Journal of the Society of Korea Industrial and Systems Engineering
Vol, 28, No. 3, pp.142 ~ 8, September 2005.

New Warranty Model with Varying Usage Intensity*

Che-Soong Kim

Dept of Industrial Engineering, Sangji University

사용율 변화를 고려한 새로운 보증정책 모형

김 제 숭

상지대학교 산업공학과

대부분의 일차원 보증정책에 대한 비용분석은 소비자 사용률이 모든 사용자들에게 동일한 것으로 가정하고 있다. 하지만 현실적으로는 사용률은 소비자에 따라서 다르기 때문에, 일반적인 경우 소비자 집단을 k 범주로 나눌 수 있다. 즉, 동일한 제품에 대해서 생산자는 k 범주에 속하는 다른 보증정책을 제시할 수 있고, 보증비용은 이런 옵션들 가운데 소비자가 자기의 입장에 맞는 보증기간을 선택할 수 있다. 본 논문에서는 소비자의 사용률이 변화하는 경우를 고려한 무료보증정책을 갖는 기대보증비용 모형을 분석하고, 이에 따른 수치예제를 제시한다.

Keywords : 'arranty, Varying Usage Rate, Reliability and Stochastic Model.

1. Introduction[7)]

A warranty is a contractual agreement offered by the manufacturer at the point of sale of a product. It requires the manufacturer to rectify all failures occurring within the warranty period or compensate the buyer. In this context, warranty serves as a device to protect the buyer from defective products.Offering warranty results in additional costs to manufacturers from the servicing of the warranty. However, warranties also serve as signals to inform customers about the quality of product. In this context, better warranty terms imply a higher product quality and manufacturers have tended to use warranty as a promotional tool to increase sales and revenue. Hence, from a manufacturers perspective, offering a warranty is worthwhile only if the benefits (in terms of greater sales and/or revenue) exceed the additional costs associated with the servicing of warranty. Many different types of warranty policies for both new and second-hand products have been proposed and analysed. A taxonomy for warranty policies for new products was proposed by Blischke and Murthy (1992).

For new products, the cost analysis of various one-and two-dimensional warranty policies can be found in Blischke and Murthy (1994). The cost analysis of one-dimensional policies is based on the assumption that buyers are homeogenous with respect to the usage intensity (or rate). This implies the usage rate is the same for all buyers. In contrast, two-dimensional warranty policies are characterized by a two-dimensional region with one axis representing time (or age) and the other usage. The product usage intensity across the buyer population can vary and is modelled as a random variable in the cost analysis. Two different approaches have been proposed and the details can be found in Blischke and Murthy (1994 and 1996), Kim and Murthy (1999).

In this paper we focus our attention on warranty cost analysis for products sold with one-dimensional free replacement warranty and the usage intensity varying across

* 본 연구는 과학재단 목적기초연구지원으로 수행되었음 (R05-2003-000-11799-0)

the buyer population. A typical example that reflects this is the following. The usage intensity (in terms of load and frequency of usage per week) of a domestic washing machine varies depending on the size of the family. This is also true for many other domestic and industrial products. The product degradation and failure depends on the usage intensity and this in turn has an impact on the expected warranty cost. This needs to be taken into account in determining the sale price and reliability decisions at the design stage. The usage intensity can be modelled either as a continuous, or as a discrete, random variable. We will discuss both of these and develop some simple models for warranty cost analysis that are analytically tractable.

2. New Model Formulation

The following notation is used in this section.

W:Warranty period

$C(W)$	:	xpected warranty cost
$F(t;\theta)$	:	Failure distribution function with scale parameter
$\bar{F}(t;\theta)$	:	rvival function associated with $F(t;\theta)$
$f(t;\theta)$	:	ilure density function associated with $F(t;\theta)$
$r(t;\theta)$	:	ilure rate function
$\theta(u)$	:	ale parameter of the Weibull distribution as function of usage rate (u)
β	:	ape parameter of $F(t;\theta)$
α	:	rameter of Gamma distribution
δ,ϕ	:	ad parameters
U	:	sage rate (random variable)
u_{min}	:	inimum usage rate of the light user
u_{max}	:	aximum usage rate of the heavy user
$G(u)$	:	sage distribution function
$g(u)$	:	sage intensity function
$E[N(t)]$	:	xpected value of N(t)
C_r	:	xpected cost of each repair
C_m	:	xpected cost of each replacement

2.1 Item Failures

Let $F(t;\theta)$ be the failure distribution function for the product. θ is the scale parameter and is a function of the usage intensity as will be discussed later in the section. The product is sold with a free replacement warranty period that requires the manufacturer to rectify all failures over the warranty period at no cost to the buyer. The first failure is a random variable with distribution function $F(t;\theta)$. Subsequent failures depend on the type of rectification action used.

Case 1. The product is non-repairable. As a result, a failed item under warranty needs to be replaced with a new one. If the claims are exercised immediately and the time to replace is small relative to the mean time between failures, then it can be ignored. As a result, failures over the warranty period occur according to a renewal process associated with $F(t;\theta)$(see, Ross (1972)).

Case 2. The product is repairable. We assume that the failures are minimally repaired (see, Barlow and Hunter (1960)) and the time to repair is negligible so that it can be ignored. This implies that failures over the warranty period occur according to a non-homogeneous Poisson process with intensity function given by the failure rate function $r(t;\theta)$ associated with $F(t;\theta)$. This failure rate function is given by

$$r(t;\theta)=\frac{f(t;\theta)}{1-F(t;\theta)} \quad \cdots\cdots (1)$$

2.2 Usage Intensity

Let U denote the usage rate. This is a random variable and characterize the different usages across the buying population. We can model this in two different approaches; the usage either as a continuous variable or as a discrete variable.

Approach 1 - Continuous Variable

The usage rate is modelled a continuous random variable distributed over an interval u_{min} and u_{max} according to a distribution function $G(u)$ with density function $g(u)$. These two limits denote the minimum and maximum usage rates. Two forms of distributions will be considered in the paper.

- iiform distributions :
  $g(u) = (1/(u_{max} - u_{min}))$
- ımma distributions :
  $g(u) = (1/\Gamma(a))u^{a-1}e^{-u}, \ a \ge 0$

Note that for the uniform distributions, $u_{min} \ge 0$ and $u_{max} \le \infty$ and for the Gamma distribution, $u_{min} = 0$ and $u_{max} = \infty$.

Conditional on the usage rate $U = u$, θ is given by

$$\theta(u) = \delta(u)\,\theta_d \quad \cdots\cdots (2)$$

where θ_d is design parameter and $\delta(u)$ defines the effect of the usage rate(or load on the item) and is modelled as $\delta_i(u) = \sum_{j=1}^{i}\left(\frac{u}{u_{j-1}}\right)^{j-1}$

$$u_{i-1} \le u \le u_i, \quad i = 1,\ 2,\ \cdots k \quad \cdots\cdots (3)$$

with u_i as the additional design parameters. The design parameters depend on the design decisions and are under the control of the manufacturer. Higher values for θ_d and u_0 are resulted from better design.

Approach 2 - Discrete Variable

The usage population can be clustered into several groups. The general case is to cluster them into k groups. Let p_i denote the probability that the buyer is i group's user. Then conditional on the usage rate $U = u$, θ is given by

$$\theta(u) = \phi^{i-1}\theta_d, \qquad i = 1, 2, \cdots k, \ \phi \ge 1 \quad \cdots\cdots (4)$$

2.3 Expected Warranty Cost

Let $N(W)$ denote the number of failures over the warranty period for an item sold with warranty period W. We first consider the case where the item is non-repairable so that a failed item is replaced with a new one. Let C_m denote the cost of each replacement. Then, the expected warranty cost per unit is given by

$$C(W) = C_m E[N(W)] \quad \cdots\cdots (5)$$

For repairable item, the failed item is repaired minimally. Let C_r denote the cost of each minimal repair. Then, the expected warranty cost per unit is given by

$$C(W) = C_r E[N(W)] \quad \cdots\cdots (6)$$

3. Warranty Model Analysis I

In this section we obtain expressions for the expected warranty cost per unit with usage rate being modelled as a continuous random variable.

3.1 Non-repairable Product

For non-repairable product, each failure over warranty is replaced with a new one. Since replacements are instantaneous, failures over the warranty period occur according to a renewal process. We use the conditional approach to obtain an expression for the expected warranty cost per unit.

Let $N(W;\theta(u))$ denotes the number of failures over warranty conditional on $U = u$. The expected value of $N(W;\theta(u))$ is given by

$$E[N(W;\theta(u))] = M(W;\theta(u)) \quad \cdots\cdots (7)$$

where $M(W;\theta(u))$ is the conditional renewal function associated with the distribution function $F(W;\theta(u))$ and is given by

$$M(W;\theta(u)) = F(W;\theta(u)) + \int_0^W M(W;\theta(u))\,dF(W;\theta(u)) \quad \cdots\cdots (8)$$

using the formula for conditional expectation, the expected number of failures over warranty, $E[N(W)]$, is given by

$$E[N(W)] = \int_{u_{min}}^{u_{max}} M(W;\theta(u))\,dG(u) \quad \cdots\cdots (9)$$

The expected warranty cost per unit sale is obtained from (5) using (9).

3.2 Repairable Product

Item failure over the warranty period, conditional on $U=u$, occur according to a non-homogeneous Poisson process with intensity function given by $r(t;\theta(u))$. As a result, the conditional expected value of the number of failures is given by

$$E[N(W;\theta(u))] = \int_0^W r(t;\theta(u))dt \quad \cdots\cdots (10)$$

Using the formula for conditional expectation argument, the expected number of failures over the warranty period is given by

$$E[N(W)] = \int_{u_{min}}^{u_{max}} \int_0^W r(t;\theta(u))dt\,dG(u) \quad \cdots\cdots (11)$$

The expected warranty cost per unit is obtained from (6) using (11).

In general, it is not possible to derive analytical expressions for the expected warranty costs. In this case, one needs to use some computational schemes to obtain cost estimates. However, if for the non-repairable product, the failure distribution follows an exponential distribution and for the repairable product, the failure distribution follows a Weibull distributions, then the expected warranty costs can be derived analytically. In the next section, we discuss some special cases for which it is possible to derive analytical expressions.

3.3 Special Cases

The usage population can be clustered into several groups. The general case is to cluster them into k groups. In this example, the simplest case is to cluster them into 3 groups, i. e., the light users, medium users and heavy users.

Example 1 : Non-Repairable Product with Gamma Usage Rate Distributions

The usage rate is given by a Gamma distribution with parameter a, therefore $g(x) = \frac{1}{\Gamma(a)} x^{a-1} e^{-x},\ x \geq 0$.

Also if $F(t;\theta)$ is an exponential with parameter $\theta(u)$, then from (9) we have

$$E[N(W)] = W\theta_d \sum_{i=1}^{k} \prod_{j=1}^{i} \left(\frac{1}{u_{j-1}}\right)^{j-1} \frac{\Gamma(a+i(i-1)/2)}{\Gamma(a)} \times$$
$$\left[G\left(u_i;a+\frac{i(i-1)}{2}\right) - G\left(u_{i-1};a+\frac{i(i-1)}{2}\right)\right] \quad \cdots\cdots (12)$$

Let $\theta_d=0.1$, $u_{min}=0$, $u_{max}=5$, $\beta=1$, $W=2$ and the unit for usage rate is 10^4km/*year*. Table 1 shows $E[N(W)]$ for different combinations of u_1, u_2 and a.

Note that as a increases $\theta(u)$ also increases. As a result, the expected number of failures for the same warranty period increases. $\theta(u)$ decreases as u also increases and as a result the expected number of failures decreases.

<Table 1> $E[N(W)]$ for different combinations of u_1, u_2 and a

		$u_1=0.5$	$u_1=1.0$	$u_1=1.5$	$u_1=2.0$	$u_1=2.5$
$u_2=1.0$	$a=1$	13.7118	-	-	-	-
	$a=2$	53.7814	-	-	-	-
$u_2=1.5$	$a=1$	2.7314	0.6908	-	-	-
	$a=2$	10.6412	2.6630	-	-	-
$u_2=2.0$	$a=1$	0.8972	0.2324	0.1112	-	-
	$a=2$	3.4028	0.8534	0.3834	-	-
$u_2=2.5$	$a=1$	0.4084	0.1102	0.0568	0.0392	-
	$a=2$	1.4502	0.3654	0.1664	0.0982	-
$u_2=3.0$	$a=1$	0.2432	0.0688	0.0386	0.0290	0.0250
	$a=2$	0.7750	0.1964	0.0914	0.0560	0.0406

Example 2 : Repairable Product with Gamma Usage Rate Distributions

$F(t;\theta)$ is a Weibull distribution with parameter $\theta(u)$ and β thus $F(t;\theta) = 1-e^{-(\theta(u)t)^{\beta}}$. Then from (11) we have the expected number of failures over warranty is given by

$$E[N(W)] = (W\theta_d)^{\beta} \sum_{i=1}^{k} \prod_{j=1}^{i} \left(\frac{1}{u_{j-1}}\right)^{(j-1)\beta}$$
$$\times\left[\frac{\Gamma(a+i(i-1)\beta/2)}{\Gamma(a)}\right]\times$$

$$\left[G\left(u_i; \alpha+\frac{\lambda(i-1)\beta}{2}\right)-G\left(u_{i-1}; \alpha+\frac{\lambda(i-1)\beta}{2}\right)\right] \cdots (13)$$

Let $\beta=2$ and the remaining parameters are as in Example 1. Table 2 shows $E[N(W)]$ for different combinations of u_1, u_2 and α.

<Table 2> $E[N(W)]$ for different combinations of u_1, u_2 and α.

		$u_1=0.5$	$u_1=1.0$	$u_1=1.5$	$u_1=2.0$	$u_1=2.5$
$u_2=1.0$	$\alpha=1$	3.7332	-	-	-	-
	$\alpha=2$	10.814	-	-	-	-
$u_2=1.5$	$\alpha=1$	1.8668	1.0380	-	-	-
	$\alpha=2$	4.9348	2.5024	-	-	-
$u_2=2.0$	$\alpha=1$	1.2708	0.7400	0.5880	-	-
	$\alpha=2$	2.9632	1.5164	1.0584	-	-
$u_2=2.5$	$\alpha=1$	1.0336	0.6212	0.5088	0.4640	-
	$\alpha=2$	2.1276	1.0988	0.7796	0.6368	-
$u_2=3.0$	$\alpha=1$	0.9296	0.5696	0.4744	0.4380	0.4220
	$\alpha=2$	1.7356	0.9028	0.6492	0.5388	0.4836

Note that as β increases $\theta(u)$ also increases. As a result, the expected number of failures for the same warranty period increases.

4. Warranty Model Analysis II

In this section we obtain expressions for the expected warranty cost per unit with usage rate being modeled as a discrete random variable.

4.1 Non-repairable Product

It is easily shown that the expected number of failures over warranty is given by,

$$E[N(W)] = \sum_{i=1}^{k} p_i M(W;\theta_i) \cdots (14)$$

where $M(W;\theta_i)$ is the renewal function associated with $F(t;\theta_i)$. When the failure distribution is exponential, $E[N(W)]$ is given by

$$E[N(W)] = W\theta_d \sum_{i=1}^{k} p_i \phi^{i-1} \cdots (15)$$

4.2 Repairable Product

It is easily seen that the expected number of failures over warranty is given by

$$E[N(W)] = \sum_{i=1}^{k} p_i \int_0^W r(t;\theta_i)dt \cdots (16)$$

When $E[N(W)]$ is a Weibull distribution, the expected number of failures over the warranty period is given by

$$E[N(W)] = (W\theta_d)^{\beta} \sum_{i=1}^{k} p_i \phi^{\beta(i-1)} \cdots (17)$$

4.3 Special Cases

Example 3 : Non-repairable Product

Let $\theta_d=0.1$, $\beta=1$, $W=2$, $u_l=1$, $u_m=3$ and $u_h=4$ (the mean usage per year the light user is $1(\times 10^4)$, for the medium user is $3(\times 10^4)$ and for the heavy user is $4(\times 10^4)$. Table 3 shows $E[N(W)]$ for different combinations of ϕ and p_i.

<Table 3> $E[N(W)]$ for different combinations of ϕ and p_i.

		$E[N(W)]$	$\phi=2.0$	$\phi=2.5$	$\phi=3.0$
$p_1=0.3$	$p_2=0.3$	0.0600	0.1580	0.3556	0,7080
	$p_2=0.4$	0.0544	0.1340	0.2904	0.5640
	$p_2=0.5$	0.0488	0,1100	0.2248	0.4200
$p_1=0.4$	$p_2=0.3$	0.0518	0.1280	0.2798	0.5480
	$p_2=0.4$	0.0462	0.1040	0.2142	0.4040
	$p_2=0.5$	0.0406	0.0800	0.1486	0.2600
$p_1=0.5$	$p_2=0.3$	0.0438	0.0980	0.2038	0.3880
	$p_2=0.4$	0.0382	0.0740	0.1382	0.2440
	$p_2=0.5$	0.0326	0.0500	0.0726	0.1000

The result show that as ϕ increases. this increases the number of failure over warranty period. As p_i increases, the expected number of failure decreases as to be expected since the buyer is more likely to be a light user.

Example 4 : Repairable Product

Let $\beta=2$ and remaining parameters are as in Example 3. Table 4 shows $E[N(W)]$ for different combinations of ϕ and p_i.

The result show that as ϕ increases. this increases the number of failure over warranty period. As p_i increases, the expected number of failure decreases as to be expected since the buyer is more likely to be a light user.

<Table 4> $E[N(W)]$ for different combinations of ϕ and p_i

		$\phi=1.5$	$\phi=2.0$	$\phi=2.5$	$\phi=3.0$
$p_1=0.3$	$p_2=0.3$	0.6600	1.0000	1.4200	1.9200
	$p_2=0.4$	0.6300	0.9200	1.5200	1.6800
	$p_2=0.5$	0.6000	0.8400	1.1200	1.4400
$p_1=0.4$	$p_2=0.3$	0.6100	0.8800	1.2100	1.6000
	$p_2=0.4$	0.5800	0.8000	1.0600	1.3600
	$p_2=0.5$	0.5500	0.7200	0.9100	1.1200
$p_1=0.5$	$p_2=0.3$	0.5600	0,7600	1.0000	1.2800
	$p_2=0.4$	0.5300	0.6800	0.8500	1.0400
	$p_2=0.5$	0.5000	0.6000	0.7000	0.8000

5. Implications for Manufacturer

The expected cost of servicing the warranty for a high intensity user is more than that for a light user. The warranty cost analysis carried out in earlier sections yield a cost based on averaging over different intensity users. If the manufacturer offers the same warranty terms to all buyers and determines the sale price based on the average across different usage intensities, it is unfair to low intensity users as they receive less benefit than heavy intensity users. This can be overcome in several different ways and we discuss this briefly in this section.

One option for the manufacturer is to sell the product with the same warranty period but at different prices (higher price for high intensity user). Another option is the same sale price but different warranty terms (longer warranty period for light users). These options, though may seem fair from the buyer perspective, would lead to a new problem for the manufacturer if buyers do not reveal their usage intensity and there is no way for the manufacturer to assess it for a potential buyer. The high intensity user might be tempted not to disclose its usage in order to get the benefit of cheaper price or longer warranty period thus creating a moral hazard problem. Also, this type of differentiation can lead to adverse selection problem with buyers making wrong selections due to lack of information, knowledge etc. The adverse selection and moral hazard problems have received some attention -- Lutz (1989) deals with a static formulation and Murthy & Padmanabhan (1995) deals with a dynamic situation.

6. Conclusions

Warranty cost model for one dimensional warranties assumes that the usage intensity is the same for all buyers. But in real life the usage intensity varies across the population of buyers. In the paper we deals with models to study the expected warranty cost for products with free replacement warranty with varying usage intensity. We have confined our discussion to the free replacement warranty policy. New incentive-based policies can help reduce some of the moral hazard issues and also the adverse selection problem. One such policy is a combination policy where the buyer incurs no cost for rectifying failure in the early stages of the warranty and an increasing share of the cost later in the warranty period. Also, the manufacturer might offer extra period of warranty coverage at the expire of the initial warranty if the buyer makes the right selection. This would discourage high intensity users from claiming to be light intensity users and the extra coverage for light intensity users. This would compensate for the unfairness. Some of these problems are currently under investigation by the authors.

References

[1] Barlow, R.E. and Hunter, L.C,,)ptimum Preventive Maintenance Policies", *Operations Research*, 8, 90-100, 1960

[2] Blischke, W.R. and Murthy, D.N.P., "Product Warranty Management - I : taxonomy for warranty policies", *European Journal of Operational Research*, 62, 127-148, 1992

[3] Blischke, W.R. and Murthy, D.N.P., Warranty Cost Analysis, Marcel Dekker, New York, 1994

[4] Kim, C.S., I. Djamaludin and Murthy, D.N.P., "Warranty Cost Analysis Varying Usage Intensity", *Proceedings of the 15th National Conference of the Austrian Society for Operations Research*, July 1999, Australia, 1999

[5] Loss S.M., Stochastic Process, John Wiley & Sons, 1972

[6] Lutz, N.A., "Warranties as a Signals Under Consumer Moral Hazard", *Journal of Economics*, 20, 239-255, 1989

[7] Murthy, D.N.P. and Padmanabhan, V., "A Dynamic Model of Product Warranty with Consumer Moral Hazard", *working paper, Graduate School of Business,* Stanford University, Stanford, CA., 1995

[8] Murthy, D.N.P. and Blischke, W.R, "Product Warranty Management 2 : n Integrated Framework for Study", *European Journal of Operational Research*, 62, 261-281, 1992a