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MOD M NORMALITY OF β-EXPANSIONS

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ABSTRACT. If $\beta > 1$, then every non-negative number x has a β-expansion, i.e.,

$$x = \epsilon_0(x) + \frac{\epsilon_1(x)}{\beta} + \frac{\epsilon_2(x)}{\beta} + \cdots$$

where $\epsilon_0(x) = [x]$, $\epsilon_1(x) = [\beta(x)]$, $\epsilon_2(x) = [\beta((\beta x))]$, and so on ($[x]$ denotes the integral part and (x) the fractional part of the real number x). Let T be a transformation on $[0, 1)$ defined by $x \to (\beta x)$. It is well known that the relative frequency of $k \in \{0, 1, \cdots, [\beta]\}$ in β-expansion of x is described by the T-invariant absolutely continuous measure μ_β. In this paper, we show the mod M normality of the sequence $\{\epsilon_n(x)\}$.

1. INTRODUCTION

Let $(X, \mathcal{B}, \mu)$ be a probability space and T a measure preserving transformation on X. A transformation T on X is called ergodic if the constant function is the only T-invariant function and it is called weakly mixing if the constant function is the only eigenfunction with respect to T. A measure preserving transformation T is called an exact transformation if $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}$ is the trivial σ-algebra consisting of empty set and whole set modulo measure zero sets. So exact transformation are as far from being invertible as possible. Recall that if a transformation is exact then that transformation is weakly mixing [10].

Let $X = \{\, x \,|\, 0 \le x < 1\}$ be the compact group of real numbers modulo 1, and let $\theta \in X$ be irrational. The numbers $j\theta, j = 0, \pm 1, \cdots$, comprise a dense subgroup of X. For each interval $I \subset X$ and $n > 0$ define $S_n = S_n(\theta, I)$ to be the numbers of integers $j, 1 \le j \le n$, such that $j\theta \in I$. By Kronecker-Weyl theorem $\lim_{n\to\infty} \frac{S_n}{n} = \mu(I)$, where μ is Lebesgue measure on X. Veech [9] is interested in the behavior of the sequence $\{d_n\}$ of parities of $\{S_n\}$. That is, d_n is 0 or 1 as S_n is even or odd. i.e., He investigates the existence of the limit

$$\mu_\theta(I) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n,$$

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and he shows that a necessary and sufficient condition for $\mu_\theta(I)$ to exist for every interval $I \subset X$ is that θ has bounded partial quotients and show that d_n is evenly distributed if the length of the interval is not an integral multiple of θ modulo 1.

For a given $\beta > 1$, consider a β-transformation, T on $[0,1)$ defined by $x \to (\beta x)$.

In this paper, we are interested in the uniform distribution of the sequence $d_n \in \{0, \cdots, M-1\}$ defined by

$$d_n(x) \equiv \sum_{k=0}^{n-1} \mathbf{1}_E(T^k x) \pmod{M},$$

for β-transformations on the interval where $\mathbf{1}_E(x)$ is an indicator function of finite union of intervals, E. If E is the form of $[\frac{k}{\beta}, \frac{k+1}{\beta})$ when $\frac{k+1}{\beta} < 1$ or $[\frac{k}{\beta}, 1)$ otherwise, then the distributions of $d_n(x)$ are just the mod M normality of the sequence $\{\epsilon_n(x)\}$.

In [1], Ahn and Choe consider the case when transformations defined defined by $x \mapsto (Lx)$ with $L \in \mathbb{N}$ on $X = [0,1)$ and $M = 2$, and show that the sequence $\{d_n\}$ is evenly distributed if $\exp(\pi i \mathbf{1}_E(x))$ has finite L-adic discontinuity points $\frac{1}{L} \le t_1 < \cdots < t_n \le 1$. Recently, Choe, Hamachi and Nakada[4] show that $\{d_n\}$ is evenly distributed for more general sets and that $\mathbb{Z}_2$-extension induced by $\phi(x) = \exp(\pi i \mathbf{1}_B(x))$ where $\mathbf{1}_B$ is a characteristic function of B, is ergodic. In this paper, we show that for all β-transformations on the unit interval, the sequence $\{d_n\}$ is uniformly distributed and that the corresponding compact group extension of β-transformation is weakly mixing. Hence the compact group extension by $\phi(x)$ is exact.

2. Properties of β-expansions

In this section, we recall the basic properties of β-expansions. For more related results, see [3, 7, 8]. For every $\beta > 1$, there is a unique T_β-invariant absolutely continuous normalized measure μ_β and T_β is an exact transformation on $(X, \mathcal{B}, \mu_\beta)$. By the Radon-Nikodym theorem, there is a measurable function $h_\beta(x)$, essentially unique, such that

$$\mu_\beta(E) = \int_E h_\beta(x)\,dx.$$

We call $h_\beta(x)$ as the density function of μ_β.

Theorem 1. *Let T be the transformation on $[0,1)$ defined by $T(x) = (\beta x)$. Then the density function of T-invariant absolutely continuous measure $h_\beta(x)$ is described by*

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{n, T^n(1) > x}^{\infty} \frac{1}{\beta^n} \quad \text{where} \quad F(\beta) = \int_0^1 \left(\sum_{n, T^n(1) > x}^{\infty} \frac{1}{\beta^n} \right) dx.$$

Proof. Since $\mu_\beta E = \mu_\beta T^{-1} E$ for all Lebesgue measurable set E,

$$\mu_\beta[a,b) = \int_a^b h_\beta(x)\,dx = \mu_\beta T^{-1}[a,b) = \sum_{m=0}^{[\beta - b]} h_\beta\left(\frac{x+m}{\beta}\right) \qquad \text{a.e.}$$

if $[\beta - b]$ is the largest integer m for which both $\frac{a+m}{\beta}$ and $\frac{b+m}{\beta}$ are less than 1, and either $a < (\beta)$ and $b < (\beta)$ or $a > (\beta)$ and $b > (\beta)$. In this case,

$$\beta\Big(\int_a^b \frac{h_\beta(x)}{b-a}\,dx\Big) = \sum_{m=0}^{[\beta-b]}\Big(\Big(\int_{\frac{a+m}{\beta}}^{\frac{b+m}{\beta}} h_\beta(x)\,dx\Big)\Big/\Big(\frac{b+m}{\beta} - \frac{a+m}{\beta}\Big)\Big).$$

Hence we show that the density function of μ_β, $h_\beta(x)$ satisfies the following.

$$\beta h_\beta(x) = \sum_{Ty=x} h_\beta(y) = \sum_{m=0}^{[\beta-x]} h_\beta\Big(\frac{x+m}{\beta}\Big).$$

Now we will show that $h_\beta(x)$ defined by

$$h_\beta(x) = \sum_{n,T^n(1)>x}^{\infty} \frac{1}{\beta^n},$$

satisfies $\beta h_\beta(x) = \sum_{m=0}^{[\beta-x]} h_\beta(\frac{x+m}{\beta})$.

Let

$$a_{n,m} = \begin{cases} 1, & \text{if } \dfrac{x+m}{\beta} < T^n(1), \\ 0, & \text{otherwise} \end{cases}.$$

and

$$a_{n,m} = \begin{cases} 1, & \text{if } x < T^n(1), \\ 0, & \text{otherwise.} \end{cases}.$$

Note that $\sum_{m=0}^{[\beta-x]} a_{n,m}$ is the number of m among $0, 1, \cdots, [\beta - x]$ with $\frac{x+m}{\beta} < T^n(1)$. Hence

$$\sum_{m=0}^{[\beta-x]} a_{n,m} = \begin{cases} [\beta T_\beta^n(1)] + 1, & \text{if } x < (\beta T^n(1)) = T^{n+1}(1) \\ [\beta T_\beta^n(1)],. & \text{otherwise.} \end{cases}.$$

Thus

$$\begin{aligned}\sum_{m=0}^{[\beta-x]} h_\beta\Big(\frac{x+m}{\beta}\Big) &= \sum_{m=0}^{[\beta-x]}\sum_{n=0}^{\infty} \frac{a_{n,m}}{\beta^n} = \sum_{n=0}^{\infty}\sum_{m=0}^{[\beta-x]} \frac{a_{n,m}}{\beta^n} = \sum_{n=0}^{\infty} \frac{[\beta T_\beta^n(1)] + a_{n+1}}{\beta^n} \\ &= \sum_{n=0}^{\infty} \frac{\epsilon_n(\beta)}{\beta^n} + \beta\sum_{n=0}^{\infty} \frac{a_{n+1}}{\beta^{n+1}} = \beta + \beta\sum_{n=0}^{\infty} \frac{a_n}{\beta^n} - \beta a_0 \\ &= \beta h_\beta(x).\end{aligned}$$

Finally, consider the following formula,

$$F(\beta) = \int_0^1 \Big(\sum_{n, T^n(1) > x}^{\infty} \frac{1}{\beta^n} \Big)\, dx = \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{a_n(x)}{\beta^n} \Big)\, dx = \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{a_n(x)}{\beta^n} \Big)\, dx$$
$$= \sum_{n=0}^{\infty} \frac{1}{\beta^n} \int_0^1 a_n(x)\, dx = \sum_{n=0}^{\infty} \frac{T^n(1)}{\beta^n}$$

So we have

$$1 \le F(\beta) \le 1 + \frac{1}{\beta} + \frac{1}{\beta^2} + \cdots = \frac{\beta}{\beta - 1}.$$

Hence $h_\beta(x)$ is well defined. By the ergodicity of T, the proof is completed. □

REMARK 1. We call those β which have recurrent tails, i.e., $\epsilon_{n+k}(\beta) = \epsilon_n(\beta)$ for all $n \ge N$ in their β-expansions, as *β-numbers*. Those with zero tails, we call *simple β-number*. It is easy to know that $h_\beta(x)$ is a step function with finite discontinuity if and only if β has a recurrent tail in its β-expansion by the previous Theorem.

EXAMPLE 1. Let $\beta = \dfrac{\sqrt{5}+1}{2}$. Then $(\beta) = \beta - 1 = \frac{1}{\beta}$. So $T^2(1) = 1$. Hence by the formula of the previous Theorem,

$$h_\beta(x) = \begin{cases} \dfrac{5+3\sqrt{5}}{2} & \text{for} \quad 0 \le x < \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{5+\sqrt{5}}{2}, & \text{for} \quad \dfrac{\sqrt{5}-1}{2} \le x < 1. \end{cases}.$$

3. MOD M NORMALITY AND COBOUNDARY EQUATIONS OF β-TRANSFORMATIONS

Let G be a finite subgroup of the circle group $\mathbb{T}$ generated by $\exp(\frac{2\pi i}{M})$. To investigate the sequence $\{d_n(x)\}$, we consider the behavior of the sequence $\exp(\frac{2\pi i}{M} d_n(x))$ and check whether this sequence is uniformly distributed on compact group G generated by $\exp(\frac{2\pi i}{M})$. Weyl's criterion on uniform distribution says that the sequence $\exp(\frac{2\pi i}{M} d_n(x))$ is uniformly distributed if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp^k\Big(\frac{2\pi i}{M} d_n(x)\Big) = 0$$

for all $1 \le k \le M-1$ [5].

We investigate the problem from the viewpoint of spectral theory. Let (X, μ) be a probability space and T an ergodic measure preserving transformation on X, which is not necessarily invertible. Let $\phi(x)$ be a G-valued function defined by $\phi(x) = \exp(\frac{2\pi i}{M} \mathbf{1}_E(x))$. Consider the skew product transformation T_ϕ on $X \times G$ defined by $T_\phi(x, g) = (Tx, \phi(x)g)$. Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1}^{N}\exp^k\left(\frac{2\pi i}{M}d_n(x)\right)\cdot z^k=\lim_{N\to\infty}\frac{1}{N}\sum_{1}^{N}(U_{T_\phi})^n f(x,z)$$

where U_{T_ϕ} is an isometry on $L^2(X\times G)$ induced by T_ϕ and $f(x,z)=z^k$. Hence if T_ϕ is ergodic, then $\lim_{N\to\infty}\frac{1}{N}\sum_1^N\exp^k(\frac{2\pi i}{M}d_n(x))=0$ by an application of the Birkhoff's Ergodic theorem to $f(x,z)=z^k$. Recall that the dual group of G consists of the trivial homomorphism 1 and γ_k defined by $\gamma_k(z)=z^k$ for $1\le k\le M-1$. Hence $L^2(X\times G)=\bigoplus_{k=0}^{L-1}L^2(X)\cdot z^k$ and each $L^2(X)\cdot z^k$ is an invariant subspace of U_{T_ϕ}. If $f(x,z)$ is an eigen-function with eigenvalue λ then $f(x,z)=\sum_{k=0}^{L-1}f_k(x)\cdot z^k$ and

$$U_{T_\phi}f(x,z)=\sum_{k=0}^{L-1}\phi^k(x)f_k(Tx)\cdot z^k.$$

Thus $\phi^k(x)f_k(Tx)=\lambda f_k(x)$ for each k. Hence to check the ergodicity we only need to know whether there exist $0\le k\le M-1$ and $f(x)$ such that $\phi^k(x)f(Tx)=\underline{f(x)}$.

Recall that a function $f(x)$ is called a *quasicoboundary* if $f(x)=\lambda\cdot\overline{q(x)}q(Tx)$, $|q(x)|=1$, $|\lambda|=1$ a.e. on X. Specially if $\lambda=1$ then $f(x)$ is called a coboundary. Similarly a real valued function $g(x)$ is called an additive quasicoboundary if $g(x)=k+q(Tx)-q(x)$ $k\in\mathbb{R}$. Hence if $g(x)$ is an additive quasicoboundary then $f(x)\equiv\exp(2\pi i g(x))$ is a quasicoboundary.

Proposition 1. *Let T be an ergodic transformation on X and $\phi(x)$ be a G-valued function. Let T_ϕ be the skew product transformation defined by $T_\phi(x,g)=(Tx,\phi(x)\cdot g)$ on $X\times G$. If $\phi(x)h(Tx)=h(x)$, then there exists a G-valued function $q(x)$ such that the following diagram commutes*

$$\begin{array}{ccc} X\times G & \xrightarrow{T_\phi} & X\times G \\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle Q} \\ X\times G & \xrightarrow{S} & X\times G \end{array}$$

where $Q(x,g)=(x,q(x)g)$ and $S(x,g)=(Tx,g)$. Hence T_ϕ has M ergodic components.

Proof. Since $(\phi(x))^M=1$, $(\phi(x))^M(h(Tx))^M=(h(x))^M$ is equivalent to $(h(Tx))^M=(h(x))^M$. So we may assume that $(h(x))^M=1$ by the ergodicity of T. Hence there exist a G-valued function $q(x)$ such that $\phi(x)q(Tx)=q(x)$. For this $q(x)$, it turns out that the diagram commutes by easy consideration. □

Lemma 1. *Let τ be a piecewise twice continuously differentiable function such that $\inf_{x\in J_1}|\tau'(x)|>1$ where $J_1=\{x\in X,\tau'(x)\text{exists}\}$. If the number of discontinuity points of τ or τ' is finite, then there is a finite collection of sets $L_1,\cdots,L_n$ and a set of invariant function $\{f_1,\cdots,f_n\}$ such that*

(1) *each $L_i(1\le i\le n)$ is a finite union of closed intervals;*
(2) *$L_i\cap L_j$ contains at most a finite number of points when $i\ne j$;*

(3) $f_i(x) = 0$ *for* $x \notin L_i, 1 \le i \le n$, *and* $f_i(x) > 0$ *for a.e.* x *in* L_i;
(4) $\int_{L_i} f_i(x)dx = 1$ *for* $1 \le i \le n$;
(5) *every* τ *invariant function can be written as* $f = \sum_{i=1}^{n} a_i f_i$ *with suitable chosen* $\{a_i\}$.

Proof. For the proof, See [2, 6]. □

Proposition 2. *For the* β*-transformation, if a* G*-valued function* $\phi(x)$ *is a step function with finite discontinuity points and* $\phi(x)h(Tx) = h(x)$, *then there exist* G*-valued function* $q(x)$ *which is a step function with finite discontinuity points and* $\phi(x)q(Tx) = q(x)$. *Hence we may assume that* $h(x)$ *is a* G*-valued step function with finite discontinuity points.*

Proof. Assume that $\phi(x)h(Tx) = h(x)$. Without loss of generality assume that $X = [0,1)$. Since $X \times G = \bigcup_{k=0}^{M-1}\{X \times \exp(\frac{2k\pi i}{M})\}$, let's identify $\{X \times \exp(\frac{2k\pi i}{M})\}$ with the unit interval $[k, k+1)$, $0 \le k < M$. Since $\phi(x)$ is a G-valued step function with finite discontinuity points, we can regard T_ϕ as a piecewise continuous map on $[0, M)$ satisfying the condition of Lemma 1. So there exist a G-valued function $q(x)$ which is a step function with finite discontinuity points by Lemma 1 and Proposition 1. □

To investigate the mod M normality of β-transformation, we consider a function $\phi(x) = \exp(\frac{2\pi i}{M}\mathbf{1}_E(x))$. In the following two Lemmas, we consider more general functions $\phi(x)$ with finite discontinuity points.

Proposition 3. *For* β*-transformation, if a* G*-valued nonconstant function* $\phi(x)$ *is a step function with finite discontinuity points* $\frac{1}{\beta} \le t_1 < \cdots < t_n < 1$, *then* $\phi(x)$ *is not coboundary.*

Proof. Assume that $\phi(x)h(Tx) = h(x)$. Since $\phi(x)$ is a step function with finite discontinuity points, $h(x)$ is also a step function with finite discontinuity points. Hence there exist $0 < r \le \frac{1}{\beta}$ such that $h(x)$ is constant on $[0, r)$. There exist $x_0 \in (0, r)$ such that βx_0 is also in $(0, r)$. Since $\phi(x)h(Tx) = h(x)$, we have $\phi(x_0)h(Tx_0) = h(x_0)$. Hence $\phi(x_0) = 1$ on $[0, \frac{1}{\beta})$. Hence $h(Tx) = h(x)$ for all $x \in [0, \frac{1}{\beta})$. Since $T[0, \frac{1}{\beta}) = [0, 1)$, $h(x)$ has to be constant, i.e., $\phi(x) \equiv 1$. This contradicts to the assumption of $\phi(x)$. □

EXAMPLE 2. Let $\beta = \dfrac{\sqrt{5}+1}{2}$. For the transformation on $[0,1)$ defined by $x \to (\beta x)$, let's consider the following. Let $I = [\frac{1.5}{\beta}, 1)$, $F = \bigcup_{k=0}^{\infty} \dfrac{1}{\beta^k} I$ and $E = F \vartriangle T^{-1}F$. Then $\phi(x) = \exp(\pi i \mathbf{1}_E(x))$ is a coboundary even if the discontinuity points of $\phi(x)$ are contained in $[\dfrac{1}{\beta}, 1)$ where the cobounding function is $h(x) = \exp(\pi i \mathbf{1}_F(x))$. Hence the assumption of finite discontinuity points on $\phi(x)$ can't be omitted.

Let $\phi(x) = \exp(\frac{2\pi i}{M}\mathbf{1}_E(x))$ where E is the form of $[\frac{k}{\beta}, \frac{k+1}{\beta})$ when $\frac{k+1}{\beta} < 1$ or $[\frac{k}{\beta}, 1)$ otherwise. By the previous Propositon, $\phi(x)$ is not coboundary. Hence we have the following Theorem.

Theorem 2. *Let $\beta > 1$ be given and we have a β-expansion for nonnegative number x, i.e.,*

$$x = \epsilon_0(x) + \frac{\epsilon_1(x)}{\beta} + \frac{\epsilon_2(x)}{\beta} + \cdots$$

where $\epsilon_0(x) = [x]$, $\epsilon_1(x) = [\beta(x)]$, $\epsilon_2(x) = [\beta((\beta x))]$, and so on ($[x]$ denotes the integral part and (x) the fractional part of the real number x). Then the sequence $\{\epsilon_n(x)\}$ satisfies the mod M normality almost everywhere.

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