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FUNCTION APPROXIMATION OVER TRIANGULAR DOMAIN USING CONSTRAINED Legendre POLYNOMIALS

YOUNG JOON AHN

ABSTRACT. We present a relation between the orthogonality of the constrained Legendre polynomials over the triangular domain and the BB (Bézier -Bernstein) coefficients of the polynomials using the equivalence of orthogonal complements. Using it we also show that the best constrained degree reduction of polynomials in BB form equals the best approximation of weighted Euclidean norm of coefficients of given polynomial in BB form from the coefficients of polynomials of lower degree in BB form.

1. INTRODUCTION

Degree reduction of Bézier curves is one of the important problems in CAGD (Computer Aided Geometric Design) or CAD/CAM. In general, degree reduction cannot be done exactly so that it invokes approximation problems. Thus many efforts and proposals for dealing with the problems have been made in the recent twenty years or so. They are classified by different norm in which the distance between polynomials is measured, e.g., in L_∞-norm [7, 13], in L_2-norm [15, 16, 17], in L_1-norm [11] or in L_p-norm [5, 10], etc. Furthermore, the constrained degree reduction of Bézier curves with C^{a-1}-continuity at both end points is developed in many previous literature [1, 2, 4, 6, 8, 12, 14, 17, 19, 20].

Recently, Lutterkort et al. [18] showed that the orthogonal complement of a subspace in the polynomial space of degree n over the triangular domain with respect to the L_2-inner product and the Euclidean inner product of BB coefficients are equal. Using this fact they also showed that the best degree reduction of polynomial f of degree n over the triangular domain in L_2-norm is equivalent to the best approximation of the vector of BB coefficients of f from all vector of BB coefficients of degree elevated polynomials of degree less than n in the Euclidean norm of the vector. We follow their results in the case of constrained degree reduction over the triangular domain. We first show that the orthogonal complement of a subspace in the constrained polynomial space of degree n over the triangular domain with respect to L_2-inner product and the

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weighted Euclidean inner product of BB coefficients are equal for some weights. Using the fact we also show that the best constrained degree reduction of f of degree n over the triangular domain in L_2-norm is equal to the best approximation of the vector of coefficients from all vectors of coefficients of degree elevated polynomials with the constraint in weighted Euclidean norm of vectors. Finally we present a relation between the orthogonality of the constrained Legendre polynomials over the triangular domain and the BB (Bézier -Bernstein) coefficients of the polynomials using the equivalence of orthogonal complements. The relation plays an important role to construct the constrained Legendre polynomials in BB form.

The outline of this paper is as follows. In Section 2, we explain the constrained polynomial space over triangular domain and the constrained Legendre polynomials. In Section 3, we show that the orthogonal complement of a subspace in the constrained polynomial space of degree n over triangular domain with respect to L_2-inner product and the weighted Euclidean inner product of BB coefficients are equal. In Section 4, we present the properties of the best constrained degree reduction in BB form and the constrained Legendre polynomials over triangular domain in BB form.

2. Constrained Legendre Polynomials over triangular domain

In this section we consider the constrained Legendre polynomial of degree n over triangular domain. Let T be a triangle in the plane, defined by vertices $\mathbf{p}_k = (x_k, y_k)$ for $k = 0, 1, 2$. If these vertices are not collinear, any point $\mathbf{p} \in T$ can be written uniquely in terms of *barycentric coordinates* u, v, w where $u + v + w = 1$, with respect to T:

$$\mathbf{p} = u\mathbf{p}_0 + v\mathbf{p}_1 + w\mathbf{p}_2.$$

Let $\mathbb{P}_n$ be the linear space of polynomials of degree less than or equal to n. It is convenient to introduce the compact notation $\alpha = (\alpha_1, \alpha_2)$ to denote doubles of nonnegative integers, and we write $|\alpha| = \alpha_1 + \alpha_2$. The Bernstein basis of degree n over T is denoted by

$$B^n_\alpha(u, v) = \frac{n!}{\alpha_1!\alpha_2!(n - \alpha_1 - \alpha_2)!} u^{\alpha_1} v^{\alpha_2} (1 - u - v)^{(n-\alpha_1-\alpha_2)}, \qquad |\alpha| \le n.$$

Thus $\mathbb{P}_n$ has exactly $(n+1)(n+2)/2$ basis functions. We collect the basis functions in triangular arrays of size n

$$B^n := [B^n_\alpha]_{|\alpha| \le n}$$

and with $b = [b_\alpha]_{|\alpha| \le n}$ a simplicial array of reals we write polynomials in BB form as

$$B^n b = \sum_{|\alpha| \le n} B^n_\alpha b_\alpha.$$

For the nonnegative integer $a \le n/3$, $\mathbb{P}^a_m$ is denoted by the linear space of the constrained polynomials of degree less than or equal to m over the triangle T as follows:

$$\mathbb{P}^a_m = \{B^n b \in \mathbb{P}_m : \ b_\alpha = 0 \text{ for } \alpha \in J^a_n\}.$$

where I_n^a and J_n^a are the sets of double index α such that

$$\begin{aligned} I_n^a &= \{|\alpha| \le n \ : \ \alpha_1 \ge a,\ \alpha_2 \ge a,\ |\alpha| \le n-a\} \\ J_n^a &= \{|\alpha| \le n \ : \ \alpha \notin I_n^a\} \end{aligned}$$

Then $\mathbb{P}_m^a$ has exactly $(k+1)(k+2)/2$ basis functions, where $k = m-3a$. Farouki et al.[9] constructed the basis of linearly independent and mutually orthogonal polynomials $L_{m,i}$, say *Legendre polynomials*, with hierachical ordering in BB form over T. We also consider the constrained Legendre polynomials $L_{m,i}^a$ with hierachical ordering in BB form over T. For example $a = 1$,

1 degree 3 basis functions $L_{3,0}^1$
2 degree 4 basis functions $L_{4,0}^1$, $L_{4,1}^1$
... ...
$m-2$ degree n basis functions $L_{m,0}^1, \cdots L_{m,m-3}^1$

We present a relation between the orthogonality of the constrained Legendre polynomials and the BB (Bézier -Bernstein) coefficients of the polynomials in section 4 using the equivalence of orthogonal complements in section 3.

3. Equivalence of orthogonal complements

For $a \le m/3$, let

$$\mathbb{Q}_m^a \ = \ \{f(u,v) \in \mathbb{P}_m : f(i,j) = 0 \text{ for } i,j,n-i-j = 0,\cdots,a-1\},$$

which was also introduced for one variable by Ahn et al. [3]. Note that $\mathbb{P}_m = \mathbb{P}_m^0 = \mathbb{Q}_m^0$. We consider the Lagrange polynomials characterized by

$$Q_\alpha^n(\beta) = \delta_{\alpha,\beta}, \qquad |\alpha|, |\beta| \le n.$$

Peters and Reif [18] was already introduced the notation of the Lagrange polynomials Q_α^n. We collect the basis functions in triangular arrays of size n

$$Q^n := [Q_\alpha^n]_{|\alpha| \le n}$$

with $b = [b_\alpha]_{|\alpha| \le n}$ a simplicial array of reals we write polynomials in Lagrange form as

$$Q^n b = \sum_{|\alpha| \le n} Q_\alpha^n b_\alpha.$$

The Lagrange form is used to relate a discrete polynomial dependence of the coefficients on the array index to a continuous polynomial. For example, if the coefficients $b_\alpha = (\alpha_1\alpha_2 + \alpha_2)\alpha_1\alpha_2(n-\alpha_1-\alpha_2)$ depend quintically on the index α, then $Q^5(u,v)b = (uv+v)uv(n-u-v)$ is the corresponding quintic polynomial. The following lemma is an extension of Lemma 2.1 in Lutterkort et al. [17], Lemma 3.1 in Ahn et al. [3] and Lemma 2.1 in Peters and Reif [18].

Lemma 3.1. *A polynomial $B^n b$ is of degree $\le m$ with $b_\alpha = 0$ for $\alpha \in J_n^a$ if and only if the triangular array of coefficients is a polynomial of degree $\le m$ with zeros at (i,j), $i,j,n-i-j = 0,\cdots,a-1$ in its index, i.e.,*

$$B^n b \in \mathbb{P}_m^a \Leftrightarrow Q^n b \in \mathbb{Q}_m^a.$$

Proof. It is well-known [18] that

$$B^n b \in \mathbb{P}_m^0 \Leftrightarrow Q^n b \in \mathbb{Q}_m^0.$$

Note that if the coefficients b_α depend on the index α, then $Q^n(u,v)b$ is also the corresponding polynomial. Thus $b_\alpha = 0$ for $\alpha \in J_n^a$ is equivalent to $Q^n(i,j)b = 0$ for $i,j,n-i-j = 0,\cdots,a-1$. Hence we have $B^n b \in \mathbb{P}_m^a$ if and only if $Q^n b \in \mathbb{Q}_m^a$. □

Theorem 3.2. *The orthogonal complements of $\mathbb{P}_m^a$ in $\mathbb{P}_n^a$ with respect to the L_2-inner product*

$$\langle f,g\rangle_L := \int\int_T f(x)g(x)dx \tag{3.1}$$

and the weighted Euclidean inner product of the BB coefficients

$$\langle B^n b, B^n c\rangle_W := \sum_{\alpha\in I_n^a} b_\alpha c_\alpha w_\alpha \tag{3.2}$$

where

$$w_\alpha := \begin{cases} \dfrac{\binom{2\alpha_1}{\alpha_1}\binom{2\alpha_2}{\alpha_2}\binom{2(n-|\alpha|)}{n-|\alpha|}}{\binom{2\alpha_1}{\alpha_1-a}\binom{2\alpha_2}{\alpha_2-a}\binom{2(n-|\alpha|)}{n-|\alpha|-a}} & (\alpha\in I_n^a) \\ 1 & (\alpha\in J_n^a) \end{cases}$$

are equal.

Proof. Denote the orthogonal complement of $\mathbb{P}_m^a$ in $\mathbb{P}_n^a$ with respect to the weighted Euclidean inner product by $\mathbb{P}_{m,n}^a$, and let $\{B^n q^\alpha : m-3a < |\alpha| \le n-3a\}$ be some basis of this space. By equality of dimensions it suffices to show that $\mathbb{P}_{m,n}^a$ is contained in the orthogonal complement with respect to the L_2-inner product, i.e., the polynomials $B^n w^\alpha$ have to be L_2-orthogonal to all polynomials in $\mathbb{P}_m^a$,

$$\langle B^n q^\alpha, u^{a+\beta_1}v^{a+\beta_2}(1-u-v)^a\rangle_L = 0, \qquad (0\le|\beta|\le m-3a<|\alpha|\le n-3a).$$

Defining the triangular array p_α^β by

$$p_\alpha^\beta := \frac{1}{w_\alpha}\int\int_T B_\alpha^n(u,v)u^{a+\beta_1}v^{a+\beta_2}(1-u-v)^a dA,$$

clearly we have

$$\langle B^n q^\alpha, u^{a+\beta_1}v^{a+\beta_2}(1-u-v)^a\rangle_L = \langle B^n q^\alpha, B^n p^\beta\rangle_W.$$

By definition, the latter expression vanishes if and only if $B^n p^\beta \in \mathbb{P}_m^a$, and by Lemma 3.1, this is equivalent to $Q^n p^\beta \in \mathbb{Q}_m^a$. In other words, we have to show that p_α^β is a polynomial in α of degree $\le m$ with zeros at $\alpha \in J_n^a$, for all β with $|\beta| \le m-3a$. Using the formula

$$\int\int_T B_\alpha^n(u,v)dA = \frac{1}{(n+1)(n+2)}.$$

we have

$$\begin{aligned} p_\alpha^\beta &= \frac{1}{w_\alpha}\int\int_T B_\alpha^n(u,v)u^{a+\beta_1}v^{a+\beta_2}(1-u-v)^a dA \\ &= \frac{1}{w_\alpha}\frac{\binom{n}{\alpha}}{\binom{n+|\beta|+3a}{\alpha+\beta+(a,a)}}\int\int_T B_{n+|\beta|+3a}^{\alpha+\beta+(a,a)}(u,v)dA \\ &= \frac{1}{w_\alpha}\frac{\binom{n}{\alpha}}{\binom{n+|\beta|+3a}{\alpha+\beta+(a,a)}}\frac{1}{(n+|\beta|+3a+1)(n+|\beta|+3a+2)} \end{aligned}$$

Now,

$$\begin{aligned} \frac{1}{w_\alpha} &= \frac{\binom{2\alpha_1}{\alpha_1-a}\binom{2\alpha_2}{\alpha_2-a}\binom{2(n-|\alpha|)}{n-|\alpha|-a}}{\binom{2\alpha_1}{\alpha_1}\binom{2\alpha_2}{\alpha_2}\binom{2(n-|\alpha|)}{n-|\alpha|}} \\ &= \frac{\alpha_1!^2\alpha_2!^2(n-|\alpha|)!^2}{(\alpha_1-a)!(\alpha_1+a)!(\alpha_2-a)!(\alpha_2+a)!(n-|\alpha|-a)!(n-|\alpha|+a)!} \end{aligned}$$

and

$$\frac{\binom{n}{\alpha}}{\binom{n+|\beta|+3a}{\alpha+\beta+(a,a)}} = \frac{n!}{(n+|\beta|+3a)!}\frac{(\alpha_1+\beta_1+a)!(\alpha_2+\beta_2+a)!(n-|\alpha|+a)!}{\alpha_1!\alpha_2!(n-|\alpha|)!}$$

so that

$$\begin{aligned} p_\alpha^\beta &= \frac{n!}{(n+|\beta|+3a+2)!}\times\frac{\alpha_1!}{(\alpha_1-a)!}\frac{\alpha_2!}{(\alpha_2-a)!}\frac{(n-|\alpha|)!}{(n-|\alpha|-a)!} \\ &\quad\times\frac{(\alpha_1+\beta_1+a)!}{(\alpha_1+a)!}\frac{(\alpha_2+\beta_2+a)!}{(\alpha_2+a)!} \\ &= \frac{n!}{(n+|\beta|+3a+2)!}\times\prod_{l_1=1}^{a}(\alpha_1-a+l_1)\prod_{l_2=1}^{a}(\alpha_2-a+l_2)\prod_{l_3=1}^{a}(n-|\alpha|-a+l_3) \\ &\quad\times\prod_{r_1=1}^{\beta_1}(\alpha_1+a+r_1)\prod_{r_2=1}^{\beta_2}(\alpha_2+a+r_2). \end{aligned}$$

Thus p_α^β is a polynomial of degree $\le m$ and $p_\alpha^\beta = 0$ for $\alpha \in J_n^a$ i.e., $\alpha_1 = 0,\cdots,a-1$, $\alpha_2 = 0,\cdots,a-1$ and $n-|\alpha| = 0,\cdots,a-1$. □

In particular, in one dimensional case, i.e., $d=1$, the weights are

$$w_\alpha = \frac{\binom{2\alpha_1}{\alpha_1}\binom{2(n-\alpha_1)}{n-\alpha_1}}{\binom{2\alpha_1}{\alpha_1-a}\binom{2(n-\alpha_1)}{n-\alpha_1-a}} = \frac{\frac{(2\alpha_1)!}{\alpha_1!^2}\frac{(2(n-\alpha_1))!}{(n-\alpha_1)!^2}}{\frac{(2\alpha_1)!}{(\alpha_1-a)!(\alpha_1+a)!}\frac{(2(n-\alpha_1))!}{(n-\alpha_1-a)!(n-\alpha_1-a)!}} = \frac{\binom{n}{\alpha_1}^2}{\binom{n}{\alpha_1-a}\binom{n}{\alpha_1+a}}$$

which are the same weights given by Ahn. et al. [3].

4. Properties of constrained Legendre polynomials in BB form

Theorem 4.1. *Given a polynomial $B^n b$ of degree n, the approximation problem*

$$\min_{p\in\mathbb{P}_m}\{\|B^n b - p\| : p = B^n c \in \mathbb{P}_m,\ b_\alpha = c_\alpha \quad for\ \alpha \in J_n^a\}$$

has the same minimizer for the norm induced either by the L_2-inner product (3.1) or the weighted Euclidean inner product (3.2).

Proof. Let $f^a = B^n d \in \mathbb{P}_m$ be a polynomial of degree m satisfying

$$b_\alpha = d_\alpha$$

for $\alpha \in J_n^a$. Then the polynomial $B^n b - f^a \in \mathbb{P}_n^a$ can be decomposed uniquely according to

$$B^n b - f^a = p^a + q^a, \qquad p^a \in \mathbb{P}_m^a, \qquad q^a \in \mathbb{P}_{m,n}^a$$

and, by the orthogonality, p^a is the minimizer of

$$\min_{p^a\in\mathbb{P}_m^a} \|B^n b - f^a - p^a\|$$

for both norm. For all $p = B^n c \in \mathbb{P}_m$ satisfying

$$b_\alpha = c_\alpha$$

for $\alpha \in J_n^a$, we have

$$\|B^n b - p\| = \|B^n b - f^a - (p - f^a)\| \geq |B^n b - f^a - p^a\|$$

since $p - f^a \in \mathbb{P}_m^a$. Thus $p = p^a + f^a \in \mathbb{P}_m$ is the wanted solution for both norms. □

Corollary 4.2. *Denote by $\mathcal{P}_{m,n}^a$ the linear operator mapping polynomials $B^n b \in \mathbb{P}_n$ to their best constrained L_2-norm or weighted Euclidean approximant $p \in \mathbb{P}$. Then*

$$\mathcal{P}_{m,n}^a = \mathcal{P}_{m,\ell}^a \mathcal{P}_{\ell,n}^a, \qquad m \leq \ell \leq n.$$

For $m \geq 1$, consider the space $\mathcal{L}_m$ of degree-m polynomials that are orthogonal to all polynomials of degree $< m$ over T:

$$\mathcal{L}_m^a = \{p \in \mathbb{P}_m^a : p \perp \mathbb{P}_{m-1}^a\}.$$

The following theorem is an extension of Lemma 4 in Farouki et al. [9].

Theorem 4.3. *Let* $p = B^n c \in \mathbb{P}_n^a$. *Then we have*

$$p \in \mathcal{L}_n^a \quad \Leftrightarrow \quad \sum_{\alpha \in I_n^a} b_\alpha c_\alpha w_\alpha = 0 \ \text{ for all } \ q = B^n b \in \mathbb{P}_{n-1}^a$$

Proof. By definition of $\mathcal{L}_n^a$, $p \in \mathcal{L}_n^a$ if and only if $p \perp q$ for all $q \in \mathbb{P}_{n-1}^a$. It is equivalent to

$$\int \int_T pq dA = 0.$$

By Theorem 3.2, These are equivalent to

$$\sum_{\alpha \in I_n^a} c_\alpha b_\alpha w_\alpha = 0.$$

□

Let $f(u, v)$ be a given C^{a-1} function in T. If $q(u, v)$ is chosen in $\mathbb{P}_n$ so that q is the C^{a-1}-interpolation of f at the boundaries, and $P \in \mathbb{P}_n^a$ satisfies

$$\min_{P \in \mathbb{P}_n^a} \int \int_T [(f(u, v) - q(u, v)) - P(u, v)]^2 dA,$$

then the polynomial $q(u, v) - P(u, v)$ is a constrained n-the degree polynomial approximation of $f(u, v)$. If the constrained Legendre polynomials are constructed, then $P(u, v)$ can be simply obtained as

$$P(u, v) = \sum_{r=0}^{n} \sum_{i=0}^{r} \ell_{r,i} L_{r,i}(u, v)$$

with

$$\ell_{r,i} = \frac{< L_{r,i}, f - q >}{< L_{r,i}, L_{r,i} >}.$$

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Young Joon Ahn:
DEPARTMENT OF MATHEMATICS EDUCATION, CHOSUN UNIVERSITY, GWANGJU, 501–759, SOUTH KOREA
E-mail: ahn@chosun.ac.kr