Direct Gradient Descent Control and Sontag's Formula on Asymptotic Stability of General Nonlinear Control System

J. Naiborhu, S. M. Nababan, R. Saragih, and I. Pranoto

Abstract: In this paper, we study the problem of stabilizing a general nonlinear control system by means of gradient descent control method which is a dynamic feedback control law. In this method, the general nonlinear control system can be considered as an affine nonlinear control systems. Then by using Sontag's formula we investigate the stability (asymptotic) of the general nonlinear control system.

Keywords: Direct gradient descent control, Sontag's formula, asymptotic stability.

1. INTRODUCTION

Stabilization of nonlinear control system has been a subject of research by many authors. In [1], Tsinias proved that Lyapunov conditions were sufficient for asymptotic stabilization of affine nonlinear systems. Praly, d'Andrea-Novel and Coron [2] designed a state feedback control law based on control Lyapunov function. Wei Lin [3] developed sufficient conditions for general nonlinear control system to be asymptotically stabilizable via smooth state feedback. For more information and additional details on nonlinear feedback stabilization, the reader may refer to a survey due to Sontag [4].

The purpose of this paper is to apply a new method "*direct gradient descent control method*" for stabilizing asymptotically the general nonlinear control systems at an equilibrium point which is stable (asymptotically) if input is zero. In other word, input of the system is used to improve the stability performance of the system. The main idea of this method is as follows. Consider the equilibrium point as a desired value that we want to reach. Then define a performance index as a squared error function. Design a control law to decrease the performance index via gradient descent method.

The paper is organized as follows. In Section 2, we state the direct gradient descent control method for a nonlinear system, while in Section 3 we apply the gradient descent control to the nonlinear control system and investigate the stability (asymptotic) of system by using affine nonlinear control system technique(Sontag's formula). Section 4 shows three illustrative examples and Section 5 states concluding remarks.

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2. DIRECT GRADIENT DESCENT CONTROL

Consider a nonlinear control system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the state vector and $u(t) \in R^m$ is the control vector.

The aim of control is to decrease a performance index $F(x(t),u(t))$ at any time t along the trajectory of system (1), and so our problem is formulated as follows:

$$\underset{u(t)}{\text{decrease}} \ F(x(t), u(t)) \tag{2}$$

$$\text{subject to } \dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0. \tag{3}$$

As the class of admissible controls, we consider a space U_t consisting of m-dimensional-vector valued functions which are differentiable $[t_0,t]$, and define the following inner product :

$$< \bar{u}, \bar{v} > = \int_{t_0}^{t} u(\tau)^T v(\tau) d\tau, \tag{4}$$

where $\bar{u}$ and $\bar{v} \in U_t$.

In unconstrained optimization problem, problem (2)-(3) becomes

$$\underset{u(t)}{\text{decrease}} \ F(x(t;\bar{u}), u(t)), \tag{5}$$

where

$$x(t;\bar{u}) = x(t_0) + \int_{t_0}^{t} f(x(\tau), u(\tau)) d\tau . \tag{6}$$

A very broad and perhaps the most important class of methods for unconstrained optimization is that which is based upon the so-called gradient descent methods. By applying gradient descent method, we obtain as a control law, the first-order ordinary differential equations

$$\dot{u}(t) = -\alpha \nabla_u F(x(t;\bar{u}), u(t)), \tag{7}$$

where α can become a constant or function of $x(t)$, $u(t)$, and t. Below, the gradient of performance index $F(x(t;\bar{u}), u(t))$ with respect to $u(u(t))$ will be derived.

Assume that

A.1 f and F are continuously differentiable on $(x,u) \in R^n \times R^m$.

A.2 f_x, f_u, F_x and F_u are Lipschitz continuous.

Definition 1: $x(t;\bar{u})$ is Gateaux differentiable, i.e. for an arbitrary $\bar{s} \in U_t$,

$$\delta_{\bar{s}} x(t;\bar{u}) = \frac{d}{d\varepsilon} x(t:\bar{u} + \varepsilon\bar{s})\,|_{\varepsilon=0} \tag{8}$$

exists.

Theorem 1: Functional $x(t;\bar{u})$, defined by (6), is Gateaux differentiable with

$$\delta_{\bar{s}} x(t;\bar{u}) = \int_{t_0}^{t} \nabla x(t;\bar{u})(\tau)^T s(\tau) d\tau, \tag{9}$$

where

$$\nabla x(t;\bar{u})(\tau) = f_u(x(\tau;u), u(\tau))^T \Phi(t,\tau)^T, \quad t_0 \le \tau \le t. \tag{10}$$

Proof: Integrating (1) from t_0 to t with $\bar{u} + \varepsilon\bar{s}$ given, we have

$$x(t;\bar{u} + \varepsilon\bar{s}) = x(t_0) + \int_{t_0}^{t} f(x(\tau;\bar{u} + \varepsilon\bar{s}), u(\tau) + \varepsilon s(\tau)) d\tau. \tag{11}$$

Differentiate (11) w.r.t. ε and let $\varepsilon = 0$, and then differentiate it w.r.t. t then finally we obtain

$$\frac{d}{dt}\frac{d}{d\varepsilon} x(t;\bar{u} + \varepsilon\bar{s})\,|_{\varepsilon=0} = f_x(x(t;\bar{u}), u(t)) \frac{d}{d\varepsilon} x(t;\bar{u} + \varepsilon\bar{s})\,|_{\varepsilon=0} + f_u(x(t;\bar{u}), u(t)) s(t). \tag{12}$$

Since (12) is a time-varying linear differential equation w.r.t $\delta_{\bar{s}} x(t;\bar{u}) = \frac{d}{d\varepsilon} x(t;\bar{u} + \varepsilon\bar{s})\,|_{\varepsilon=0}$, its solution is given by

$$\delta_{\bar{s}} x(t;\bar{u}) = \int_{t_0}^{t} \Phi(t,\tau) f_u(x(\tau,u), u(\tau)) s(\tau) d\tau, \tag{13}$$

where $\Phi(t,\tau)$ is a continuous transition matrix function which have properties

$$\frac{\partial}{\partial t}\Phi(t,\tau) = f_x(x(t;\bar{u}), u(t))\Phi(t,\tau),\ t_0 \le \tau \le t,$$

$$\Phi(\tau,\tau) = I.$$

According to (4), $\delta_{\bar{s}} x(t;\bar{u})$ can be rewritten in the inner product form

$$\delta_{\bar{s}} x(t;\bar{u}) = < \nabla x(t;\bar{u}), \bar{s} >,$$

where

$$\nabla x(t;\bar{u})(\tau) = f_u(x(\tau;u), u(\tau))^T \Phi(t,\tau)^T,\ t_0 \le \tau \le t \tag{14}$$

such that

$$\delta_{\bar{s}} x(t;\bar{u}) = \int_{t_0}^{t} \nabla x(t;\bar{u})(\tau)^T s(\tau) d\tau. \tag{15}$$

□

Theorem 2: By defining $\nabla_u x(t;\bar{u}) = \nabla x(t;\bar{u})(t)$, gradient of objective function $F(x(t;\bar{u}), u)$ with respect to u at time t is

$$\nabla_u F(x(t;\bar{u}), u) = f_u(x(t;\bar{u}), u(t))^T F_x(x(t;\bar{u}), u(t))^T + F_u(x(t;\bar{u}), u(t))^T. \tag{16}$$

Proof: By chain rule,

$$\nabla_u F(x(t;\bar{u}), u) = (F_x(x(t,\bar{u}), u)\nabla_u x(t;\bar{u}))^T + (F_u(x(t;\bar{u}), u))^T. \tag{17}$$

From (10) we obtain

$$\nabla x(t;\bar{u})(t) = f_u(x(t;\bar{u}), u(t))^T \Phi(t,t)^T + f_u(x(t;\bar{u}), u(t))^T. \tag{18}$$

By substitution equation (18) into equation (17) we obtain

$$\nabla_u F(x(t;\bar{u}), u) = f_u(x(t;\bar{u}), u(t))^T F_x(x(t;\bar{u}), u(t))^T + F_u(x(t;\bar{u}), u(t))^T. \tag{19}$$

□

Thus, the dynamic control (7) becomes

$$\dot{u}(t) = -\alpha(x(t), u(t))\left\{f_u(x(t;\bar{u}), u(t))^T F_x(x(t,\bar{u}), u(t))^T + F_u(x(t;\bar{u}), u(t))^T\right\}, \tag{20}$$

where $\alpha(x(t), u(t))$ is a matrix function. This control law is called as the direct gradient descent control [5,7].

Remark 1: The same result which is stated in Theorem 2 can be found in [7] but the way of proof is different.

3. PERFORMANCE IMPROVEMENT

In this section we apply the direct gradient descent control (20) to stabilize the nonlinear control system

$$\dot{x}(t) = f(x(t), u(t)), \tag{21}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input. Without loss of generality, let $(0,0) \in R^n \times R$ be an equilibrium point of (21), $f(0,0) = 0$.

Assume that system (21) satisfies the following assumption.

A.3 There exists a function $V{:}R^n \to R$, with $V(0) = 0$, which is continuous, positive definite and radially unbounded such that the unforced dynamic system of (21), namely $\dot{x}(t) = f(x(t),0)$ is *globally asymptotically stable*, i.e., $V_x(x(t))\, f(x(t),0) < 0,\ \ x \neq 0$.

First we define the performance index

$$F(x(t), u(t)) = V(x(t)) + u(t)^T R u(t), \tag{22}$$

where R is a matrix constant, $R>0$. Then we determine the value of $\alpha(x(t), u(t))$ by Sontag's formula such that the extended nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \tag{23}$$

$$\dot{u}(t) = -\alpha(x(t), u(t)) \nabla_u F(x,u), \quad u(t_0) = u_0 \tag{24}$$

is asymptotically stable about $(x(t),u(t))=(0,0)$. The most important thing is to guarantee the existence of $\alpha(x(t), u(t))$.

Remark 2: Consider (22). If $u(t) = 0$ then $F(x(t), u(t)) = V(x(t))$. In other words performance index becomes Lyapunov function and so we don't need to design control input $u(t)$ for only stabilizing system. With $u(t) = 0$ we can not do anything to increase the rate of convergence. By adding $u(t)$ to the system, however, we have freedom to accelerate the rate of convergence.

Remark 3: It would appear that when $x = 0$ is globally asymptotically stable as assumed by assumption **A.3**, then the global stabilization of the whole system should not be difficult. The following simple example [8] show that this is not so. Consider system

$$\dot{x} = -x + ux^2. \tag{25}$$

Disturbed by exponentially decaying input

$$u = u(0)e^{-\gamma t}. \tag{26}$$

The substitution of (26) with $\gamma = 1$ into (25) yields the equation

$$\dot{x} = -x + u(0)e^{-\gamma t}x^2. \tag{27}$$

Whose explicit solution is

$$x(t) = \frac{2x(0)}{x(0)u(0)e^{-t} + [2 - x(0)u(0)]e^{t}}. \tag{28}$$

It is easy to see that if $x(0)u(0) < 2$, then $x(t)$ will converge to zero as $t \to \infty$. However, if $x(0)u(0) > 2$, then at the time

$$t_{esc} = \frac{1}{2}\ln\frac{x(0)u(0)}{x(0)u(0) - 2}$$

the difference of the two exponential terms in the denominator becomes zero, that is

$$|x(t)| \to \infty \ \ as \ \ t \to t_{esc}.$$

Thus, disturbed by an exponentially decaying input $u(t)$, the nonlinear system (25) can become unstable, or even worse: its state may escape to infinity in finite time.

Before we prove the existence of $\alpha(x,u)$, below we recall the Sontag's formula which will be used to investigate the stability of the extended systems (23)-(24).

Sontag's formula for multi input affine nonlinear control system [4,8]

A smooth positive and radially unbounded function $V^a{:}R^n \to R_+$ is called a control Lyapunov function (clf) for (21) if

$$\inf_{u \in R^m} \left\{ V_x^a(x) f(x,u) \right\} < 0, \qquad \forall\, x \neq 0. \tag{29}$$

Consider the affine nonlinear control system

$$\dot{x} = f(x) + g(x)u, \quad x \in R^n, \quad u \in R^m. \tag{30}$$

In componentwise form, (30) becomes

$$\dot{x}_i = f_i(x) + g_{i1}(x)u_1 + \cdots + g_{im}(x)u_m, \quad i = 1,\cdots,n. \tag{31}$$

Let $\hat{g}_j(x) = [g_{1j}(x), \cdots, g_{nj}(x)]^T$, then (31) becomes

$$\dot{x} = f(x) + \hat{g}_1(x)u_1 + \cdots + \hat{g}_m(x)u_m. \tag{32}$$

If $V^a(x)$ is a clf for (32), then a particular stabilizing control law $k_i(x)$, smooth for all $x \neq 0$, is

$$u_j = k_j(x) = \begin{cases} -\left(V_x^a(x)\hat{g}_j(x)\right)\Big(V_x^a(x)f(x) \\ \quad + \left(\left(V_x^a(x)f(x)\right)^2 + \left\|V_x^a(x)g(x)\right\|^2\right)^{\frac{1}{2}}\Big) \\ \quad \times\left(\left\|V_x^a(x)g(x)\right\|\right)^{-1}; & \left\|V_x^a(x)g(x)\right\| \neq 0 \\ 0; & \left\|V_x^a(x)g(x)\right\| = 0 \end{cases} \tag{33}$$

where $j = 1,\cdots,m$.

In vector form, the general Sontag's formula is

$$u = k(x) = \begin{cases} -\left(V_x^a(x)g(x)\right)^T\left(V_x^a(x)f(x)\right. \\ +\left(\left(V_x^a(x)f(x)\right)^2+\left\|V_x^a(x)g(x)\right\|^2\right)^{\frac{1}{2}}\Bigg) \\ \times\left(\left\|V_x^a(x)g(x)\right\|\right)^{-1}; \quad \left\|V_x^a(x)g(x)\right\| \neq 0 \\ 0; \quad \left\|V_x^a(x)g(x)\right\| = 0. \end{cases} \tag{34}$$

Existence of $\alpha(x,u)$

In the following we show that the existence of $\alpha(x,u)$ is guaranteed.

Consider the performance index $F(x,u)$ as a control Lyapunov function candidate for system (23)-(24). Calculate time derivative of the performance index $F(x,u)$ along the trajectory of system (23)-(24) to obtain

$$\begin{aligned}\dot{F}(x,u) &= F_x(x,u)\dot{x} + F_u(x,u)\dot{u} \\ &= F_x(x,u)f(x,u) - F_u(x,u)\alpha(x,u)\nabla_u F(x,u).\end{aligned} \tag{35}$$

Our objective is to find $\alpha(x,u)$ such that $\dot{F}(x,u) < 0$. If $F_u(x(t),u(t)) \neq 0$ and $\nabla_u F(x,u) \neq 0, \forall t > t_0$, we can find $\alpha(x(t),u(t))$ such that $\dot{F}(x(t),u(t)) < 0$ by Sontag's formula. (Note that we let $\alpha(x,u)$ correspond to a stabilizing control law $k(x)$.) The most crucial step is to determine $\alpha(x,u)$ at a time $t = t_c$, $t_c > t_o$ such that $F_u(x(t_c),u(t_c)) = 0$ or $\nabla_u F(x,u)(t_c) = 0$. For this case, we can not choose $\alpha(x(t_c),u(t_c))$ to make $\dot{F}(x(t_c),u(t_c)) < 0$ because the value of $\dot{F}(x(t_c),u(t_c))$ depends only on the value of $F_x(x(t_c),u(t_c))$ $f(x(t_c),u(t_c))$. Hence, we need an assumption to assure the condition $F_x(x(t_c),u(t_c))\ f(x(t_c),u(t_c)) < 0$ if $F_u(x(t_c),u(t_c))=0$ or $\nabla_u F(x,u)(t_c) = 0$ and then we can choose any value of $\alpha(x(t_c),u(t_c))$, for instance $\alpha(x(t_c),u(t_c)) = 0$.

Now we check when $F_u(x(t_c),u(t_c)) = 0$ or $\nabla\varphi^{t_c}[u](t_c) = 0$ happens. From (22) can be shown that $F_u(x(t_c),u(t_c)) = 0$ at $u(t_c) = 0$, and from the understanding of the gradient descent, $\nabla_u F(x,u)(t_c)$ =0 is achieved at $u(t_c) = 0$. Thus $F_u(x(t_c),u(t_c)) = 0$ or $\nabla_u F(x,u)(t_c)$ =0 at $u(t_c)$=0. Then if $F_u(x(t_c),$ $u(t_c))$=0 or $\nabla_u F(x,u)(t_c) = 0$ then $u(t_c)$ in $F_x(x(t_c),$ $u(t_c))\ \dot{x}(t_c)$ can be substituted with 0, and becomes $F_x(x(t_c),0)f(x(t_c),0)$. From equation (22), $F_x(x(t_c),0)$ $f(x(t_c),0)$ = $V_x(x(t_c))f(x(t_c),0)$ and by assumption **A.3**, we have $F_x(x(t_c),0)f(x(t_c),0) < 0$. Thus, if system (23) satisfies assumption **A.3**, then the condition

$$F_x(x,u)f(x,u) < 0,$$
$$\textit{if } F_u(x,u) = 0 \textit{ or } \nabla_u F(x,u) = 0 \textit{ at a time } t$$

is satisfied.

From the above explanation, by assumption **A.3** there exists $\alpha(x,u)$ such that the performance index (22) satisfies

$$\begin{aligned}\dot{F}(x,u) &= F_x(x,u)f(x,u) \\ &\quad - F_u(x,u)\alpha(x,u)\nabla_u F(x,u) \\ &< 0.\end{aligned} \tag{36}$$

Now, let us consider the equation (23)-(24) as an affine nonlinear control system in which $\alpha(x,u)$ is regarded as a "control input". Then (23)-(24) can be expressed in the form

$$\dot{z} = a(z) + \tilde{\alpha}\, b(z) \triangleq \tilde{f}(z,\alpha), \tag{37}$$

where

$z=[x^T\ u^T]^T$, $a(z)=[f^T(x,u)\ 0]^T$, $b(z)=[0-\nabla_u F(x,u)^T]^T$, $\tilde{\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}$. From (36) and (37), we have

$$\inf_{\alpha \in R_{r\times r}} \left\{F_z(z)\tilde{f}(z,\alpha)\right\} < 0, \quad \forall z \neq 0. \tag{38}$$

Hence, the performance index (22) is a control Lyapunov function for system (23)-(24).

In the following, we find $\alpha(x,u)$ by general Sontag's formula. In this paper we consider $\alpha(x,u)$ as a diagonal matrix $m\times m$, i.e., $\alpha_{ij} = 0$, if $i \neq j$; $i,j = 1,\cdots,m$. Let $\nabla_{u_j} F(x,u)$ is element of vector $\nabla_u F(x,u)$, $j = 1,\cdots m$, then (37) can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} f(x,u) \\ 0_{r\times 1} \end{bmatrix} + \alpha_{11} \begin{bmatrix} 0_{n\times 1} \\ -\nabla_{u_1} F(x,u) \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \alpha_{rr} \begin{bmatrix} 0_{n\times 1} \\ 0 \\ \vdots \\ 0 \\ -\nabla_{u_m} F(x,u) \end{bmatrix}. \tag{39}$$

By applying the general Sontag's formula (33) we can find α_{jj} as follows.

$\alpha_{jj} = k_j(x,u)$ where

$$k_j(x,u) = \left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)\left(F_x(x,u)f(x,u) + \left(\left(F_x(x,u)f(x,u)\right)^2 + \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^2\right)^{\frac{1}{2}}\right) \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^{-1} \quad (40)$$

if

$$\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2 \neq 0$$

and $k_j(x,u) = 0$ if

$$\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2 = 0,$$

$j = 1,\cdots,r$.

Let $K(x,u)$ is a diagonal matrix $r \times r$ with $k_j(x,u)$; $j=1,\cdots,r$ as diagonal element. By substituting $\alpha(x,u)= K(x,u)$ in (40) into (35), we have

$$\begin{aligned}\dot{F}(x,u)\Big|_{\alpha(x,u)=K(x,u)} &= F_x(x,u)f(x,u) - \sum_{j=1}^{m} k_j(x,u)F_{u_j}(x,u)\nabla_{u_j}F(x,u) \\ &= F_x(x,u)f(x,u) - \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right) \\ &\quad \times\left(F_x(x,u)f(x,u) + \left(\left(F_x(x,u)f(x,u)\right)^2 + \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^2\right)^{\frac{1}{2}}\right) \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^{-1} \quad (41)\\ &= F_x(x,u)f(x,u) - \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)\left(F_x(x,u)f(x,u) + \left(\left(F_x(x,u)f(x,u)\right)^2 + \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^2\right)^{\frac{1}{2}}\right) \\ &\quad \times\left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^{-1} \\ &= -\left(\left(F_x(x,u)f(x,u)\right)^2 + \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^2\right)^{\frac{1}{2}},\end{aligned}$$

which guarantees the trivial solution ($x = 0$; $u = 0$) of system

$$\dot{x} = f(x,u), \quad (42)$$

$$\dot{u} = -K(x,u)\nabla_u F(x,u) \quad (43)$$

is asymptotically stable.

To accelerate decreasing of performance index $F(x,u)$, we modify α_{jj} in (40) becomes

$\alpha_{jj} = \bar{k}_j(x,u)$ where

$$\bar{k}_j(x,u) = \left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)\left(F_x(x,u)f(x,u) + \gamma\left(\left(F_x(x,u)f(x,u)\right)^2 + \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^2\right)^{\frac{1}{2}}\right) \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^{-1} \quad (44)$$

if

$$\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2 \neq 0$$

and $\bar{k}_j(x,u) = 0$ if

$$\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2 = 0,$$

$j = 1,\cdots,r$ and γ, $\gamma \geq 0$. γ is called as speed parameter.

By substituting $\alpha(x,u) = \bar{K}(x,u)$ in (44) into (35), we have

$$\dot{F}(x,u)\Big|_{\alpha(x,u)=K(x,u)} = -\gamma\left(\left(F_x(x,u)f(x,u)\right)^2 + \left(\sum_{j=1}^{m}\left(F_{u_j}(x,u)\nabla_{u_j}F(x,u)\right)^2\right)^2\right)^{\frac{1}{2}}. \quad (45)$$

Thus, by taking $\alpha_{jj}, j = 1,\cdots,r$, the extended system (23)-(24) becomes asymptotically stable.

In [5], for single input we have been proved that the system (23)-(24) is still asymptotically stable, even if $a(x,u)$ is a positive constant. Furthermore, in case $a(x,u)$ is a positive constant the assumption **A.3**, "globally asymptotically stable" can be replace by assumption "globally stable". Under an appropriate assumption the extended system

$$\dot{x} = f(x,u)\,, \tag{46}$$

$$\dot{u} = -\alpha\nabla_u F(x,u) \tag{47}$$

is still globally asymptotically stable (see [6] for proofs).

4. EXAMPLES

In this section, we give three examples to show how our technique is applied.

Example 1: Consider a single input nonlinear control system

$$\dot{x}_1 = -x_1 + x_2^2\,,$$

$$\dot{x}_2 = -x_2 + u.$$

Take the equilibrium point for this system is (0,0,0). This system is stable asymptotically if $u = 0$. Based on lyapunov design, the input u is equal to $-x_1x_2$ that make system is still asymptotically stable.

To calculate the direct gradient descent control, define the performance index

$$F(x,u) = \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right). \tag{48}$$

Then we have the gradient of performance index

$$\nabla_u F(x(t;\overline{u}),u) = x_2 + u,$$

the direct gradient descent control (DGDC)

$$\dot{u} = -\alpha(x,u)(x_1^2 + u),$$

with

$$K(x,u) = \frac{a(x) + \gamma\sqrt{(a(x))^2 + (b(x))^2}}{b(x)}\,, \tag{49}$$

where

$$a(x) = -x_1^2 + x_1x_2^2 - x_2^2 + x_2u,$$

$$b(x) = u(x_2 + u).$$

Simulation Initial Condition : $x_1(0) = 1, x_2(0) = -1$, $u(0) = 0.5$. Simulation results are shown in Fig. 1 for $x_1(t)$ and in Fig. 2 for $x_2(t)$ with $\gamma = 1.123$. ($x_ia(t) = x_i(t)$ with DGDC, $x_ib(t) = x_i(t)$ with $u=-x_1(t)x_2(t)$, $x_ic(t) = x_i(t)$ with $u(t) = 0$.)

Example 2: Consider a single input nonlinear control system

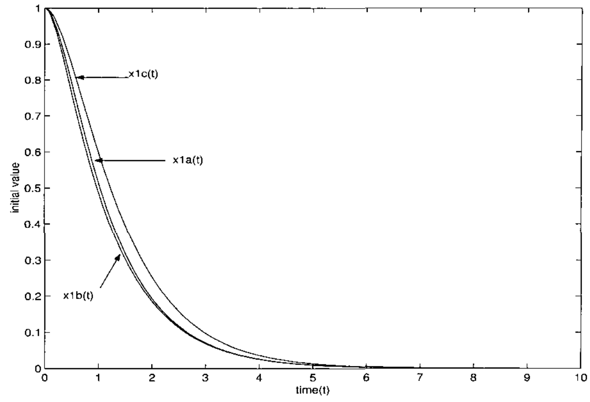

Fig. 1. $x_1(t)$.

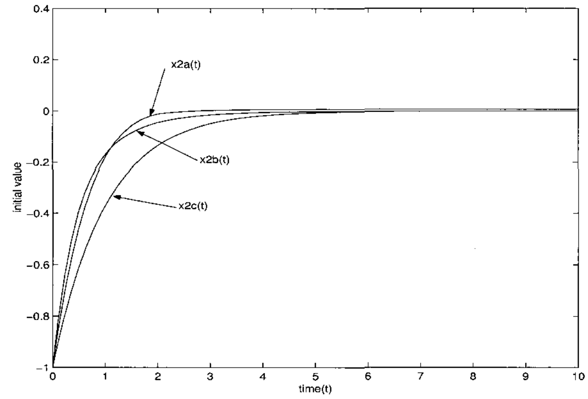

Fig. 2. $x_2(t)$.

$$\dot{x}_1 = -x_2 - x_1u\,,$$

$$\dot{x}_2 = x_1 - x_2^3.$$

Take the equilibrium point for this system is (0,0,0). This system is stable if $u(t)$=0.

Define the Performance index :

$$F(x,u) = \frac{1}{2}\left(x_1^2 + x_2^2 + u^2\right). \tag{50}$$

Next we find the gradient of performance index, gradient descent control and $\alpha(x,u)$ respectively.

The gradient of performance index :

$$\nabla_u F(x(t;\overline{u}),u) = x_1^2 + u.$$

The gradient descent control algorithm :

$$\dot{u} = -\alpha(x,u)(x_1^2 + u).$$

$$K(x,u) = \frac{-x_1^2u - x_2^4 + \gamma\sqrt{(-x_1^2u - x_2^4)^2 + (u(x_1^2+u))^2}}{u(x_1^2+u)}$$

Simulation Initial Condition : $x_1(0) = 0.8$; $x_2(0) = 0.9$, $u(0) = 0.5$. Simulation results are shown in Fig. 3 for $u(t) = 0$ and in Fig. 4 for system which gradient descent control is applied with $\gamma = 0.1$.

Example 3: Consider the Brockett integrator[9] as described by the following equation :

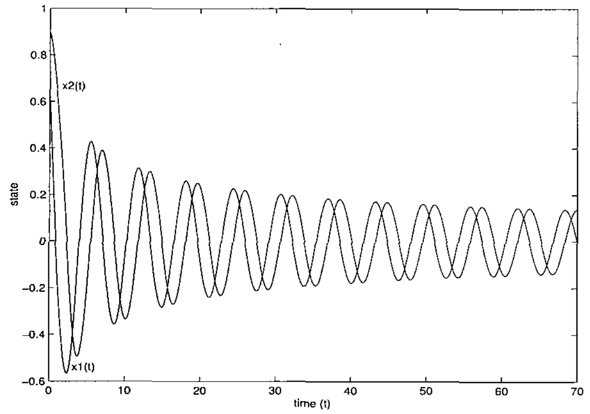

Fig. 3. $u(t)$=0.

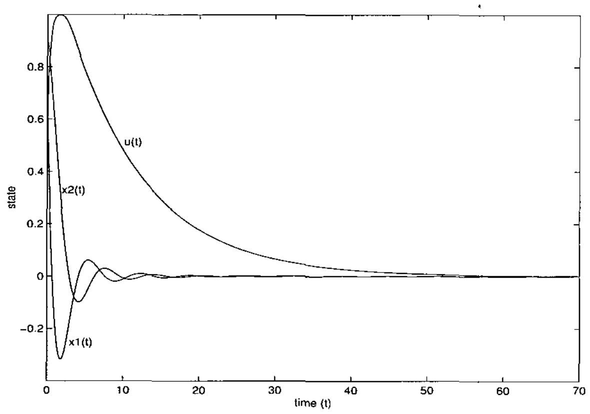

Fig. 4. $\alpha(x(t),u(t))=K(x(t),u(t))$ and $\gamma=0.1$.

$$
\begin{aligned}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= x_1u_2 - x_2u_1.
\end{aligned} \tag{51}
$$

Besides uncontrollable eigenvalue at the origin, this system is an example to show that the smooth feedback fails [9] to stabilize that system asymptotically. In this paper, we stabilize (51) by dynamic feedback control (43).

Let the performance index

$$F(x,u) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + u_1^2 + u_2^2). \tag{52}$$

By applying the direct gradient descent control to nonlinear system (51) we have the extended system

$$
\begin{aligned}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= x_1u_2 - x_2u_1, \\
\dot{u}_1 &= -k_1(x,u)(x_1 - x_2x_3 + u_1), \\
\dot{u}_2 &= -k_2(x,u)(x_2 + x_1x_3 + u_2),
\end{aligned} \tag{53}
$$

with

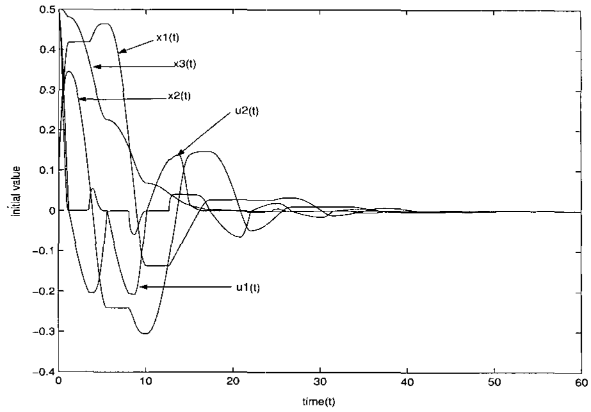

Fig. 5. $\alpha(x(t),u(t)) = K(x(t),u(t))$ and $\gamma = 0.801$.

$$k_1(x,u) = \frac{a(z+\gamma\sqrt{z^2+(a^2+b^2)^2}}{a^2+b^2},$$

$$k_2(x,u) = \frac{b(z+\gamma\sqrt{z^2+(a^2+b^2)^2}}{a^2+b^2},$$

where

$$a = x_4(x_1 - x_2x_3 + x_4),\ b = x_5(x_2 + x_1x_3 + u_2),$$
$$z = x_1x_4 + x_2x_5 + x_3(x_1 - x_2x_3 + x_4).$$

Simulation Initial Condition : $x_1(0) = 0.1$; $x_2(0) = 0.1$; $x_3(0) = 0.5$; $u_1(0) = 0.5$; $u_2(0) = 0.5$ Simulation results are shown in Fig. 5 for system which gradient descent control is applied with $\gamma = 0.801$.

5. CONCLUSIONS

We studied the direct gradient descent control for stabilizing the single input general nonlinear control systems when the unforced system is asymptotically stable. Direct gradient descent control and original system form a new system (extended system) which can be considered as affine nonlinear control system with $\alpha(x,u)$ as a control variable. By using Sontag's formula, $\alpha(x,u)$ can be calculated such that the extended system become asymptotically stable.

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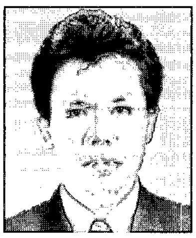

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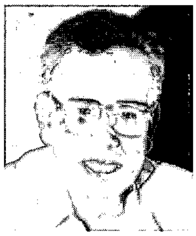

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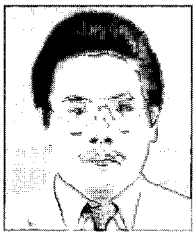

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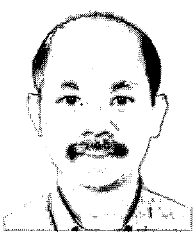

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