+d)(p+d) > n+d+1$ according to the strictly properness or properness of the 1st order dynamic elements $e_1(s), \ldots, e_d(s)$.

In [10,Lemma 3 and Lemma 4], it is proved that if $d = 1$, "strictly proper" dynamic element is allowable without intrinsic column rank reduction of $L^{aug.sub}$, but if $d \geq 2$, "proper" dynamic elements should be selected preventing intrinsic column rank reduction of $L^{aug.sub}$. Thus, in the case of $d \geq 2$, their augmented SOF linear systems are to be "proper".

Therefore, the new sufficient condition of generic pole-assignment by minimum d-th order DOF (without any *none* pole-assignability by rank deficiency of Plücker sub-matrices of L^{aug}) is obtained by $(m+d)(p+d) > n+d$ and $rank\, L^{aug.sub} = n+d$ when $d = 1$, or obtained by $(m+d)(p+d) > n+d+1$ and $rank\, L^{aug.sub'} = n+d+1$ when $d \geq 2$. □

From the Remark 3, it is natural question whether the *generic* pole-assignment conditions by m-DOF in (18a) and (18b) of this Theorem are also to be *complete* pole-assignment conditions by m-DOF, or not.

6.2. Construction algorithm of L^{aug} in $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$

From the S.Q.R. in (3) and (4), the (dynamic) Grassmann space parameter L^{aug} in $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ can be constructed in the very similar way with the construction algorithm of (static) Grassmann space parameter L in $L\boldsymbol{k} = \boldsymbol{a}$ presented in Section 3.2.

Step 1: (*Determine properness or strictly properness of 1st order dynamic elements $e_1(s)$, ..., $e_d(s)$ for minimum d-th order DOF*)

Step 2 ~ Step 5: Same procedures with the Step 1 ~ Step 4 in Section 3.2 (for the construction algorithm of (static) Grassmann space parameter L).

Example 3: In the SOF none pole-assignable system in Example 1,

$$G(s) = \begin{bmatrix} \frac{(s-1)^2}{s^3} & 0 & 0 \\ 0 & \frac{(s+2)^2}{s^3+2} & 0 \\ 0 & 0 & \frac{s+3}{s^2+2} \end{bmatrix}$$

let's construct the m-DOF compensator $H(s)^{min}$ for pole-assignment in arbitrary pole positions $(s+1)^9 = 0$.

Step 1: Let $d = 1$, then $(3+1)(3+1) > 8+1$, i.e., well-satisfied! Hence let's set a 1st order "strictly proper" dynamic element by

$$e_1(s) = 1/(s\text{-}2)$$

whose denominator's value, s-2, is relative prime with numerators and denominators with other transfer functions. Then augmented transfer function matrix $G(s)^{aug}$ of augmented linear systems is obtained by

$$G(s)^{aug} = \begin{bmatrix} \frac{(s-1)^2}{s^3} & 0 & 0 & 0 \\ 0 & \frac{(s+2)^2}{s^3+2} & 0 & 0 \\ 0 & 0 & \frac{s+3}{s^2+2} & 0 \\ 0 & 0 & 0 & \frac{1}{s-2} \end{bmatrix}. \tag{19}$$

Step 2 ~ Step 4: Same procedures with Step 1 ~ Step 3 shown in Example 1 for construction of L.

Step 5: Through Step 1 and Step 4, the Grassmann space parameter L^{aug} in $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ is constructed by

$$\left[\begin{array}{c|ccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 7 & -2 & -6 & 2 & 0 & -1 & -2 & 5 & 4 & 3 & 1 & 1 & 1 & 1 & 0 \\ -4 & -8 & -4 & 2 & 2 & -5 & -7 & 3 & 2 & 6 & 0 & 3 & 2 & 1 & 7 & 1 \\ 4 & 2 & -8 & 2 & 0 & 2 & 15 & -2 & -20 & 8 & 2 & -7 & -1 & -5 & 16 & 5 \\ -8 & 10 & -16 & -12 & 4 & -2 & -8 & -2 & -24 & 8 & 6 & -19 & 0 & 5 & 12 & 3 \\ 0 & -20 & 0 & 0 & 0 & -4 & -14 & 6 & 0 & 0 & 0 & 18 & -2 & 2 & 0 & -13 \\ 0 & 20 & 0 & 0 & 0 & 24 & 26 & -8 & 0 & 0 & 0 & 28 & -8 & -10 & 0 & -8 \\ 0 & -8 & 0 & 0 & 0 & -16 & -12 & 4 & 0 & 0 & 0 & -24 & 8 & 6 & 0 & 12 \end{array}\right]$$

$L^{aug.sub}$

$$\begin{bmatrix} 1 \\ k_{11} \\ k_{22} \\ k_{33} \\ k_{44} \\ k_{j1} \\ k_{j2} \\ k_{j3} \\ k_{j4} \\ k_{j5} \\ k_{j6} \\ k_{j7} \\ k_{j8} \\ k_{j9} \\ k_{j10} \\ k_{j11} \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 36 \\ 84 \\ 126 \\ 126 \\ 84 \\ 36 \\ 9 \\ 1 \end{bmatrix}, \tag{20a}$$

where the interacting gains $k_{j1}, \ldots, k_{j10}$ are determined by

$$k_{j1} = \begin{vmatrix} k_{11} & k_{21} \\ k_{12} & k_{22} \end{vmatrix}, \quad k_{j2} = \begin{vmatrix} k_{11} & k_{31} \\ k_{13} & k_{33} \end{vmatrix}, \quad k_{j3} = \begin{vmatrix} k_{11} & k_{41} \\ k_{14} & k_{44} \end{vmatrix},$$

$$\cdots\cdots$$

$$k_{j11} = \begin{vmatrix} k_{11} & k_{21} & k_{31} & k_{41} \\ k_{12} & k_{22} & k_{32} & k_{42} \\ k_{13} & k_{23} & k_{33} & k_{43} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{vmatrix}. \tag{20b}$$

In the rank test of the Plücker submatrix, it is shown that

$$rank\ L^{aug.\ sub} = 9\ (= n+d = 8+1).$$

Thus, the generic pole-assignment condition in (18a) is satisfied !

6.3. Construction algorithm of m-DOF

The m-DOF compensator (or controller) $H(s)^{min}$ is constructed through following further 3 steps of Step 6 ~ Step 8.

Step 6 (*Pre-assignment of surplus variables for stable poles of m-DOF*):

From the induced sufficient condition $(m+d)$ $(p+d) > n+d$, the degree of gain-variable freedom is obtained by

$(m+d)(p+d) - (n+d) = (3+1)(3+1) - (8+1) = 7.$

Hence from the general gain formula (G.G.F.) in (6), the pole of minimum 1st order DOF, $H(s)^{min}$ can be pre-assigned by the zero of an equation

$$1 - e_1(s)W_1^{ij}(s) = 0,$$

where $W_1^{ij}(s) = k_{44}$ for all $i, j = 1, 2, 3$.

Let the desired pole of $H(s)^{min}$ be "− 2" in LHP. Then one surplus variable k_{44} is pre-assigned by "4" from $(s-2) - 1 \cdot (-k_{44}) = s+2$.

Step 7 (*Pre-assignment of other surplus variables in good pre-assignable area*):

Considering the good pre-assignable areas of surplus gain-variables of 3-input, 3-output system in Fig.3, the (possible) good pre-assignable areas of surplus gain-variables in augmented 4-input, 4-output system can be variously figured like Fig. 4(a), Fig. 4(b), … , in below.

From Fig. 4(a) and Fig. 4(b), it is easily observed that 6 gain-variables in any upper or lower triangular position in the matrix K^{aug} shall fully prevent any non-deterministic (singular) computations. So we shall pre-assign the variables in upper part by value "1" like

$(k_{31}, k_{32}) = (1, 1)$ over $(k_{13}, k_{23}, k_{31}, k_{32})$,
$(k_{42}, k_{43}) = (1, 1)$ over $(k_{24}, k_{34}, k_{42}, k_{43})$,
$(k_{21}, k_{41}) = (1, 1)$ over $(k_{12}, k_{14}, k_{21}, k_{41})$.

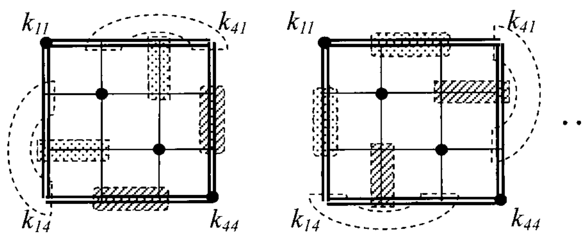


Fig. 4. Various good pre-assignable areas in augmented (3+1)-input, (3+1)-output system.

Applying "NSolve" in MATHEMATCA 5.0 software to the eqns., $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ in (20a) and correspondent NRs in (20b), the 26 number of real solutions of K^{aug} are obtained. Hence, a best solution (which has lowest ratio between maximal values and minimum values) is selected by

$$K^{aug} = \begin{bmatrix} 2.790665812976755 & 1 & 1 & 1 \\ 3.963140666380242 & 3.148255874359452 & 1 & 1 \\ 4.314877112434576 & 2.741207399335048 & 1.061078312663792 & 1 \\ -1.135732011608804 & -1.756083538098622 & -2.845735498678207 & 4 \end{bmatrix}. \quad (21)$$

Step 8 (*Construction of stable m-DOF*):

From the general gain formula (G.G.F.) in (6), each element of m-DOF, $H(s)^{min}$ is constructed by

$$H_{11}(s)^{min} = k_{11} - \left(k_{14} \frac{\frac{1}{s-2}}{1 + \frac{1}{s-2} k_{44}} k_{41} \right)$$
$$= \frac{2.790665812976755\, s + 6.717063637562314}{s+2},$$

$$H_{12}(s)^{min} = k_{12} - \left(k_{14} \frac{\frac{1}{s-2}}{1 + \frac{1}{s-2} k_{44}} k_{42} \right)$$
$$= \frac{3.963140666380242\, s + 9.682364870859107}{s+2},$$

. . . .

$$H_{33}(s)^{min} = k_{33} - \left(k_{34} \frac{\frac{1}{s-2}}{1 + \frac{1}{s-2} k_{44}} k_{43} \right)$$
$$= \frac{1.0610783126637924\, s + 4.967892124005791}{s+2}.$$

In this way, the m-DOF, $H(s)^{min}$ for pole-assignment of $G(s)$ in arbitrary pole positions $(s+1)^9 = 0$ is constructed by

$$H(s)^{min} = \begin{bmatrix} \frac{2.790665812976755s+6.717063637562314}{s+2} & \frac{s+3.756083538098622}{s+2} & \frac{s+4.8457354986782075}{s+2} \\ \frac{3.963140666380242s+9.682364870859107}{s+2} & \frac{3.148255874359452s+8.052595286817526}{s+2} & \frac{s+4.8457354986782075}{s+2} \\ \frac{4.314877112434576s+9.765486236477956}{s+2} & \frac{2.741207399335048s+7.238498336768718}{s+2} & \frac{1.0610783126637924s+4.967892124005791}{s+2} \end{bmatrix} \quad (22)$$

Remark 7: This systematic construction algorithm of m-DOF, $H(s)^{min}$ under the frame of Grassmann space parameter L^{aug} in $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ is comparable with the recent best parametric construction algorithm by Söylemez and Munro in 2002 [23] (which is less-systematic than ours, without using the (basis–free Grassmannian) SOF and m-DOF parameters, L and L^{aug}[11,Theorem 4.1]). But this parametric construction algorithm still need further studies for practical applications on following inquiries: *1*) Are the sufficient conditions of generic pole-assignment in Theorem in Section 6.1 also to be the sufficient conditions of *complete* pole-assignment by SOF and m-DOF ? *2*) What are the sufficient conditions of good pre-assignment of the surplus gain-variables in the degree freedoms mp - n and $(m+d)(p+d)$ - $(n+d)$ (or $(m+d)(p+d)$ - $(n+d+1)$)[21,24,25] ?

Remark 8: As seen in (21), the values of the m-DOF for 3-input, 3-output system with 16 digit high precision are impractical ones, and even the rigorous values of 4 ~ 6 digit high precision do not assign the system poles at the exact pole positions by heavy nonlinear multiplications of the total 3rd order interacting gains k_{j7}, ... , k_{j10} and total 4th order interacting gain k_{j11}. In other words, the real coefficient parameters $\boldsymbol{a}^{aug}$ (= $L^{aug}\boldsymbol{k}^{aug}$) for the system poles are so sensitive over the slight variations of OF gains k_{11}, ..., k_{mp} in MIMO systems. So we can say that the SOF compensation in $\min\{m, p\} \geq 4$ systems and the m-DOF compensation in $\min\{m, p\} \geq 3$ systems for pole-assignments under (Grassmannian) full-rank feedback configuration *are actually impractical ones* by the extremely vulnerable sensitivity problem over their OF gains, as mentioned in [8].

7. CONCLUSIONS

Major outcomes of this paper are summarized as follows.

1) A general construction algorithm of the Grassmann space parameters (so-called, the Plücker matrices) "L" and "L^{aug}" in $L\boldsymbol{k} = \boldsymbol{a}$ and $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ is presented, respectively, for numerical checking of necessary conditions of complete pole-assignments and for discernment of deterministic computation conditions of real solutions in linear OF systems.

2) Through the construction of the Grassmann space parameter L in $L\boldsymbol{k} = \boldsymbol{a}$, it is shown that certain generically SOF pole-assignable strictly proper $mp > n$ system is actually none pole-assignable over any

real and complex OF.

3) From 1), it is also shown in strictly proper linear systems that deterministic or non-deterministic (singular) computations of pole-assignable real OF gain matrices K in $L\boldsymbol{k} = \boldsymbol{a}$ depend upon the selection manner of the surplus gain-variables in the degree freedom $mp - n$.

4) From 1), it is shown that the none pole-assignable system by any real and complex OF in 2) can be arbitrary pole-assignable system via (real) m-DOF under full-rank of some submatrix of L^{aug} and deterministic computation of the augmented real OF gain matrices K^{aug} in $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$.

5) From 1), it is shown that a mathematical interesting issue, concrete construction algorithm of 2nd-order quadratic relations (QRs) as the Grassmann variety in Plücker coordinates [16-18] can be deduced from arbitrary-order nonlinear relations (NRs) in the case of SOF linear systems — which relations are formulated by all kinds of $N\times N$ sub-determinants of $m\times p$ SOF matrix K (where N = 2, ... , min{m, p}) (see the Remark 1 and Remark 2).

APPEDIX A
(Terminologies related with Grassmannian)

Grassmann space. Consider the set of all complex d-dimensional planes in $\mathbb{C}^n$. These can be identified with equivalent classes $[\boldsymbol{x}]$ $(:= [x^1, \ldots, x^d])$ of $(d\text{-}1)$-dimensional linearly independent planes in $\mathbb{C}^n$, where d-pairs $\{(x_1^{\,1}, x_2^{\,1}, \ldots, x_n^{\,1}), \ldots, (x_1^{\,d}, x_2^{\,d}, \ldots, x_n^{\,d})\}$ being equivalent with $\{(y_1^{\,1}, y_2^{\,1}, \ldots, y_n^{\,1}), \ldots, (y_1^{\,d}, y_2^{\,d}, \ldots, y_n^{\,d})\}$ are regarded as being equivalent if they span the same d-dimensional subspace. This set of d-dimensional subspaces is called Grassmannian (or also, *Grassmann space*), *Grass*(d, n).

Grassmann variety in Plücker embedding. The Grassmann space can be thought of as a generalization of projective space. And also we can consider a map (called Plücker map) which sends a d-dimensional plane (simply, d-plane) $\pi = \mathbb{C}\{v_1, \ldots, v_d\} \subset \mathbb{C}^n$ to multivector $v_1 \wedge \ldots \wedge v_d$ like

$$p: \ Grass(d, n) \ \rightarrow \ \mathbb{P}(\wedge^d\mathbb{C}^n) = \mathbb{P}^\sigma$$

(where $\sigma = \binom{n}{d} - 1$). This map p has nonzero differential and is known embedding. The image of this map is provided by certain equations called *quadratic Plücker relations*(QRs), and the algebraic projective variety of these equations are named by *Grassmann variety*, and the coordinates in the projective space $\mathbb{P}^\sigma$ are named by *Plücker coordinates* (or Grassmann coordinates).

Schubert condition and Schubert variety. Let $A_0 \subset A_1 \subset \ldots \subset A_d$ be a strictly increasing sequence(or flag) of $(d+1)$ linear spaces in $\mathbb{P}^n$. A d-plane L in $\mathbb{P}^n$ is said to satisfy the *Schubert condition* defined by this sequence if $\dim(L \cap A_i) \le i$ for all i. The set of all such d-planes L corresponds to a subset of *Grass*(d, n), which is denoted by $\Omega(A_0, A_1, \ldots, A_d)$ called *Schubert variety*.

Grassmann representative. If W is any nonzero m-dimensional subspace of V, then any nonzero decomposable element (in the exterior product of m vectors in W),

$$x_1 \wedge \ldots \wedge x_m, \quad x_i \in W, \quad i = 1, \ldots, m$$

is called *a Grassmann representative* for W.

APPENDIX B (Binet-Cauchy theorem)

Suppose $A \in M_{n,p}(\mathrm{F})$, $B \in M_{p,m}(\mathrm{F})$, and $C = AB \in M_{n,m}(\mathrm{F})$.

If $1 \le r \le \min\{n, m, p\}$, $\alpha \in Q_{r,n} = \binom{n}{r}$,

and $\beta \in Q_{rm} = \binom{m}{r}$,

then, $$det\,(C[\alpha|\beta]) = \sum_{\omega \in Q_{r,p}} det\,(A[\alpha|\omega])\; det\,(B[\beta|\omega])$$

(where $C[\alpha|\beta]$ indicates submatrix of C lying in rows α and columns β, and $Q_{r,p}$ indicates totality of strictly increasing sequences of r integers chosen from 1, ... p) [17].

APPENDIX C (Proof of Lemma 2)

The total number T of all $N\times N$ submatrices (where N = 1, ... , min{m, p}) in a $m\times p$ matrix is described by (A.1a), and is computed by (A.1b) as follows:

$$T = \sum_{N=1}^{\min\{m,p\}} \binom{m}{N} \times \binom{p}{N} \qquad \text{(A.1a)}$$

$$= \binom{m+p}{m} - 1. \qquad \text{(A.1b)}$$

The equality between (A.1a) and (A.1b) is derived from following rationale.

Consider all $N\times N$ submatrices by "all combinative selected sets" of N number of rows (1, 2, ...) and N number of columns (1′, 2′, ...) in the (m+p) number of set, {1, 2, ... , m; 1′, 2′, ... , p′}(where N = 1, ... , min{m, p}).

i) $m \le p$ case: See the Fig. A.1 below. In thin rectangular enclosure, a selection of set, {4, ... , m; 1′, 2′, 3′} is equally counted by the selection of set, {1, 2, 3; 1′, 2′, 3′} which implies a 3×3 submatrix. And in thick rectangular enclosure, a selection of set, {3, ... , m-1; 3′, (p-2)′, p′} is equally counted by the selection of set, {1, 2, m; 3′, (p-2)′, p′} of a 3×3 submatrix.

In this way, all $N\times N$ submatrices are selected by all combinative selections of m number of set in

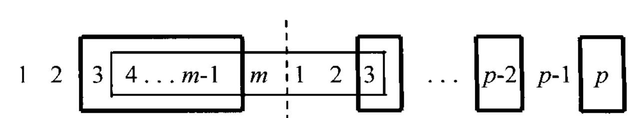


Fig. A.1. Two (kinds of) selections for 3×3 submatrices.

the $(m+p)$ number of set, $\{1, 2, \dots, m; 1', 2', \dots, p'\}$, except only one selection of set $\{1, 2, \dots, m\}$ which does not formulate a submatrix.

ii) $p \le m$ case: By the same way with *i*), one can select all $N \times N$ submatrices by all combinative selections of p number of set in the $(m+p)$ number of set, $\{1, 2, \dots, m; 1', 2', \dots, p'\}$, except only one selection of set $\{1', 2', \dots, p'\}$ which does not formulate a submatrix.

The outcomes of *i*) and *ii*) prove the equality between (A.1a) and (A.1b). Therefore, the total number r of all $N \times N$ submatrices (where $N = 2, \dots, \min\{m, p\}$) in a $m \times p$ matrix is obtained by

$$r = \binom{m+p}{m} - mp - 1. \qquad \square$$

REFERENCES

[1] C. I. Byrnes, "Pole-assignment by output feedback," *Lecture Notes in Control and Infor. Siences,* vol. 135, pp. 31-78, Spring-Verlag, Berlin, Heideberg, New York, 1989.

[2] D. S. Bernstein, "Some open problems in matrix theory arising in linear systems and control," *Linear Algebra and its Applications*, pp. 409-432, 1992.

[3] H. Kimura, "Pole assignment by output feedback: A longstanding open problem," *Proc. of Conf. Decision Control*, vol. 12, pp. 2101-2105, 1994.

[4] V. Blondel, M. Gevers, and A. Linquist, "Survey on the state of systems and control," *European J. Control*, vol. 1, pp. 5-23, 1995.

[5] V. L. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback: A survey," *Automatica,* vol. 33, pp. 125-137, 1997.

[6] J. Rosenthal and J. C. Willems, "Open problems in the area of pole placement," *Open problems in mathematical systems and control (Blondel, Sontag, Vidyasagar and Willems(Eds): Springer)*, vol. 37, pp. 181-191, 1999.

[7] R. Hermann and C. F. Martin, "Application of algebraic geometry to system theory. Part-I," *IEEE Trans. on Automatic Control,* vol. 22, pp. 19-25, 1977.

[8] Y. Yang and A. L. Tits, "Generic pole assignment may produce very fragile designs," *Proc. of the 37rd conf. on Decision and Control,* Tampa, FL, USA, pp. 1745-1746, 1998.

[9] L. Carotenuto, G. Franze, and P. Muraca, "Some results on the genericity of the pole assignment problem," *System and Control Letters,* vol. 42, pp. 291-298, 2001.

[10] S.-W. Kim, "Mismatching problem between generic pole-assignabilities by static output feedback and dynamic output feedback in linear system," *International Journal of Control, Automation, and System,* vol. 3, no. 1, pp. 56-69, 2005.

[11] C. Giannakopoulos and N. Karcanias, "Pole assignment of strictly and proper linear system by constant output feedback," *International Journal of Control,* vol. 42, pp. 543-565, 1985.

[12] N. Karcanias and C. Giannakopoulos, "Grassmann invariants, almost zeros and the determinantal zeros, pole assignment problems of linear multivariable systems," *International Journal of Control*, vol. 40, pp. 673-698, 1984.

[13] J. C. Willems and W. H. Hesselink, "Generic properties of the pole assignment problem," *Proc. IFAC*, Helsinki, Finland, pp.1725-1729, 1978.

[14] A. S. Morse, W. A. Wolovich, and B. D. O. Anderson, "Generic pole assignment: Preliminary results," *IEEE Trans. on Automatic Control,* vol. 28, no. 4, pp. 503-506, 1983.

[15] X. Wang, "Pole placement by static output feedback," *Journal of mathematical systems, Estimation and Control,* vol. 2, no. 2, pp. 205-218, 1992.

[16] J. V. Chipalkatti, "Notes on Grassmannians and Schubert varieties," *Queen's papers in Pure and Applied Math.* 13, no. 119, 2001.

[17] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities,* Dover Publications, Inc., 1969.

[18] S. L. Kleiman and D. Laksov, "Schubert calculus," *America Math. Monthly*, vol. 79, pp. 1061-1082, 1972.

[19] A. G. J. McFarlane and N. Karcanias, "Pole and zeros of linear multivariable systems: The algebraic, geometric and complex-variable theory," *International Journal of Control*, vol. 24, pp. 33-74, 1976.

[20] M. W. Hirsch, *Differential Topology*, Springer-Verlag, 1980.

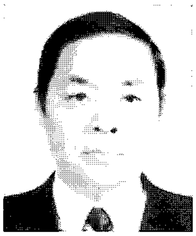
[21] S. Billey and V. Lakshmibai, *Singular Loci of Schubert Varieties*, Birkhäuser, 2000.

[22] J. Rosenthal and X. Wang, "Output feedback pole placement with dynamic compensators," *IEEE Trans. on Automatic Control*, vol. 41, pp. 830-843, 1996.

[23] M. T. Söylemez and N. Munro, "A parametric solution to the pole assignment using dynamic output feedback," *IEEE Trans. on Automatic Control,* vol. 46, pp. 711-723, 2001.

[24] B. Huber and J. Verschelde, "Pieri homotopies for problems in enumerative geometry applied to pole placement in linear systems control," *SIAM J. Contr. Opt.,* vol. 38, pp. 1265-1287, 2000.

[25] S. Billey and A. Postnikov, "Smoothness of Schubert varieties via patterns in root subsystems," *Advances in Applied Mathematics*, vol. 34, pp. 447-466, 2005.



Su-Woon Kim received the B.S. degree in Electrical Engineering (with submajor in Industrial Education) in 1974 and M.S. degree in Electrical Engineering in 1979, respectively from Seoul National University. He served as an instructor from 1980 to 1983 in Ulsan University, and received Ph.D. degree in Control Science and Dynamic Systems from University of Minnesota, Minneapolis in 1996, and then engaged in post-doctoral research at the same school. Since 2003, he has been with Department of Electrical Engineering at Cheju National University. His research interests include mathematical system theory and linear system theory (especially, related problems with pole-assignment and stabilization).