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Parameter Estimations in the Complementary Weibull Reliability Model

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Abstract. The Bayes estimators of the parameters included in the complementary Weibull reliability model are obtained. In the process of deriving Bayes estimators, the scale and shape parameters of the complementary Weibull distribution are considered to be independent random variables having prior exponential distributions. The maximum likelihood estimators of the desired parameters are derived. Further, the least square estimators are obtained in closed forms. Simulation study is made using Monte Carlo method to make a comparison among the obtained estimators. The comparison is made by computing the root mean squared errors associated to each point estimation. Based on the numerical study, the Bayes procedure seems better than the maximum likelihood and least square procedures in the sense of having smaller root mean squared errors.

Key Words : *Maximum likelihood procedure, Bayes procedure, least square method.*

1. INTRODUCTION

In 1972, Ciechanwicz wrote a paper on a lifetime model called *"the generalized gamma distribution"* having the shape parameter of negative value. The Weibull distribution of negative shape parameter can be considered as a special case of this

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distribution. Ciechanwicz has also derived the maximum likelihood estimators of the included parameters for complete samples. Twenty years later a lifetime model having the same analytical form called *"the complementary Weibull distribution"* (CWD) has been studied by Derapella (1993). Derapella (1993) has introduced two examples related to electronic component reliability and an example related to biomedical statistics.

At first sight the CWD is very similar to the Weibull distribution. It has the same pairs of parameters, namely the scale and shape parameters. Zain-Din (1994) has studied three versions of the CWD. These versions are: (*i*) the basic version (two parameters), (*ii*) the simplified version (one parameter), (*iii*) the generalized version (three parameters). He has derived the maximum likelihood estimator(s) of the considered parameter(s).

The main object of this article is to obtain the Bayes and least square estimators of the parameters included in the basic version. The maximum likelihood estimators of the parameters are obtained also. The estimators are obtained by using the data of type II censoring testing without replacement. Both Bayes and maximum likelihood estimators have no closed forms. Therefore, the numerical techniques should be used to calculate the estimators. On the other hand, the least square estimators of the parameters are obtained in closed forms. The performance of these estimators are compared. The comparison is made possible by writing programs on computer using Turbo Pascal and mathcad. In such computerized work a large simulation study using Monte Carlo method is used. The criterion of comparison is made possible by computing the root mean squared errors associated to the estimators using each method.

The paper is organized as follows. Section 2 introduces the complementary Weibull reliability model. The general formulation of the likelihood function of the observations and the maximum likelihood estimators are presented in section 3. Section 4 is devoted to the Bayes analysis. The least square estimators are given in section 5. Section 6 presents the simulation study and conclusion.

2. THE COMPLEMENTARY WEIBULL RELIABILITY MODEL

The reliability measures of the complementary Weibull reliability model are introduced in what follows.
The reliability function, $R(x|\alpha,\beta)$, is

$$R(x|\alpha,\beta)=\begin{cases}1-\exp\left(-\alpha x^{-\beta}\right), & x>0\\ 1, & x\le 0.\end{cases} \tag{2.1}$$

where $\alpha,\beta>0$.
The cumulative failure distribution function, $F(x|\alpha,\beta)$, is

$$F(x|\alpha,\beta)=\begin{cases}\exp\left(-\alpha x^{-\beta}\right), & x>0\\ 0, & x\le 0.\end{cases} \tag{2.2}$$

The probability density function, $f(x|\alpha,\beta)$, is

$$f(x|\alpha,\beta) = \alpha\,\beta\,x^{-\beta-1}\exp\left(-\alpha x^{-\beta}\right),\; x > 0,\; \alpha,\beta > 0 \tag{2.3}$$

Finally, the hazard rate function, $h(x|\alpha,\beta)$, is

$$h(x|\alpha,\beta) = \alpha\,\beta\,x^{-(\beta+1)}\left[\exp\left(\alpha x^{-\beta}\right)-1\right]^{-1},x > 0,\; \alpha,\beta > 0. \tag{2.4}$$

The hazard rate function is plotted in Figures 1 and 2 at different values of the scale and shape parameters. Figure 1 shows the case when $\beta = 1.5$ and $\alpha = 0.2,\ 1,\ 1.8,\ 4$ while figure 2 shows the case when $\alpha = 1$ and $\beta = 0.1,\ 0.3,\ 0.5,\ 1,\ 2,\ 2.5,\ 3$.

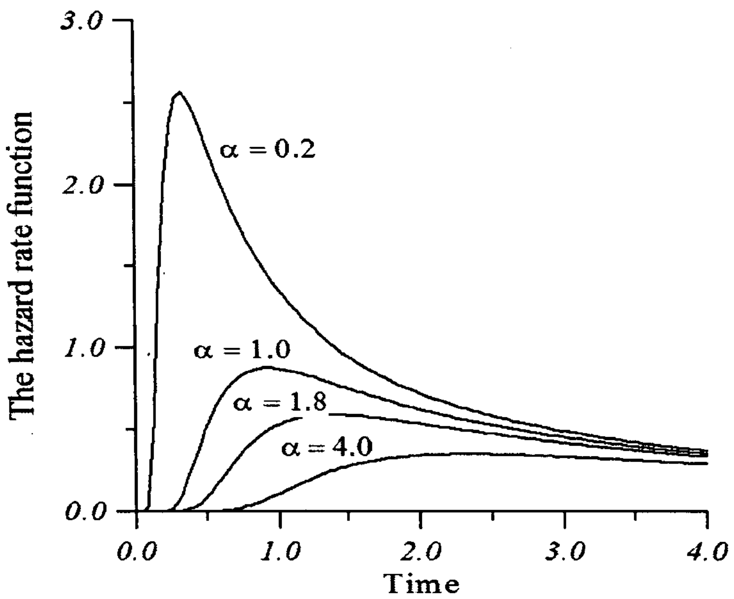

Figure 1. The hazard rate function when $\beta = 1.5$ and $\alpha = 0.2,\ 1.0,\ 1.8,\ 4.0$.

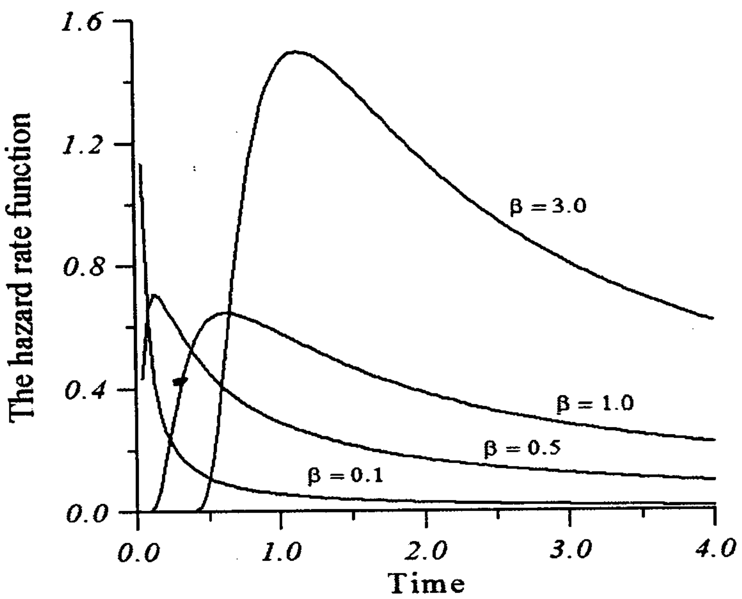

Figure 2. The hazard rate function when $\alpha = 1.0$ and $\beta = 0.1,\ 0.5,\ 1.0,\ 3.0$.

3. THE MAXIMUM LIKELIHOOD ESTIMATORS

Henceforth we shall consider the data of type II censoring testing without replacement. In such type of data, it is assumed that n identical items are put on the life test. The testing process is terminated at the time of r^{th} item failure. The number of observations r is decided before the data are collected. Assume the r times to failures are $x_1, x_2, \ldots, x_r$. Let x denote the information obtained from the life testing. It means the number of all items to be tested n, the number of failed items r, and their times to failure. That is $x = \{n, r, x_1, x_2, \ldots, x_r\}$. It is assumed also that the life time of each item has the distribution given by (2.2).

The likelihood function of x is, see Lawless (1982),

$$l(x) = \{R(x_r|\alpha,\beta)\}^{(n-r)} \left\{\prod_{i=1}^{r} f(x_i|\alpha,\beta)\right\}.$$

That is,

$$l(x) = (\alpha\beta)^r \left(\prod_{i=1}^{r} x_i\right)^{-(\beta+1)} \left\{1 - \exp\left(-\alpha x_r^{-\beta}\right)\right\}^{(n-r)} \exp\left(-\alpha \sum_{i=1}^{r} x_i^{-\beta}\right). \quad (3.1)$$

The maximum likelihood estimators of α and can be obtained by maximizing the likelihood function. In what follows we shall obtain the maximum likelihood estimators of α and β. The log-likelihood, denoted $L(x)$, is

$$L(x) = r(\ln(\alpha)+\ln(\beta))-\alpha\sum_{i=1}^{r} x_i^{-\beta}-(\beta+1)\sum_{i=1}^{r}\ln(x_i)+(n-r)\ln\left(1-\exp\left(-\alpha x_r^{-\beta}\right)\right). \tag{3.2}$$

Calculating the partial derivatives of L and equating each to zero we can get the following system of nonlinear equations of α and β

$$\frac{r}{\alpha} - \sum_{i=1}^{r} x_i^{-\beta} + \frac{(n-r)x_r^{-\beta}}{-1+\exp\left(\alpha x_r^{-\beta}\right)} = 0,$$

$$\frac{r}{\beta} - \sum_{i=1}^{r}\ln(x_i) + \alpha\sum_{i=1}^{r} x_i^{-\beta}\ln(x_i) - \frac{(n-r)\alpha x_r^{-\beta}\ln(x_r)}{-1+\exp\left(\alpha x_r^{-\beta}\right)} = 0.$$

To find out the maximum likelihood estimators of α and β, we have to solve the above system on equations with respect to α and β. But this system has no closed form solution. Then we could use a numerical method, such as Newton's method, to obtain the solution.

4. THE BAYES ANALYSIS

Assuming that the parameters α and are independent random variables having exponential distributions with known parameters a, b, respectively, as prior distributions. That is, the joint prior probability density function of (α, β) is

$$g(\alpha,\beta) = a\,b\,\exp(-a\alpha - b\beta), \;\; \alpha,\beta,a,b > 0. \tag{4.1}$$

Combining the prior joint probability density function $g(\alpha,\beta)$ with the likelihood function $l(x)$, given by (3.1), the posterior joint probability density function of (α, β), according to Bayes' theorem, becomes

$$g(\alpha,\beta|x) = \frac{1}{I_0}(\alpha\beta)^r\Big[\prod_{i=1}^{r} x_i\Big]^{-(\beta+1)}\Big\{1-\exp\big(-\alpha x_r^{-\beta}\big)\Big\}^{(n-r)}\times$$

$$\exp\Big(-\alpha(a+\sum_{i=1}^{r} x_i^{-\beta})\Big)e^{-b\beta}, \tag{4.2}$$

where I_0 is the normalized factor given

$$I_0 = \int_0^\infty\int_0^\infty(\alpha\beta)^r\Big[\prod_{i=1}^{r} x_i\Big]^{-(\beta+1)}\Big\{1-\exp\big(-\alpha x_r^{-\beta}\big)\Big\}^{(n-r)}\times$$

$$\exp\Big(-\alpha(a+\sum_{i=1}^{r} x_i^{-\beta})\Big)e^{-b\beta}d\alpha\, d\beta\,. \tag{4.3}$$

The integral I_0 has no closed form solution. Therefore, we have to use the numerical techniques to find the solution.
Under the squared error loss function, the Bayes estimator of an unknown parameter is defined as the posterior mean. Then the Bayes estimators of α and β are

$$\hat{\alpha}_B = \frac{I_\alpha}{I_0} \text{ and } \hat{\beta}_B = \frac{I_\beta}{I_0}, \tag{4.4}$$

where I_α and I_β are

$$I_\alpha = \int_0^\infty \int_0^\infty \alpha(\alpha\beta)^r \Big[\prod_{i=1}^{r} x_i\Big]^{-(\beta+1)} \Big\{1-\exp(-\alpha x_r^{-\beta})\Big\}^{(n-r)} \times$$

$$\exp\Big(-\alpha(a+\sum_{i=1}^{r} x_i^{-\beta})\Big)e^{-b\beta}d\alpha\, d\beta, \tag{4.5}$$

$$I_\beta = \int_0^\infty \int_0^\infty \beta(\alpha\beta)^r \Big[\prod_{i=1}^{r} x_i\Big]^{-(\beta+1)} \Big\{1-\exp(-\alpha x_r^{-\beta})\Big\}^{(n-r)} \times$$

$$\exp\Big(-\alpha(a+\sum_{i=1}^{r} x_i^{-\beta})\Big)e^{-b\beta}d\alpha\, d\beta. \tag{4.6}$$

The above integrals I_α and I_β have no closed form solutions. Then to find out the Bayes estimators we have to use numerical methods to integrate I_α and I_β.

5. THE LEAST SQUARE ESTIMATORS

In this section we shall derive the least square estimators (LSEs) of α and β. The following theorem gives the LSEs of α and β in closed forms.

Theorem 5.1. Given the observed lifetimes $x_1 < x_2 < \ldots < x_r$ in a certain censored sample from the complementary Weibull distribution indexed by the parameters α and β. Then the LSEs of α, β, denoted $\hat{\alpha}_R$, $\hat{\beta}_R$ respectively, are given by

$$\hat{\alpha}_R = \exp\left\{\frac{\sum_{i=1}^r y_i \sum_{i=1}^r (\ln(x_i))^2 - \sum_{i=1}^r \ln(x_i)\sum_{i=1}^r y_i \ln(x_i)}{r\sum_{i=1}^r (\ln(x_i))^2 - \left(\sum_{i=1}^r \ln(x_i)\right)^2}\right\} \tag{5.1}$$

and

$$\hat{\beta}_R = \frac{\sum_{i=1}^r y_i \sum_{i=1}^r \ln(x_i) - r\sum_{i=1}^r y_i \ln(x_i)}{r\sum_{i=1}^r (\ln(x_i))^2 - \left(\sum_{i=1}^r \ln(x_i)\right)^2} \tag{5.2}$$

where $y_i = \ln\Big(-\ln\big(\frac{i-0.5}{r}\big)\Big)$, $i = 1, 2, \ldots, r$.

Proof. Using the complementary Weibull distribution function (2.2), we have

$$\ln[-\ln(F(x)] = \ln\alpha - \beta\ln x.$$

Based on the observed lifetimes $x_1, x_2, \ldots, x_r$, let $\hat{F}(x_i)$ be some convenient estimate of $F(x_i)$. We may let $\hat{F}(x_i) = \frac{i-0.5}{r}$. Let also $y_i = \ln[-\ln\hat{F}(t)]$. The LSEs of α and can be obtained by minimizing the following function with respect to α and

$$Q = \sum_{i=1}^{r}[y_i - (\ln\alpha - \beta\ln x_i)]^2. \tag{5.3}$$

The partial derivatives of Q with respect to α and β are respectively

$$\frac{\partial Q}{\partial\alpha} = -\frac{2}{\alpha}\sum_{i=1}^{r}[y_i - (\ln\alpha - \beta\ln x_i)],$$
$$\frac{\partial Q}{\partial\beta} = 2\sum_{i=1}^{r}[y_i - (\ln\alpha - \beta\ln x_i)]\ln(x_i).$$

Setting $\frac{\partial Q}{\partial\alpha} = 0$ and $\frac{\partial Q}{\partial\beta} = 0$, we shall get the following system

$$\sum_{i=1}^{r}[y_i - (\ln\alpha - \beta\ln x_i)] = 0,$$
$$\sum_{i=1}^{r}[y_i - (\ln\alpha - \beta\ln x_i)]\ln(x_i) = 0.$$

Solving the above system with respect to α and β, we can reach the proof.

6. SIMULATION STUDY AND CONCLUSION

A large simulation study using Monte Carlo method has been made in order to generate numerical lifetime data from CWD and do that follows:

1. Computing the point estimations of the CWD parameters using Bayes, maximum likelihood, and least square methods according to the previous theoretical results.

2. Compare the performance of each procedure to calculate the point estimations of the considered parameters.

3. How the value of r effects on the accuracy of the estimators using each procedure.

The comparison among the used procedures are made possible in two different methods:

1. by computing the percentage error, say PE, associated to the point estimation using the respective procedure. The PE associated to the point estimate, say $\hat{\theta}$, of the parameter, say θ, is defined as:

$$\text{PE}_\phi = \frac{|\hat{\theta} - \text{Exact value of } \theta|}{\text{Exact value of } \theta} \times 100\% \tag{6.1}$$

   where ϕ means the respective used procedure. That is, $\phi = B$ for Bayes, $\phi = M$ for maximum likelihood and $\phi = R$ for the least square procedure.

2. by computing the root mean squared errors, say RMSE's, associated to the point estimator using each procedure. Of course, the point estimation $\hat{\theta}_1$ will be better than $\hat{\theta}_2$ if $RMES(\hat{\theta}_1)$ is smaller than $RMES(\hat{\theta}_2)$. It is notable to mention that, the RMSE's gives an estimator of the expected loss associated to the given estimator of the considered parameter.

In what follows, we shall introduce two examples. In the first, data of type II censoring from the complementary Weibull distribution are generated. Then these data are used to calculate the point estimations of α and β, using each procedure. The second is introduced to: (i) make a comparison among performances of the used procedures to obtain the estimators of the considered parameters, (ii) investigate the influence of the values of (n, r) on the accuracy of the point estimators using each procedure.

The values of the prior parameters a and b can be randomly generated according to the following steps:

1. Specify values for α and β, say α_0 and β_0.
2. Generate random numbers u_1 and u_2 on the interval (0, 1).
3. Since α_0 and β_0 are values of the exponentially distributed random variables α and β having the respective distribution functions $F_\alpha(\alpha) = 1 - \exp\{-a\,\alpha\}$, $a, \alpha > 0$ and $F_\beta(\beta) = 1 - \exp\{-b\,\beta\}$, $b, \beta > 0$, so the random valued for a and b can be obtained by using the following forms:

$$a = \frac{1}{\alpha_0} \ln(1 - u_1), \quad b = \frac{1}{\beta_0} \ln(1 - u_2). \tag{6.2}$$

Example 1. In this example the information $x = \{10, 5; 0.715, 0.718, 1.056, 1.4, 1.416\}$ are generated from the complementary Weibull distribution with $\alpha = 1$ and $\beta = 1.5$. According to the previous theoretical results, we could compute the point estimations and their respective percentage errors. The calculated results are summarized on table 1. Note that, in the case of obtaining the Bayes estimators, the parameters a and b of the prior distributions of α and β are randomly generated to be $a = 1.748$ and $b = 1.165$, respectively.

Table 1. Point estimation and associated PE.

The Method	$\hat{\alpha}$	PE	$\hat{\beta}$	PE
LS	0.619	38.12 %	3.17	111.31 %
ML	1.453	45.33 %	1.968	31.21 %
Bayes	0.971	2.92 %	1.446	3.57 %

Based on the results booked on Table 1, it seems that the Bayes procedure is better than the maximum likelihood and least square procedures in the sense of having smaller percentage errors associated with the point estimations of the parameters α and β.

Example 2. To make a comparison among the used procedures and investigate how the value of failed items r effects on the accuracy of each, this example will be introduced. To do such task, a large simulation study on computer using Monte Carlo method has been done. In such study, the following scheme has been followed.

1. Specifying the tested and failed items n and r, respectively.
2. Determine the values of the scale and shape parameters of the CWD, α and β, respectively.
3. Generating the censored information $x = \{n,\ r;\ x_1, x_2,\ \ldots,\ x_r\}$ from the CWD given by (2.2).
4. Generating random values of the prior parameters a and b.
5. Using the generated data obtained in step 3, we calculate:
   (a) The MLE's of α and β denoted $\hat{\alpha}_M$ and $\hat{\beta}_M$, respectively. In this case, the Turbo Pascal function *newtsys.pas* from Marciniak, et al (1991) is used to solve a system of two nonlinear equations.
   (b) The LSE's of α and β denoted $\hat{\alpha}_R$ and $\hat{\beta}_R$, respectively.
6. Using the generated data obtained in steps 3 and 4, the Bayes estimators $\hat{\alpha}_B$ and $\hat{\beta}_B$ are computed.
7. The steps 3-6 are repeated 1000 times
8. The root mean squared errors associated with each estimators of $(\alpha,\ \beta)$ is calculated according to the following formula:

$$RMSE_\phi = \sqrt{\frac{\sum_{i=1}^{1000}\left[(\hat{\alpha}_i - \alpha)^2 + \left(\hat{\beta}_i - \beta\right)^2\right]}{1000}} \tag{6.3}$$

   where ϕ denotes the used procedure to find $(\hat{\alpha},\ \hat{\beta})$. That is, $\phi = M,\ L,\ B$ for maximum likelihood, least square and Bayes.

9. The above steps are made when $(\alpha, \beta) = (2, 1.5)$, $(2.5, 2.5)$ and $(4, 1.5)$.
10. Repeat the entire process when $r = 3, 4, \ldots, 10$, at fixed $n = 10$

Table 2 summarizes the root mean squared errors associated with the estimators of (α, β).

Table 2. The RMSE's associated with the estimators of the pair (α, β).

r	$\alpha = 2.0,\ \beta = 1.5$			$\alpha = 2.5,\ \beta = 2.5$			$\alpha = 4.0,\ \beta = 1.5$		
	ML	*LS*	*Bayes*	*ML*	*LS*	*Bayes*	*ML*	*LS*	*Bayes*
3	172.2	3.221	1.278	110.1	5.831	2.069	323.6	4.534	3.238
4	22.86	2.338	1.357	42.18	3.598	2.155	89.72	4.243	3.337
5	3.581	2.096	1.408	4.498	2.860	2.199	22.72	4.060	3.430
6	2.792	2.023	1.458	3.301	2.763	2.255	7.334	4.007	3.533
7	2.911	1.946	1.531	2.588	2.618	2.264	6.893	3.966	3.665
8	2.389	1.930	1.595	2.791	2.580	2.297	7.151	3.953	3.746
9	2.883	1.900	1.704	4.076	2.570	2.373	7.464	3.945	3.822
10	2.070	1.889	1.791	3.143	2.510	2.435	15.17	3.935	3.868

Based on the results summarized on Table 2, we could conclude that:

1. the $RMSE_M$ is greater than $RMSE_L$, which is greater than $RMSE_B$ for each investigated r.
2. the $RMSE_M$ is not monotonic*(neither decreasing nor increasing)* function of r,
3. the $RMSE_L$ is monotonically decreasing with r, while the $RMSE_B$ is monotonically increasing with r,
4. the difference between the $RMSE_L$ and $RMSE_B$ is small and monotonically decreasing with r. Or, we can say that this difference is monotonically increasing with the difference $n - r$.

Based on the above analysis we could say that, the Bayes procedure is better than the least square in the sense of having smaller RMSE's, especially in the cases for which the censored data are used and the number of failed items is relatively small. Further, the least square estimators is better than the maximum likelihood estimators in the sense of *(i)* having respective smaller RMSE's, *(ii)* having closed form and easier required calculations.

In spite of the Bayes procedure seems better than the least square procedure, in the sense of having smaller RMSE's, the LSE's have three advantages: *(i)* the prior information is not required, *(ii)* the LSE's have closed forms, and *(iii)* the estimators could be calculated using a handle calculator.

REFERENCES

Ciechanowicz, K. (1972). Rozklad Webulla z ujmnym parametrem jako moel matematyczny rozkldu trwalosci. *Praca Instytut Automatyki PAN, z.* **90**, *Warszawa, Polska.*

Derapella, A. (1993). The Complementary Weibull distribution Unknown or only Forgotten ?. *Quality and Reliability Engineering International,* **9**, *383-384.*

Lawless J. F. (1982). *Statistical Models and Methods for Lifetime Data.* John Wiley & Sons.

Marciniak, A., Grequlec, D. and Kaczmarek, J. (1991). *Basic Numerical Procedures in Turbo Pascal for Your PC.* 61-799, Poznan, Poland.

Zain-Din, M. (1994). The Complementary Weibull Distribution and its Generalization. *Ph.D. Dissertation, Institute of Electronics, Gdansk Polytechnic, Poland.*